New fully symmetric and rotationally symmetric cubature rules on the triangle using minimal orthonormal bases

Stefanos-Aldo Papanicolopulos

Abstract
Cubature rules on the triangle have been extensively studied, as they are of great practical interest in numerical analysis. In most cases, the process by which new rules are obtained does not preclude the existence of similar rules with better characteristics. There is therefore clear interest in searching for better cubature rules.

Here we present a number of new cubature rules on the triangle, exhibiting full or rotational symmetry, that improve on those available in the literature. These rules were obtained by determining and implementing minimal orthonormal polynomial bases that can express the symmetries of the cubature rules. As shown in a specific benchmark example, this results in significantly better performance of the employed algorithm. This is the first time that a large number of rotationally symmetric rules are obtained, with these rules being in most cases of good quality and having less points than the best available fully symmetric rules.

Keywords: Cubature, triangle, fully symmetric rules, rotationally symmetric rules, symmetric polynomials
2000 MSC: Primary 65D32, Secondary 65D30

1. Introduction
Cubature, that is the numerical computation of a multiple integral, is an important method of numerical analysis, as it is of great practical interest in different applications involving integration. An extensive literature therefore exists on this topic [see e.g. 1, 2], including also compilations of specific cubature rules [3].

The present paper considers cubature rules on the triangle. This is perhaps the most studied cubature domain, with a correspondingly large body of literature a selection of which is presented here. While rules of degree up to 20, thus covering most cases of practical interest, where progressively developed by 1985 [1, 4, 5, 6], this is still an active field [7, 8, 9, 10, 11, 12, 13]. This happens for two distinct reasons, the first being that different applications require different properties of the cubature rules; the previously cited work focuses only on fully symmetric rules (which are also the easier to determine), while only a few works consider rotationally symmetric [14] or asymmetric [15, 16] rules. The second reason explaining the interest in researching new cubature rules is that almost all rules in the literature have been determined numerically using an iterative procedure, therefore the possibility that a “better” rule (matching some given requirements) may exist, for example one having fewer points (see [17] for a lower bound on the number of points for given degree). For fully symmetric rules, the fact that the “best” existing rules for degree up to 14 have indeed the minimal possible number of points was recently proved using solutions based on algebraic solving [12].

In this paper we focus on the iterative algorithm for obtaining fully symmetric cubature rules on the triangle initially proposed by Zhang et al. [10] and recently refined by Witherden and Vincent [13]. A main feature of [13] (which had already been used in [15]) is the use of an orthonormal basis instead of the typical monomial basis usually employed. Further improving upon this point, we describe here a minimal orthonormal basis for fully symmetric rules and then extend this basis to also cover the case of rules with only rotational symmetry. This results in a number of new rules that improve upon those found in the literature, especially for the rotationally symmetric case.

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The structure of the paper is as follows: after this introduction, Section 2 summarises the required theoretical background. Section 3 presents orthonormal bases for the fully symmetric case, including minimal bases, in terms of the typically used orthonormal polynomials while Section 4 presents the minimal basis in terms of symmetric polynomials. In Section 5 we extend the minimal basis to obtain a minimal basis for cubature rules with rotational symmetry. A summary of the numerical results is presented and discussed in Section 6, while the conclusions of the paper are stated in Section 7.

2. Theoretical background

A cubature rule approximates the integral of a function $f$ on a domain $\Omega$ (normalised by the domain’s area $A$) as
the weighted sum of the function’s value evaluated at a set of $n$ points $x_i$,

$$\sum_{i}^{n} w_i f(x_i) \approx \frac{1}{A} \int_{\Omega} f(x) dx \quad (1)$$

The cubature rule is of polynomial degree $\phi$ if equation (1) is exact for all polynomials of degree up to $\phi$ but not exact for at least one polynomial of degree $\phi + 1$.

Since equation (1) is linear in the function $f$, we only need to ensure that it is exact for a basis of the polynomials of degree $\phi$. The simplest such basis in two dimensions is the set of monomials $x^i y^j$ in the Cartesian coordinates $x$ and $y$ with $i + j \leq \phi$, but for the triangle another simple basis is the set of monomials $L_1 L_2 L_3^{\phi-i-j}$ expressed in term of the areal (or barycentric) coordinates $L_1$, $L_2$ and $L_3$ (with all exponents being non-negative).

In two dimensions each point contributes three unknowns (two coordinates and a weight), therefore setting in equation (1) $f$ as each of the basis polynomials for degree $\phi$ results in a polynomial system of $(\phi + 1)(\phi + 2)/2$ equations in $3n_k$ variables. The solution of this system yields the cubature point coordinates and weights defining the cubature rule.

In the general (asymmetric) case it can be quite difficult to solve the above-mentioned system even for moderate values of $\phi$, therefore some symmetry condition is imposed on the cubature points to reduce the number of unknowns. As mentioned in the introduction, these symmetries may also be a requirement of the application being considered; on the triangle, for example, full symmetry ensures that the computed approximate value of the integral is independent of the order in which the vertices are numbered.

For cubature rules on the triangle, the most commonly used symmetry is full symmetry, where if a point with areal coordinates $(L_1, L_2, L_3)$ appears in the rule, then all points resulting from permutation of the areal coordinates also appear. Depending on the number of distinct values of the areal coordinates we therefore obtain different symmetry orbits (for 1, 2 or 3 distinct values we get orbits of type 0, 1 or 2 which have 1, 3 or 6 points and contribute 1, 2 or 3 unknowns to the system of equations). Full symmetry allows for a significant reduction in the number of unknowns (roughly by a factor of 6 for larger values of $\phi$) and, through appropriate considerations, for a corresponding decrease in the number of equations.

The disadvantage of full symmetry is that in most cases it does not lead to the cubature rule with the minimal number of points for a given degree and quality. It is possible to get rules with fewer points, while still reducing the number of equations and unknowns, by requiring only rotational symmetry. In this case instead of considering all the permutations of the areal coordinates $(L_1, L_2, L_3)$ we only consider the even permutations. This results in two types of orbits: type-0 with only one point (the centroid) and type-1 with three points, therefore the number of unknowns is approximately twice that of the fully symmetric case.

3. Orthonormal bases on the triangle

3.1. A full orthonormal basis

While the monomials (in either the Cartesian or the areal coordinates) described in Section 2 are the simplest basis polynomials, they lead at higher degrees to polynomial systems which are poorly conditioned, therefore the use of an orthonormal basis has been proposed [15, 13].
A standard set of orthonormal basis polynomials on the triangle has been proposed in the literature [18, 19, 20], which we can write in the form

\[ \psi_{ij}(x) = \tilde{P}_i(d/s)\hat{P}^{(2i+1/2)}_j(1 - 2s)s' \]  

where \( \hat{P}^{(2\alpha+1)}_j = \sqrt{2n + \alpha + 1}P^{(2\alpha+1)}_j \) are scaled Jacobi polynomials and the values \( d \) and \( s \) depend on the coordinates.

Specific expressions for the \( \psi_{ij}(x) \) (and therefore for \( d \) and \( s \) in Cartesian coordinates) are given in the literature by specifying a reference triangle. Using areal coordinates, however, for which we obtain

\[ \text{Indicating by } n_\omega(\phi) \text{ the cardinality of the degree-}\phi \text{ basis and by } m_\omega(\omega) \text{ the number of basis polynomials of degree } \omega, \text{ for the full basis we easily see that} \]

\[ n_\phi(\phi) = \frac{(\phi + 1)(\phi + 2)}{2}, \quad m_\phi(\omega) = \omega + 1 \]

3.2. Objective orthonormal bases for fully symmetric rules

While a full basis is needed to represent all polynomials of degree \( \phi \), a reduced basis can be used when considering fully symmetric cubature rules, as this restricts the form of the system of polynomial equations to be solved. Witherden and Vincent [13] propose an “objective” basis, that is a subset of the full basis that can still represent the polynomial fully symmetric cubature rules, as this restricts the form of the system of polynomial equations to be solved. Witherden

\[ s = L_2 + L_1, \quad d = L_2 - L_1 \]

An interesting property of the basis polynomials expressed in terms of \( d \) and \( s \) is that the \( \psi_{ij} \) are the Gram-Schmidt orthonormalisation of the monomials \( d(−s)^i \) taken in increasing graded lexicographic order.

3.2. Objective orthonormal bases for fully symmetric rules

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\[ \hat{P}^d_{\omega} = \{\psi_{ij}(x) \mid 0 \leq i \leq \phi, \quad 0 \leq j \leq \phi - i\} \]

In equation (6), due to the limits on \( j \), the actual limits on \( i \) are 0 \( \leq i \leq \lfloor \phi/2 \rfloor \), and it is therefore easy to show that

\[ n_\omega(\phi) = \left[ \frac{\phi(\phi + 1)}{4} \right] - \frac{1}{2}n_\phi(\phi), \quad m_\omega(\omega) = 1 + \lfloor \omega/2 \rfloor \]

As already noted in [13], this objective basis is not optimal as its modes are not completely independent. Indeed, while the basis \( \hat{P}^d_{\omega} \) is an objective basis, it is interesting to note that there is no obvious reason why the specific \( \psi_{ij} \) polynomials were omitted. It is actually possible to have other objective bases with the same cardinality that use another subset of the full basis, such as

\[ \hat{P}^e_{\omega} = \{\psi_{ij}(x) \mid 0 \leq i \leq \phi, \quad 0 \leq j \leq \min(i, \phi - i)\} \]

To further reduce the cardinality of the basis, we first note that for a symmetric orbit we will be adding the polynomials \( \psi_{ij}(d, s) \) and \( \psi_{ij}(−d, s) \), which correspond to points with barycentric coordinates \( (L_1, L_2, L_3) \) and \( (L_2, L_1, L_3) \). If \( i \) is odd, however, \( \psi_{ij}(d, s) \) is also odd with respect to \( d \) and therefore all \( \psi_{ij} \) with odd \( i \) can be removed from the objective basis \( \hat{P}^e_{\omega} \) to obtain the “even” basis

\[ \hat{P}^e_{\omega} = \{\psi_{ij}(x) \mid 0 \leq i \leq \lfloor \phi/2 \rfloor, \quad i \leq j \leq \phi - i, \ i \text{ even}\} \]

for which we obtain

\[ n_e(\phi) = \left[ \frac{\phi(\phi + 3)}{8} \right] - \frac{1}{4}n_\phi(\phi), \quad m_e(\omega) = 1 + \lfloor \omega/4 \rfloor \]

\(^1\)Obviously different even basis can be obtained, e.g. starting from the basis \( \hat{P}^e_{\omega} \).
Expressing the basis polynomials in terms of \( d \) and \( s \), and then \( d \) and \( s \) in terms of the areal coordinates, has therefore the advantage of making obvious the symmetry and antisymmetry of the basis polynomials with respect to exchange of two vertices.

As will also be discussed in Section 4, a symmetric basis with even lower cardinality is possible. Indeed for the minimal basis we get \[17\]
\[
\begin{align*}
n_m(\phi) &= \left\lceil \frac{(\phi + 3)^2}{12} + \frac{1}{4} \right\rceil - \frac{1}{6} n_f(\phi), \quad m_m(\omega) = 1 + \lfloor \omega/6 \rfloor - \kappa_0(\omega) \tag{11}
\end{align*}
\]
where
\[
\kappa_0(\omega) = \begin{cases} 1 & \text{if } \omega \text{ mod } a = 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}
\]

To construct a minimal basis of degree \( \phi \), it suffices to choose a subset of the even base of the same degree \( \phi \) so that the number of polynomials \( \psi_{ij} \) with \( i + j = \omega \) is given by \( m_m(\omega) \) as defined in equation (11), that is
\[
\tilde{\mathbf{P}}^\phi_m = \{ \psi_{ij}(x) \in \tilde{\mathbf{P}}^\phi \mid \#\{\psi_{ij} \mid i + j = \phi\} = m_m(\omega) \} \tag{13}
\]
While we do not provide here a proof that \( \tilde{\mathbf{P}}^\phi_m \) is indeed an objective basis, it is relatively easy to check this for given values of \( \phi \) using a computer algebra system.

It is easy to create two such minimal objective bases as
\[
\begin{align*}
\tilde{\mathbf{P}}^\phi_m &= \{ \psi_{(i\phi,\omega-2\phi)}(x) \mid 0 \leq i \leq m_m(\omega) - 1, \ 0 \leq \omega \leq \phi \} \tag{14} \\
\tilde{\mathbf{P}}^\phi_m &= \{ \psi_{(i\phi,\omega-2\phi)}(x) \mid \lfloor \omega/2 \rfloor - (m_m(\omega) - 1) \leq i \leq \lfloor \omega/2 \rfloor, \ 0 \leq \omega \leq \phi \} \tag{15}
\end{align*}
\]
which can alternatively be written as
\[
\begin{align*}
\tilde{\mathbf{P}}^\phi_m &= \{ \psi_{ij}(x) \mid 0 \leq i \leq \lfloor \phi/3 \rfloor, \ 2i \leq j \leq \phi - i, \text{ even}, \ j \neq 2i + 1 \} \tag{16} \\
\tilde{\mathbf{P}}^\phi_m &= \{ \psi_{ij}(x) \mid 0 \leq i \leq \phi, \ 0 \leq j \leq \min(\phi - i, i/2), \text{ even} \} \tag{17}
\end{align*}
\]
Other minimal objective can also be derived. For example, minimising the maximum of \( i \) and \( j \) for a given \( \phi \) (trying to reduce number of computations and round-off error), results in the following basis
\[
\begin{align*}
\tilde{\mathbf{P}}^\phi_m &= \{ \psi_{ij}(x) \mid 0 \leq i \leq 2\lfloor \phi/3 \rfloor + 2\kappa_0(\phi - 1), \ 2\lfloor i/4 \rfloor \leq j \leq \min(\phi - i, 2i), \text{ even} \} \tag{18}
\end{align*}
\]

4. An orthonormal basis for symmetric polynomials

We present here the derivation of an orthonormal basis to be used in computing fully symmetric cubature rules on the triangle, which makes full use of the imposed symmetry by using symmetric polynomials. More details on this approach can be found in [12].

We consider, without loss of generality, a type-2 orbit in a fully symmetric cubature rule on the triangle. This orbit consists of a point with areal coordinates \((L_1, L_2, L_3)\) and the five other points resulting from permutation of these coordinates (which, for the type-2 orbit, are all distinct). Using equation (1) for a polynomial \( \tilde{f}(L_1, L_2, L_3) \) in the areal coordinates, yields only sums of the form\(^2\)
\[
T_s = \tilde{f}(L_1, L_2, L_3) + \tilde{f}(L_3, L_1, L_2) + \tilde{f}(L_2, L_3, L_1) + \tilde{f}(L_1, L_3, L_2) + \tilde{f}(L_2, L_1, L_3) + \tilde{f}(L_3, L_2, L_1) \tag{19}
\]
Therefore, for fully symmetric rules, the left hand side of (1) only contains symmetric polynomials [21] in the areal coordinates. According to the fundamental theorem of symmetric polynomials, these can therefore be expressed

\(^2\)These sums appear multiplied by the weight corresponding to the orbit being considered.
as polynomials in the elementary symmetric polynomials 
\( \tilde{L}_1 = -(L_1 + L_2 + L_3), \tilde{L}_2 = L_1L_2 + L_2L_3 + L_3L_1 \) and 
\( \tilde{L}_3 = -L_1L_2L_3 \) (noting, however, that in this case \( \tilde{L}_1 = -1 \)).

It is therefore easily seen that instead of considering all polynomials of degree \( \phi \), or at least all polynomials in a basis of degree \( \phi \), we only need to consider a symmetric basis consisting of the largest possible number of linearly independent polynomials in \( \tilde{L}_2 \) and \( \tilde{L}_3 \) of weighted total degree less or equal to \( \phi \) (with a weight 2 for \( \tilde{L}_2 \) and a weight 3 for \( \tilde{L}_3 \)).

The simplest such symmetric basis consists of the monomials \( \tilde{L}_2^{i} \tilde{L}_3^{j} \) with \( 2i + 3j \leq \phi \), that is

\[
\mathcal{Q}_{\phi}^s = \{ \tilde{L}_2^{i} \tilde{L}_3^{j} | 2i + 3j \leq \phi \} \tag{20}
\]

For the basis \( \mathcal{Q}_{\phi}^s \) (and indeed for any symmetric basis) we easily obtain

\[
n_s(\phi) = \left[ \frac{1}{12} \left( \frac{\phi + 3}{4} \right)^2 + \frac{1}{4} \right] \approx \frac{1}{6} n_s(\phi), \quad m_s(\omega) = 1 + [\omega/6] - \kappa_s(\omega) \tag{21}
\]

with \( \kappa_s(\omega) \) already defined in equation (12), since we already used this result in Section 3.2 for the minimal objective basis.

The monomial symmetric basis given in equation (20) is obviously not orthogonal. To obtain an orthonormal symmetric basis \( \mathcal{Q}_{\phi}^o \), we can orthonormalise the monomials in the basis \( \mathcal{Q}_{\phi}^s \). While the orthonormalisation can be done numerically, to minimise numerical errors we choose here to perform it analytically with a computer algebra system. This also allows for an efficient implementation of a multivariate Horner scheme [22]. Note that monomials must be considered in weighted lexicographic order to obtain orthonormal bases which include the bases of lower degree. Chabysheva et al. [23] have recently discussed an orthonormalisation of this type, but their use of Cartesian coordinates leads to polynomials with a significantly larger number of terms, and of higher degree.

It is important to note that the minimal objective basis and the orthonormal symmetric basis are not bases of the same polynomials. Indeed, the minimal objective basis is not a proper basis of the symmetric polynomials (and indeed does not consist of symmetric polynomials); we need to sum the values of the basis polynomials \( \phi_{ij} \) on all points of the orbit to obtain a basis for the symmetric polynomials (which is then no-longer orthogonal). The orthonormal symmetric basis, on the other hand, is a proper orthonormal basis of the symmetric polynomials.

It is not clear whether the fact that the symmetric basis is really orthonormal would by itself provide better efficiency or accuracy in obtaining results; the obvious advantage of the symmetric orthonormal basis is that it requires only a single evaluation of the basis polynomials instead of the six evaluations (for type-2 orbits) required by the minimal objective basis. On the other hand, the advantage of the minimal objective basis is that it is expressed in analytical form (in terms of the Jacobi polynomials), making it easier to implement in a computer code. Additionally, the product form of the polynomials in the objective basis allow for their more efficient evaluation.

Using either type of basis will obviously result in a polynomial system with solutions that correspond to the same set of cubature rules. It is however of theoretical interest that there can be real solutions of the polynomial system expressed in terms of the symmetric polynomials that correspond to cubature rules with real weights but complex point coordinates.

5. Rotational symmetry

We consider now a rule with rotational symmetry. The system of polynomial equations will now contain, instead of the terms \( T_s \), in equation (22), polynomials in the areal coordinates of the form

\[
T_s = \hat{f}(L_1, L_2, L_3) + \hat{f}(L_3, L_1, L_2) + \hat{f}(L_2, L_3, L_1) \tag{22}
\]

This can be written as

\[
T_s = \frac{T_s + T_a}{2} \tag{23}
\]

where \( T_s \) is the symmetric polynomial given in equation (22) and \( T_a \) is the antisymmetric polynomial

\[
T_a = f(L_1, L_2, L_3) + f(L_3, L_1, L_2) + \hat{f}(L_2, L_3, L_1) - f(L_1, L_3, L_2) - \hat{f}(L_1, L_1, L_3) - \hat{f}(L_3, L_2, L_1) \tag{24}
\]
The orthonormal basis for symmetric polynomials presented in Section 4 could also be implemented in Fortran 95. The two implementations are not directly comparable, and their relative performance requires however more extensive changes to the code. It was therefore found simpler to implement the algorithm in a new code in Fortran 95. The two implementations are not directly comparable, and their relative performance depends among others on the minimisation solver used and its parameters.

As already mentioned, $T_s$ can be expressed as a polynomial in the symmetric polynomials $L_2$ and $L_3$. The antisymmetric polynomial, on the other hand, can be expressed as the product of a symmetric polynomial (in $L_2$ and $L_3$) with the alternating polynomial $L_A$,

$$L_A = (L_1 - L_2)(L_1 - L_3)(L_2 - L_3)$$

(25)

Considering that $L_A$ is of degree 3 in the areal coordinates, we see that a rotationally symmetric basis using monomials is given by

$$Q^\phi_e = \{ L_1^2 L_2^2 L_3^2 | 2i + 3j + 3k \leq \phi, \ k \in \{0, 1\} \}$$

(26)

from which we can obtain [17]

$$n_r(\phi) = 1 + \left(\frac{\phi + 3}{6}\right) \sim \frac{1}{3} n_r(\phi), \quad m_r(\omega) = 1 + \lfloor \omega/3 \rfloor - \kappa_3(\omega)$$

(27)

As in the fully symmetric case, the monomials in the basis $Q^\phi_e$ can be orthonormalised to obtain an orthonormal rotationally symmetric basis $\tilde{Q}_e^\phi$.

It is also possible to obtain minimal objective bases for rotationally symmetric rules in terms of the basis polynomials $\psi_{ij}$. After some calculations it can be seen that these will consist of a minimal objective basis for fully symmetric rules plus a set of basis polynomials $\psi_{ij}$ with $i$ odd. The basis in equations (16) and (17) yield the following minimal objective bases for rotationally symmetric rules

$$\tilde{\varphi}_f^\phi = \{ \psi_{ij}(x) | 0 \leq i \leq \lfloor \phi/3 \rfloor, 2i \leq j \leq \phi - i, j \neq 2i + 1 \}$$

(28)

$$\tilde{\varphi}_e^\phi = \{ \psi_{ij}(x) | 0 \leq i \leq \phi, 0 \leq j \leq \min(\phi - i, [i/2] - \kappa_2(i)) \}$$

(29)

6. Results

Witherden and Vincent [13] have develop the C++ code polyquad to compute fully symmetric cubature rules (on the triangle and on other domains) using objective orthonormal bases. The objective bases for fully symmetric rules proposed in Section 3 can be easily implemented with minor modifications to the existing polyquad code.

Table 1 shows the performance of polyquad for the case of rules of degree $\phi = 15$ with 49 points, considering four different combination of orbits (of which only $[1, 4, 6]$ and $[1, 6, 5]$ actually yield a cubature rule). The performance is expressed as the number of trial rules evaluated per second, and represent the average of 20 different runs with at least 100 rules evaluated per run. While the exact values depend on the compiler and hardware used, the results in Table 1 show the relative performance of different bases, with the best results obtained by the minimal objective basis $\tilde{\varphi}_{m_2}^\phi$.

The orthonormal basis for symmetric polynomials presented in Section 4 could also be implemented in polyquad, requiring however more extensive changes to the code. It was therefore found simpler to implement the algorithm in a new code in Fortran 95. The two implementations are not directly comparable, and their relative performance depends among others on the minimisation solver used and its parameters.

As implemented, the new code performed slightly better for the problem described above ($\phi = 15, 49$ points), achieving for example a rate of 3.98 rules/sec for the $[1, 4, 6]$ rules. Profiling the code revealed that approximately 90% of the total run time was spent in the subroutine evaluating the basis polynomials. It is therefore clear that the performance obtained using the orthonormal basis for symmetric polynomials critically depends on the efficiency with

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Table 2: Number of points and quality for new fully symmetric cubature rules

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Table 3: Number of points and quality for new rotationally symmetric cubature rules

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</table>

which the basis polynomials can be evaluated. It is actually expected that appropriate optimisation of the computation of the objective basis could lead to faster evaluation than in the case of the symmetric basis.

In presenting specific rules we are interested in the “quality” of the rule, which is expressed using two letters. The first letter is ‘P’ if all weights are positive (otherwise it’s ‘N’) and the second is ‘I’ if all points of the rule lie within the triangle (otherwise it’s ‘O’). We therefore obtain PI, NI, PO and NO rules (in decreasing order of quality).

Table 2 presents the newly obtained fully symmetric rules that improve on the ones in the literature either on quality or on number of points. As mentioned in the introduction, it is already known [12] that no improved rules could be found for $\phi \leq 14$. For PI rules, no improved results were obtained for $\phi \leq 20$.

The fully symmetric case has been extensively studied in the literature, especially for degrees up to 20, therefore only a few new results were found. The implementation of the rotationally symmetric basis, on the other hand, yielded a significant number of new rules that improve in some way on the results previously available. These new rules are summarised in Table 3, starting from degree 10 as for lower degrees no improved rules were obtained. Most of the obtained rules are of PI quality. However, when NI (or PO) rules with fewer points were encountered these are also mentioned. Rules of PI quality are not presented for degrees 12 and 14 as they did not have fewer points than the best fully symmetric PI rules of the same degree (with 33 and 42 points respectively).

As already mentioned, all the rules summarised in Table 3 are new (to the author’s knowledge) and improve in some way on previously available rules. More specifically, for degrees 12, 14–19 and 21–25 we obtain rules with fewer points than any previously available rule of the same degree.\footnote{A degree-15 fully symmetric rule with 46 points is obtained in [12], but it has complex coordinates.} For degree 16 we obtain a PI rule with fewer points than previous PI rules of the same degree, but we also obtain a PO rule with fewer points. Finally, for degrees 10, 11, 13, 18 and 20 we obtain a PI rule with the same number of points already found in the literature, but the new rules are rotationally symmetric instead of asymmetric.

The coordinates and weights for the rules summarised in Tables 2 and 3, computed to double precision, are provided as ancillary material accompanying this paper. While in most cases many rules were computed for given degree, number of points and quality, only one rule of each type is presented. This rule was selected to minimise the ratio of maximum to minimum weight, avoiding however (for PI rules) rules with points almost on the boundary.

Rules of increasing degree take longer to be computed, and are of decreasing interest in practical applications. For this reason, only rules of degree up to 25 have been considered here. There is however no indication that rules of higher degree cannot be obtained using the same method, given enough computation time. It is on the other hand also possible that improved rules may be obtained even for the degrees considered here.
7. Conclusions

We have presented in this paper minimal orthonormal polynomial bases on the triangle for computing fully symmetric and rotationally symmetric cubature rules. These bases can be either “objective” bases, that is subsets of the complete polynomial basis that yield the required symmetry, or true fully/rotationally symmetric bases in terms of the symmetric elementary polynomials (and the alternating polynomial for rotational symmetry).

As these bases are minimal, they allow for more efficient computation of cubature rules. We therefore present a number of new rules that improve, in some aspects, on the rules available in the literature. Especially for the rotationally symmetric rules, a large number of new rules is obtained, most of which of PI quality. These results represent a significant improvement over the very few results of this type previously available.

Further optimisation of the implementation of the algorithm could be possible, for example by implementing a more efficient computation of the basis polynomials or by employing a different optimisation solver to solve the polynomial equations. This is currently a work in progress, as it would allow more efficient computation of rules of higher degree, should they be needed, and especially more efficient computation of cubature rules on the tetrahedron.

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References