Everything old is new again:
Quoted Domain Specific Languages

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Abstract
We describe a new approach to domain specific languages (DSLs), called Quoted DSLs (QDSLs), that resurrects two old ideas: quotation, from McCarthy’s Lisp of 1960, and the subformula property, from Gentzen’s natural deduction of 1935. Quoted terms allow the DSL to share the syntax and type system of the host language. Normalising quoted terms ensures the subformula property, which guarantees that one can use higher-order types in the source while guaranteeing first-order types in the target, and enables using types to guide fusion. We test our ideas by re-implementing Feldspar, which was originally implemented as an Embedded DSL (EDSL), as a QDSL; and we compare the QDSL and EDSL variants.

Categories and Subject Descriptors  D.1.1 [Applicative (Functional) Programming]; D.3.1 [Formal Definitions and Theory]; D.3.2 [Language Classifications]: Applicative (functional) languages

Keywords domain-specific language, DSL, EDSL, QDSL, embedded language, quotation, normalisation, subformula property

1. Introduction

Don’t throw the past away
You might need it some rainy day
Dreams can come true again
When everything old is new again

– Peter Allen and Carole Sager

Implementing domain-specific languages (DSLs) via quotation is one of the oldest ideas in computing, going back at least to McCarthy’s Lisp, which was introduced in 1960 and had macros as early as 1963. Today, a more fashionable technique is Embedded DSLs (EDSLs), which may use shallow embedding, deep embedding, or a combination of the two. Our goal in this paper is to reinvigorate the idea of building DSLs via quotation, by introducing an approach we dub Quoted DSLs (QDSLs). A key feature of QDSLs is the use of normalisation to ensure the subformula property, first proposed by Gentzen in 1935.

Imitation is the sincerest of flattery.
— Charles Caleb Colton

Cheney et al. (2013) describe a DSL for language-integrated query in F# that translates into SQL. Their technique depends on quotation, normalisation of quoted terms, and the subformula property—an approach which we here dub QDSL. They conjecture that other DSLs might benefit from the same technique, particularly those that perform staged computation, where host code at generation-time computes target code to be executed at run-time.

Generaliser starts at two. Here we test the conjecture of Cheney et al. (2013) by reimplementing the EDSL Feldspar (Axelsson et al. 2010) as a QDSL. We describe the key features of the design, and show that the performance of the two versions is comparable. We compare the QDSL and EDSL variants of Feldspar, and assess the tradeoffs between the two approaches.

Davies and Pfenning (2001) also suggest quotation as a foundation for staged computation, and note a propositions-as-types connection between quotation and a modal logic; our type $\alpha$ corresponds to their type $\Box \alpha$. They also mention in passing the utility of normalising quoted terms, although they do not mention the subformula property.

Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout.
— Gerhard Gentzen

The .NET Language-Integrated Query (LINQ) framework as used in C# and F# (Meijer et al. 2006, Sync 2006), and the Lightweight Modular Staging (LMS) framework as used in Scala (Rompf and Odersky 2010), exhibit overlap with the techniques described here. Notably, they use quotation to represent staged DSL programs, and they make use to a greater or lesser extent of normalisation. In F# LINQ quotation is indicated in the normal way (by writing quoted programs inside special symbols), while in C# LINQ and Scala LMS quotation is indicated by type inference (quoted terms are given a special type).

Our approach exploits the fact that normalised terms satisfy the subformula property, first introduced in the context of natural deduction by Gentzen (1935), and improved by Prawitz (1965).

The subformulas of a formula are its subparts; for instance, the subformulas of $A \to B$ are the formula itself and the subformulas of $A$ and $B$. The subformula property states that every proof can be put into a normal form where the only propositions that appear in the proof are subformulas of the hypotheses and conclusion of the proof. Applying the principle of Propositions as Types (Howard 1980, Wadler 2015), the subformula property states that every lambda term can be put into a normal form where the only types that appear in the term are subformulas of the types of the free variables and the type of the term itself.
The subformula property provides users of the DSL with useful guarantees, such as the following:

- they may write higher-order terms while guaranteeing to generate first-order code;
- they may write a sequence of loops over arrays while guaranteeing to generate code that fuses those loops;
- they may write intermediate terms with nested collections while guaranteeing to generate code that operates on flat data.

The first and second are used in this paper, and are key to generating C; while the first and third are used by Cheney et al. (2013) and are key to generating SQL.

The subformula property is closely related to conservativity. A conservativity result expresses that adding a feature to a system of logic, or to a programming language, does not make it more expressive. Consider intuitionistic logic with conjunction; conservativity states that adding implication to this logic proves no additional theorems that can be stated in the original logic. Such a conservativity result is an immediate consequence of the subformula property; since the hypotheses and conjunction of the proof only mention conjunction, any proof, even if it uses implication, can be put into a normal form that only uses conjunction. Equivalently, any lambda calculus term that mentions only pair types in its free variables and result, even if it uses functions, can be put in a normal form that only uses pairs. Such a result is related to the first bullet point above; see Proposition 4.4 in Section 4.

As another example, the third bullet point above corresponds to a standard conservativity result for databases, namely that nested queries are no more expressive than flat queries (Wong 1996). This conservativity result, as implied by the subformula property, is used by Cheney et al. (2013) to show that queries that use intermediate nesting can be translated to SQL, which only queries flat tables and does not support nesting of data.

The subformula property holds only for terms in normal form. Previous work, such as Cheney et al. (2013) uses call-by-name normalisation algorithm that may cause computations to be repeated. Here we present call-by-value and call-by-need normalisation algorithms that guarantee to preserve sharing of computations. We also present a sharpened version of the subformula property, which we apply to characterise the circumstances under which a QDSL may guarantee to generate first-order code.

This is the short and the long of it. — Shakespeare

The contributions of this paper are:

- To introduce QDSLs as an approach to building DSLs based on quotation, normalisation of quoted terms, and the subformula property by presenting the design of a QDSL variant of Feldspar (Section 2).
- To measure QDSL and EDSL implementations of Feldspar, and show they offer comparable performance (Section 5).
- To present normalisation algorithms for call-by-value and call-by-need that preserve sharing, and to formulate a sharpened version of the subformula property and apply it to characterise when higher-order terms normalise to first-order form (Section 4).
- To compare the QDSL variant of Feldspar with the deep and shallow embedding approach used in the EDSL variant of Feldspar, and show they offer tradeoffs with regard to ease of use (Section 5).

Section 6 describes related work, and Section 7 concludes.

Our QDSL and EDSL variants of Feldspar and benchmarks are available at [github.com/shayan-najd/QFeldspar](https://github.com/shayan-najd/QFeldspar).

2. Feldspar as a QDSL

Feldspar is an EDSL for writing signal-processing software, that generates code in C (Axelsson et al. 2010). We present a variant, QDSL Feldspar, that follows the structure of the previous design closely, but using the methods of QDSL rather than EDSL. Section 5 compares the QDSL and EDSL designs.

2.1 The top level

In QDSL Feldspar, our goal is to translate a quoted term to C code. The top-level function has the type:

\[
qdsl :: (\text{Rep } a, \text{Rep } b) \Rightarrow Ql (a \Rightarrow b) \rightarrow C
\]

Here \(Ql\) represents a Haskell term of type \(a\), its quoted representation, and \(C\) represents code in C. The top-level function

Good artists copy, great artists steal. — Picasso
expects a quoted term representing a function from type $a$ to type $b$, and returns C code that computes the function.

Not all types representable in Haskell are easily representable in C. For instance, we do not wish our target C code to manipulate higher-order functions. The argument type $a$ and result type $b$ of the main function must be representable, which is indicated by the type-class restrictions $\text{Rep } a$ and $\text{Rep } b$. Representable types include integers, floats, and pairs where the components are both representable.

\begin{verbatim}
instance Rep Int
instance Rep Float
instance (Rep a, Rep b) \Rightarrow Rep (a,b)
\end{verbatim}

2.2 A first example

Let’s begin with the “hello world” of program generation, the power function. Since division by zero is undefined, we arbitrarily choose that raising zero to a negative power yields zero. Here is an optimised power function represented using QDSL:

\begin{verbatim}
power :: Int \rightarrow \text{Qt}(\text{Float} \rightarrow \text{Float})
power 0 = 1
power n = if n < 0 then
  [\lambda x \rightarrow if x == 0 then 0 else 1 / (n\text{\textit{power} \{-n\} x})]
else if n == 0 then
  [\lambda x \rightarrow 1]
else if even n then
  [\lambda x \rightarrow \text{let } y = n\text{\textit{power} \{2\} x in } y \times y]
else
  [\lambda x \rightarrow x \times n\text{\textit{power} \{n - 1\} x}]
\end{verbatim}

The typed quasi-quoting mechanism of Template Haskell is used to indicate which code executes at which time. Unquoted code executes at generation-time while quoted code executes at run-time. Quoting is indicated by $[\ldots]$ and unquoting by $\text{\textit{Qt}}(\ldots)$.

Invoking \textit{qdsl} ($\text{\textit{power} \{-6\}}$) generates code to raise a number to its $-6$ power. Evaluating $\text{\textit{power} \{-6\}}$ yields the following:

\begin{verbatim}
[\lambda x \rightarrow if x == 0 then 0 else 1 / (\lambda x \rightarrow let \ y = (\lambda x \rightarrow x \times (\lambda x \rightarrow 1) x) \ y \times y \ y) x]
\end{verbatim}

Normalising as described in Section \[4\] with variables renamed for readability, yields the following:

\begin{verbatim}
[\lambda u \rightarrow if u == 0 then 0 else
 let v = u \times 1 in
 let w = v \times v in
 1 / (w \times w)]]
\end{verbatim}

With the exception of the top-level term, all of the overhead of lambda abstraction and function application has been removed; we explain below why this is guaranteed by the subformula property. From the normalised term it is easy to generate the final C code:

\begin{verbatim}
float prog (float u) {
  float v, float w, float r;
  if (u == 0.0) {
    r = 0.0;
  } else {
    v = (u * 1.0);
    w = (u * (v * v));
    r = (1.0f / (w * w));
  }
  return r;
}
\end{verbatim}

By default, we always generate a routine called \textit{prog}; it is easy to provide the name as an additional parameter if required.

Depending on your point of view, quotation in this form of QDSL is either desirable, because it makes manifest the staging, or undesirable because it is too noisy. QDSL enables us to “steal” the entire syntax of the host language for our DSL. In Haskell, an EDSL can use the same syntax for arithmetic operators, but must use a different syntax for equality tests and conditionals, as explained in Section \[5\].

Within the quotation brackets there appear lambda abstractions and function applications, while our intention is to generate first-order code. How can the QDSL Feldspar user be certain that such function applications do not render transformation to first-order code impossible or introduce additional runtime overhead? The answer is the subformula property.

2.3 The subformula property

Gentzen’s subformula property guarantees that any proof can be normalised so that the only formulas that appear within it are subformulas of one of the hypotheses or of the conclusion of the proof. Viewed through the lens of Propositions as Types, Gentzen’s subformula property guarantees that any term can be normalised so that the type of each of its subterms is a subformula of either the type of one of its free variables (corresponding to hypotheses) or of the term itself (corresponding to the conclusion). Here the subformulas of a type are the type itself and the subformulas of its parts, where the parts of $a \rightarrow b$ are $a$ and $b$, the parts of $(a, b)$ are $a$ and $b$, and types \textit{Int} and \textit{Float} have no parts. (See Proposition \[4.2\].)

Further, it is easy to adapt the original proof to guarantee a sharpened subformula property: any term can be normalised so that the type of each of its proper subterms is a proper subformula of either the type of one of its free variables (corresponding to hypotheses) or the term itself (corresponding to the conclusion). Here the proper subterms of a term are all subterms save for free variables and the term itself, and the proper subformulas of a type are all subformulas save for the type itself. In the example of the previous subsection, the sharpened subformula property guarantees that after normalisation a closed term of type $\text{float} \rightarrow \text{float}$ will only have proper subterms of type $\text{float}$, which is indeed true for the normalised term. (See Proposition \[4.3\].)

The subformula property depends on normalisation of terms, but complete normalisation is not always possible or desirable. The extent of normalisation may be controlled by introducing uninterpreted constants. In particular, we introduce the uninterpreted constant

\begin{verbatim}
save :: Rep a \Rightarrow a \rightarrow a
\end{verbatim}

of arity 1, which is equivalent to the identity function on representable types. Unfolding of an application $L \ M$ can be inhibited by rewriting it in the form \textit{save} $L \ M$, where $L$ and $M$ are arbitrary terms. A use of \textit{save} appears in Section \[2.6\]. In a context with recursion, we take

\begin{verbatim}
fix :: (a \rightarrow a) \rightarrow a
\end{verbatim}

as an uninterpreted constant.

2.4 A second example

In the previous code, we arbitrarily chose that raising zero to a negative power yields zero. Say that we wish to exploit the \textit{Maybe} type of Haskell to refactor the code, by separating identification of the exceptional case (negative exponent of zero) from choosing a value for this case (zero). We decompose \textit{power} into two functions \textit{power’} and \textit{power’’}, where the first returns \textit{Nothing} in the exceptional case, and the second maps \textit{Nothing} to a suitable value.

The \textit{Maybe} type is a part of the Haskell standard prelude.
Here is the refactored code.

data Maybe a = Nothing | Just a
maybe :: b → (a → b) → Maybe a → b
return :: a → Maybe a
( >>= ) :: Maybe a → (a → Maybe b) → Maybe b

Again, the subformula property plays a key role. As explained in Section 2, primitives of the language to be compiled, such as (x) and while, are treated as free variables or constants of a given arity. As described in Section 4, we can ensure that after normalisation every occurrence of while has the form

\[
\text{while } (\lambda s \to \cdots ) (\lambda s \to \cdots ) (\cdots )
\]

where the first ellipses has type Bool, and both occurrences of the bound variable s and the second and third ellipses all have the same type, that of the state of the while loop.

Unsurprisingly, and in accord with the subformula property, each occurrence of while in the normalised code will contain subterms with the type of its state. The restriction of state to representable types increases the utility of the subformula property. For instance, since we have chosen that Maybe is not a representable type, we can ensure that any top-level function without Maybe in its type will normalise to code not containing Maybe in the type of any subterm. In particular, Maybe cannot appear in the state of a while loop, which is restricted to representable types. An alternative choice is possible, as we will see in the next section.

2.6 Arrays

A key feature of Feldspar is its distinction between two types of arrays, manifest arrays, Arr, which may appear at run-time, and "pull arrays", Vec, which are eliminated by fusion at generation-time. Again, we exploit the subformula property to ensure no subterms of type Vec remain in the final program.

The type Arr of manifest arrays is simply Haskell’s array type, specialised to arrays with integer indices and zero-based indexing. The type Vec of pull arrays is defined in terms of existing types, as a pair consisting of the length of the array and a function that given an index returns the array element at that index.

data Vec a = Vec Int (Int → a)

Values of type Arr are representable, assuming that the element type is representable, while values of type Vec are not representable.

instance (Rep a) ⇒ Rep (Arr a)

For arrays, we assume the following primitive operations.

\[
\text{mkArr :: } (\lambda a \to \text{Vec } a) \to \text{Int } \to \text{Arr } a
\]

\[
\text{lnArr :: } (\lambda a \to \text{Arr } a) \to \text{Int } \to \text{Int}
\]

\[
\text{ixArr :: } (\lambda a \to \text{Arr } a) \to \text{Arr } a \to \text{Int}
\]

The first populates a manifest array of the given size using the given indexing function, the second returns the length of the array, and the third returns the array element at the given index. Array components must be representable.

We define functions to convert between the two representations in the obvious way.

\[
\text{toVec :: } \lambda a \to \text{Vec } a \to \text{Arr } a
\]

\[
\text{fromVec :: } \lambda a \to \text{Arr } a \to \text{Vec } a
\]

It is straightforward to define operations on vectors, including combining corresponding elements of two vectors, summing the elements of a vector, dot product of two vectors, and norm of a vector. When combining two vectors, the length of the result is the minimum of the lengths of the arguments.

minim :: Ord a ⇒ Int (a → a → a)

minim = […]
The third of these uses the for loop defined in Section 2.3. Our final function cannot accept Vec as input, since the Vec type is not representable, but it can accept Arr as input. For instance, if we invoke qsdl on

\[
[[\text{let } \lambda \rightarrow \text{sqrt } (\text{sum Vec } v)v]]
\]

the quoted term normalises to

\[
[[\text{let } \lambda \rightarrow \text{sqrt } (\text{sum Vec } v)v]]
\]

from which it is easy to generate C code.

The vector representation makes it easy to define any function where each vector element is computed independently, such as the examples above, vector append (appVec) and creating a vector of one element (uniVec), but is less well suited to functions with dependencies between elements, such as computing a running sum.

Types and the subformula property help us to guarantee fusion. The subformula property guarantees that all occurrences of Vec must be eliminated, while occurrences of Arr will remain. There are some situations where fusion is not beneficial, notably when an intermediate vector is accessed many times, in which case fusion will cause the elements to be recomputed. An alternative is to materialise the vector as an array with the following function.

\[
\text{memorise :: Rep a \Rightarrow Qt (Vec a \rightarrow Vec a)}
\]

\[
\text{memorise = [[\text{let } \lambda \rightarrow \text{sqrt } (\text{sum Vec } v)v]]}
\]

Here we interpose save, as defined in Section 2.3, to forestall the fusion that would otherwise occur. For example, if

\[
\text{blur :: Qt (Vec Float \rightarrow Vec Float)}
\]

\[
\text{blur = [[\text{let } \lambda \rightarrow \text{sqrt } (\text{sum Vec } v)v]]}
\]

computes the geometric mean of adjacent elements of a vector, then one may choose to compute either

\[
[[\text{let } \lambda \rightarrow \text{sqrt } (\text{sum Vec } v)v]]
\]

with different trade-offs between recomputation and memory use. Strong guarantees for fusion in combination with memorise give the programmer a simple interface which provides powerful optimisations combined with fine control over memory usage.

We have described the application of the subformula to array fusion as based on “pull arrays” (Svenningsson and Axelsen 2012), but the same technique should also apply to other techniques that support array fusion, such as “push arrays” (Claessen et al. 2012).

3. Implementation

The original EDSL Feldspar generates values of a GADT (called Dp in Section 5), with constructs that represent while and manifest arrays similar to those above. A backend then compiles values of type Dp a to C code. QDSL Feldspar provides a transformer from Qt a to Dp a, and shares the EDSL Feldspar backend.

The transformer from Qt to Dp performs the following steps.

- In any context where a constant c is not fully applied, it replaces c with λt. c t. It replaces identifiers connected to the type Maybe, such as return, (==), and maybe, by their definitions.
- It normalises the term to ensure the subformula property, using the rules of Section 4. The normaliser supports a limited set of types, including tuples, Maybe, and Vec.
- It performs simple type inference, which is used to resolve overloading. Overloading is limited to a fixed set of cases, including overloading arithmetic operators at types Int and Float.
- It traverses the term, converting Qt to Dp. It checks that only permitted primitives appear in Qt, and translates these to their corresponding representation in Dp. Permitted primitives include: (==), (<), (+), (×), and similar, plus while, makeArr, lenArr, ixArr, and save.

An unfortunate feature of typed quasiquotation in GHC is that the implementation discards all type information when creating the representation of a term. Type Qt a is a synonym for the type

\[
TH. Q (TH. TExp a)
\]

where TH denotes the library for Template Haskell. TH. Q is the quotation monad (used to look up identifiers and generate fresh names), and TH. TExp a is the parse tree for a quoted expression returning a value of type a (a wrapper for the type TH. TExp of untyped expressions, with a as a phantom variable). Thus, the translator from Qt a to Dp a is forced to re-infer all type for subterms, and for this reason we support only limited overloading, and we translate the Maybe monad as a special case rather than supporting overloading for monad operations in general.

The backend performs three transformations over Dp terms before compiling to C. First, common subexpressions are recognised and transformed to let bindings. Second, Dp terms are normalised using exactly the same rules used for normalising Qt terms, as described in Section 4. Third, Dp terms are optimised using η contraction for conditionals and arrays:

\[
\text{if } L \text{ then } M \text{ else } M \Rightarrow M
\]

\[
\text{makeArr (lenArr M) (ixArr M)} \Rightarrow M
\]

and a restricted form of linear inlining for let bindings that preserves the order of evaluation.

Figure 4 lists lines of code, benchmarks used, and performance results. The translator from Dp to C is shared by QDSL and EDSL Feldspar, and listed in a separate column. All five benchmarks run under QDSL and EDSL Feldspar generate identical C code, up to permutation of independent assignments, with identical compile and run times. The columns for QDSL and EDSL Feldspar give compile and run times for Haskell, while the columns for generated code give compile and run times for the generated C. QDSL compile times are slightly greater than EDSL, and QDSL run times range from identical to four times that of EDSL, the increase being due to normalisation time (our normaliser was not designed to be particularly efficient).

4. The subformula property

This section introduces reduction rules for normalising terms that enforce the subformula property while preserving sharing. The rules adapt to both call-by-need and call-by-value. We work with simple types. The only polymorphism in our examples corresponds to instantiating constants (such as while) at different types.
Lines of Haskell code

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<th>EDSL Feldspar</th>
<th>Generated Code</th>
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<td>1722</td>
<td>5962</td>
</tr>
<tr>
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<tr>
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Benchmarks

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Performance

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<tr>
<td>Run</td>
<td>17.85</td>
<td>17.14</td>
<td>17.85</td>
</tr>
</tbody>
</table>

Times in seconds; minimum time of ten runs.
Quad-core Intel i7-2640M CPU, 2.80 GHz, 3.7 GiB RAM.
GHC 7.8.3; GCC 4.8.2; Ubuntu 14.04 (64-bit).

Figure 1. Comparison of QDSL and EDSL Feldspar

Types, terms, and values are presented in Figure 2. Let $A$, $B$, $C$ range over types, including base types ($\iota$), functions ($A \to B$), products ($A \times B$), and sums ($A + B$). Let $L$, $M$, $N$ range over terms, and $x$, $y$, $z$ range over variables. Let $c$ range over constants, which are fully applied according to their arity, as discussed below. As constant applications are non-values, we represent literals as free variables. As usual, terms are taken as equivalent up to renaming of bound variables. Write $FV(M)$ for the set of free variables of $M$, and $N[x := M]$ for capture-avoiding substitution of $M$ for $x$ in $N$. Let $V$, $W$ range over values.

Let $\Gamma$ range over type environments, which pair variables with types, and write $\Gamma \vdash M : A$ to indicate that term $M$ has type $A$ under type environment $\Gamma$. Typing rules are standard.

Reduction rules for normalisation are presented in Figure 3. The rules are confluent, so order of application is irrelevant to the final answer, but we break them into three phases to ease the proof of strong normalisation. It is easy to confirm that all of the reduction rules preserve sharing and preserve order of evaluation.

Write $M \Rightarrow N$ to indicate that $M$ reduces to $N$ in phase $i$. Let $F$ and $G$ range over different forms of evaluation frame used in Phases 1 and 2 respectively. Write $FV(F)$ for the set of free variables of $F$, and similarly for $G$. Reductions are closed under compatible closure.

The normalisation procedure consists of exhaustively applying the reductions of Phase 1 until no more apply, then similarly for Phase 2, and finally for Phase 3. Phase 1 performs let-insertion, naming subterms, along the lines of a translation to A-normal form (Flanagan et al. [1993]) or reductions (let.1) and (let.2) in Moggi’s metalanguage for monads (Moggi [1991]). Phase 2 performs two kinds of reduction: $\beta$ rules apply when an introduction (construction) is immediately followed by an elimination (deconstruction), and $\kappa$ rules push eliminators closer to introducers to enable $\beta$ rules. Phase 3 “garbage collects” unused terms as in the call-by-need lambda calculus (Marais et al. [1998]). Phase 3 should be omitted if the intended semantics of the target language is call-by-value rather than call-by-need.

Every term has a normal form.

**Proposition 4.1 (Strong normalisation).** Each of the reduction relations $\Rightarrow_1$, $\Rightarrow_2$, and $\Rightarrow_3$, reduction sequences on well-typed terms are finite.

The only non-trivial proof is for $\Rightarrow_2$, which can be proved via a standard reducibility argument (see, for example, (Lindley [2007])). If the target language includes general recursion, normalisation should treat the fixpoint operator as an uninterpreted constant.

The subformulas of a type are the type itself and its components. For instance, the subformulas of $A \to B$ are itself and the subformulas of $A$ and $B$. The proper subformulas of a type are all its subformulas other than the type itself.

The subterms of term $M$ are the term itself and its subterms. For instance, the subterms of $\lambda x. N$ are itself and the subterms of $N$. The subterms of $L M$ are itself and the subterms of $L$ and $M$. The proper subterms of $M$ are all its subterms other than the term itself.

Constants are always fully applied; they are introduced as a separate construct to avoid consideration of irrelevant subformulas and subterms. The type of a constant $c$ of arity $k$ is written $c : A_1 \to \cdots \to A_k \to B$ and its subterms are itself and $A_1, \ldots, A_k$, and $B$ (but not $A_1 \to \cdots \to A_k \to B$ for $i > 1$). An application of a constant $c$ of arity $k$ is written $c M_1 \cdots M_k$ and its subterms are itself and $M_1, \ldots, M_k$ (but not $c M_1 \cdots M_j$ for $j < k$). Free variables are equivalent to constants of arity zero.

Terms in normal form satisfy the subformula property.

**Proposition 4.2 (Subformula property).** If $\Gamma \vdash M : A$ and $M$ is in normal form, then every subterm of $M$ has a type that is either a subformula of $A$, a subformula of a type in $\Gamma$, or a subformula of the type of a constant in $M$.

The proof follows the lines of Prawitz [1965]. The differences are that we have introduced fully applied constants (to enable the sharpened subformula property, below), and that our reduction rules introduce let, in order to ensure sharing is preserved.

Normalisation may lead to an exponential or worse blow up in the size of a term, for instance when there are nested case expressions. The benchmarks in Section 3 do not suffer from blow up, but it may be a problem in some contexts. Normalisation may be controlled by introduction of uninterpreted constants, as in Section 5. Further work is needed to understand when complete normalisation is desirable and when it is problematic.

Examination of the proof in Prawitz [1965] shows that in fact normalisation achieves a sharper property.

**Proposition 4.3 (Sharpened subformula).** If $\Gamma \vdash M : A$ and $M$ is in normal form, then every proper subterm of $M$ that is not a free variable or a subterm of a constant application has a type that is a proper subformula of $A$ or a proper subformula of a type in $\Gamma$.

We believe we are the first to formulate the sharpened version.

The sharpened subformula property says nothing about the types of subterms of constant applications, but such types are immediately apparent by recursive application of the sharpened subformula property. Given a subterm that is a constant application $c \overline{T}$, where $c$ has type $T \to B$, then the subterm itself has type $B$, each subterm $\overline{M}$ has type $A$, and every proper subterm of $\overline{M}$, that is not a free variable of $\overline{M}$, or a subterm of a constant application has a type that is a proper subformula of $A$ or a proper subformula of the type of one of its free variables.

In Section 2 we require that every top-level term passed to qdsl is suitable for translation to C after normalisation, and any DSL translating to a first-order language must impose a similar requirement. One might at first guess the required property is that every
subterm is representable, in the sense introduced in Section 2.3 but this is not quite right. The top-level term is a function from a representable type to a representable type, and the constant while expects subterms of type s → Bool and s → s, where the state s is representable. Fortunately, the property required is not hard to formulate in a general way, and is easy to ensure by applying the sharpened subformula property.

Take the representable types to be any set closed under subformulas that does not include function types. We introduce a variant of the usual notion of representability. A term of type A → B has rank \( \min(m + 1, n) \) where \( m \) is the rank of \( A \) and \( n \) is the rank of \( B \), while a term of representable type has rank 0. We say a term is first-order when every subterm is either representable, or is of the form \( \lambda x. N \) where each bound variable and the body is of representable type. The following characterises translation to a first-order language.

**Proposition 4.4 (First-order).** Consider a term of rank 1, where every free variable has rank 0 and every constant has rank at most 2. Then the term normalises to a term that is first-order.

The property follows immediately by observing that any term \( L \) of rank 1 can be rewritten to the form \( \lambda \varphi. (L \varphi) \) where each bound variable and the body has representable type, and then normalising and applying the sharpened subformula property.

In QDSL Feldspar, while is a constant with type of rank 2 and other constants have types of rank 1. Section 2.3 gives an example of a normalised term. By the proposition, each subterm has a representable type (boolean, integer, float, or a pair of an integer and float) or is a lambda abstraction with bound variables and body of representable type; and it is this property which ensures it is easy to generate C code from the term.

---

**5. Feldspar as an EDSL**

This section reviews the combination of deep and shallow embeddings required to implement Feldspar as an EDSL, and considers the trade-offs between the QDSL and EDSL approaches. Much of this section reproduces Svenningsson and Axelsson (2012).

The top-level function of EDSL Feldspar has the type:

\[
\text{edsl : } (\text{Rep } a, \text{Rep } b) \Rightarrow (\text{Dp } a \rightarrow \text{Dp } b) \rightarrow C
\]

Here \( \text{Dp } a \) is the deep representation of a term of type \( a \). The deep representation is described in detail in Section 5.3 below, and is chosen to be easy to translate to C. As before, type \( C \) represents code in C, and type class Rep restricts to representable types.

**5.1 A first example**

Here is the power function of Section 2.2 written as an EDSL:

\[
\begin{align*}
\text{power : } & \text{Int} \rightarrow \text{Dp Float} \rightarrow \text{Dp Float} \\
& \text{power } n x = \\
& \quad \text{if } n < 0 \text{ then } \\
& \quad \quad x := 0 \ ? (0, 1 \ /	ext{power } (-n) x) \\
& \quad \text{else if } n == 0 \text{ then } 1 \\
& \quad \text{else if } \text{even } n \text{ then } \\
& \quad \quad \text{let } y = \text{power } (n \div 2) \ x \ \text{in } y \times y \\
& \quad \text{else } \\
& \quad \quad x \times \text{power } (n - 1) x
\end{align*}
\]

Type \( Q \ (\text{Float} \rightarrow \text{Float}) \) in the QDSL variant becomes the type \( \text{Dp Float} \rightarrow \text{Dp Float} \) in the EDSL variant, meaning that \( \text{power } n \) accepts a representation of the argument and returns a representation of that argument raised to the \( n \)’th power.

In the EDSL variant, no quotation is required, and the code looks almost—but not quite!—like an unstaged version of power,
but with different types. Clever encoding tricks, explained later, permit declarations, function calls, arithmetic operations, and numbers to appear the same whether they are to be executed at generation-time or run-time. However, as explained later, comparison and conditionals appear differently depending on whether they are to be executed at generation-time or run-time. However, as explained later, comparison and conditionals appear differently depending on whether they are to be executed at generation-time or run-time, using $M \equiv N$ and if $L$ then $M$ else $N$ for the former but $M \equiv N$ and $L \equiv (M, N)$ for the latter.

Invoking `edsl (power (−6))` generates code to raise a number to its $-6$ power. Evaluating `power (−6) u`, where $u$ is a term representing a variable of type `Dp Float`, yields the following:

\[
(u \equiv 0) ? (0, 1 / ((u \times ((u \times 1) \times (u \times 1))) \times ((u \times ((u \times 1) \times (u \times 1)))))
\]

Applying common-subexpression elimination permits recovering the sharing structure:

\[
\begin{align*}
v & \equiv (u \times 1) \\
w & \equiv u \times (v \times w) \\
top & \equiv (u \equiv 0) ? (0, 1 / (w \times w))
\end{align*}
\]

From the above, it is easy to generate the final C code, which is identical to that in Section 2.2.

Here are points of comparison between the two approaches.

- A function $a \rightarrow b$ is embedded in QDSL as $Qt (a \rightarrow b)$, a representation of a function, and in EDSDL as $Dp a \rightarrow Dp b$, a function between representations.

- QDSL enables the host and embedded languages to appear identical. In contrast, in Haskell, EDSDL requires some term forms, such as comparison and conditionals, to differ between the host and embedded languages. Other languages, notably Scala Virtualised (Rompf et al. 2013), may support more general overloading that allows even comparison and conditionals to be identical.

- QDSL requires syntax to separate quoted and unquoted terms. In contrast, EDSDL permits the host and embedded languages to intermingle seamlessly. Depending on your point of view, explicit quotation syntax may be considered as an unnecessary distraction or as drawing a useful distinction between generation-time and run-time. If one takes the former view, the type-based approach to quotation found in C# and Scala might be preferred.

- QDSL may share the same representation for quoted terms across a range of applications; the quoted language is the host language, and does not vary with the specific domain. In contrast, EDSDL typically develops custom shallow and deep embeddings for each application; a notable exception is the LMS and Delite frameworks for Scala, which provide a deep embedding shared across several disparate DSLs (Sujeeth et al. 2013).

- QDSL yields an unwieldy term that requires normalisation. In contrast, EDSDL yields the term in normalised form in this case, though there are other situations where a normaliser is required (see Section 5.2).

- QDSL requires traversing the quoted term to ensure it only mentions permitted identifiers. In contrast, EDSDL guarantees that if a term has the right type it will translate to the target. If the requirement to eyeball code to ensure only permitted identifiers are used is considered too onerous, it should be easy to build a preprocessor that checks this property. For example, in Haskell, it is possible to incorporate such a preprocessor using MetaHaskell (Mainland 2012).

- Since QDSLs may share the same quoted terms across a range of applications, the cost of building a normaliser or a preprocessor might be amortised across multiple QDSLs for a single language. In the conclusion, we consider the design of a tool for building QDSLs that uses a shared normaliser and preprocessor.

- Once the deep embedding or the normalised quoted term is produced, generating the domain-specific code is similar for both approaches.

5.2 A second example

In Section 2.4, we exploited the `Maybe` type to refactor the code.

In EDSDL, we must use a new type, where `Maybe`, `Nothing`, `Just`, and `maybe` become `Opt`, `none`, `some`, and `option`, and return and $(\Rightarrow)$ are similar to before.

- `type Opt a = u` where $u$ is a term representing a variable of type `Dp Float`

Here we have described normalisation via rewriting, but some rewrite rules need to be repeatedly applied, as described in Section 5.3. After applying these rules, common subexpression elimination yields the same structure as in the previous subsection, from which the same C code is generated.

Here we have described normalisation via rewriting, but some rewrite rules need to be repeatedly applied, as described in Section 5.3. After applying these rules, common subexpression elimination yields the same structure as in the previous subsection, from which the same C code is generated.

Before evaluating `power` yielded a term essentially in normal form. However, here rewrite rules need to be repeatedly applied, as described in Section 5.3. After applying these rules, common subexpression elimination yields the same structure as in the previous subsection, from which the same C code is generated.

Here we have described normalisation via rewriting, but some EDSDLs achieve normalisation via smart constructors, which ensure deep terms are always in normal form (Rompf 2012), the two techniques are roughly equivalent.
Hence, an advantage of the EDSL approach—that it generates terms essentially in normal form—turns out to apply sometimes but not others. It appears to often work for functions and products, but to fail for sums. In such situations, separate normalisation is required. This is one reason why we do not consider normalisation as required by QDSL to be particularly onerous.

Here are points of comparison between the two approaches.

- Both QDSL and EDSL can exploit notational conveniences in the host language. The example here exploits Haskell do notation; the embedding of SQL in F# by Cheney et al. (2013) exploits F# sequence notation. For EDSL, exploiting do notation just requires instantiating return and (⇒) correctly. For QDSL, it is also necessary for the translator to recognise and expand do notation and to substitute appropriate instances of return and (⇒).

- As this example shows, sometimes both QDSL and EDSL may require normalisation. As mentioned previously, for QDSLs the cost of building a normaliser might be amortised across several applications. In contrast, each EDSL usually has a distinct deep representation and so requires a distinct normaliser.

5.3 The deep embedding

Recall that a value of type $Dp$ $a$ represents a term of type $a$, and is called a deep embedding.

```
data Dp a where
  LitB :: Bool → Dp Bool
  LitI :: Int → Dp Int
  LitF :: Float → Dp Float
  If :: Dp Bool → Dp a → Dp a → Dp a
  While :: (Dp a → Dp Bool) → (Dp a → Dp a) → Dp a
  Pair :: Dp a → Dp b → Dp (a, b)
  Fst :: Rep b ⇒ Dp (a, b) → Dp a
  Snd :: Rep a ⇒ Dp (a, b) → Dp b
  PrimI :: Rep a ⇒ String → Dp a → Dp b
  Prim2 :: (Rep a, Rep b) ⇒ String → Dp a → Dp b
  MkArr :: Dp Int → (Dp Int → Dp a) → Dp (Arr a)
  LnArr :: Rep a ⇒ Dp (Arr a) → Dp Int
  IxArr :: Dp (Arr a) → Dp Int → Dp a
  Save :: Dp a → Dp a
  Let :: Rep a ⇒ Dp a → (Dp a → Dp b) → Dp b
  Variable :: String → Dp a
```

Type $Dp$ represents a low level, pure functional language with a straightforward translation to C. It uses higher-order abstract syntax (HOAS) to represent constructs with variable binding (Plenning and Elliot 1988). Our code obeys the invariant that we only write $Dp$ $a$ when $Rep$ $a$ holds, that is, when type $a$ is representable.

The deep embedding has boolean, integer, and floating point literals, conditionals, while loops, pairs, primitives, arrays, and special-purpose constructs to disable normalisation, for let binding, and for variables. Constructs $LitB$, $LitI$, $LitF$ build literals; $If$ builds a conditional. While corresponds to while in Section 2.5; $Pair$, $Fst$, and $Snd$ build and decompose pairs; $PrimI$ and $Prim2$ represent primitive operations, where the string is the name of the operation; $MkArr$, $LnArr$, and $IxArr$ correspond to the array operations in Section 2.6. $Save$ corresponds to $save$ in Section 2.3; $Let$ corresponds to let binding, and $Variable$ is used when translating HOAS to C code.

5.4 Class $Syn$

We introduce a type class $Syn$ that allows us to convert shallow embeddings to and from deep embeddings.

```
class Rep (Internal a) ⇒ Syn a where
  type Internal a
  toDp :: a → Dp (Internal a)
  fromDp :: Dp (Internal a) → a
```

Type $Internal$ is a GHC type family (Chakravarty et al. 2005). Functions $toDp$ and $fromDp$ translate between the shallow embedding $a$ and the deep embedding $Dp$ ($Internal$ $a$).

The first instance of $Syn$ is $Dp$ itself, and is straightforward.

```
instance Rep a ⇒ Syn (Dp a) where
  type Internal (Dp a) = a
  toDp = id
  fromDp = id
```

Our representation of a run-time $Bool$ will have type $Dp$ $Bool$ in both the deep and shallow embeddings, and similarly for $Int$ and $Float$.

We do not code the target language using its constructs directly. Instead, for each constructor we define a corresponding "smart constructor" using class $Syn$.

```
true, false :: Dp Bool
true = LitB True
false = LitB False

(?) :: Syn a ⇒ Dp Bool → (a, a) → a
  c ? (t, e) = fromDp (If c (toDp t) (toDp e))

while :: Syn a ⇒ (a → Dp Bool) → (a → a) → a → a
  while c b i = fromDp (While (c ◦ fromDp)
                             (toDp b ◦ fromDp)
                             (toDp i))
```

Numbers are made convenient to manipulate via overloading.

```
instance Num (Dp Int) where
  a + b = Prim2 (*+) a b
  a - b = Prim2 (*-) a b
  a × b = Prim2 (*) a b
  fromInteger a = LitI (fromInteger a)
```

With this declaration, $1 + 2 :: Dp$ $Int$ evaluates to $Prim2$ $(*+)$ $((LitI \ 1) \ (LitI \ 2))$.

Permitting code executed at generation-time and run-time to appear identical. A similar declaration works for $Float$.

Comparison also benefits from smart constructors.

```
(==.) :: (Syn a, Eq (Internal a)) ⇒ a → a → Dp Bool
  a ==. b = Prim2 (*=) (toDp a) (toDp b)

(<.) :: (Syn a, Ord (Internal a)) ⇒ a → a → Dp Bool
  a <. b = Prim2 (*<) (toDp a) (toDp b)
```

Overloading cannot apply here, because Haskell requires ($==$) return a result of type $Bool$, while ($==$) returns a result of type $Dp$ $Bool$, and similarly for ($<$).

Here is how to compute the minimum of two values.

```
minim :: (Syn a, Ord (Internal a)) ⇒ a → a → a
  minim x y = (x <. y) ? (x, y)
```

5.5 Embedding pairs

Host language pairs in the shallow embedding correspond to target language pairs in the deep embedding.
instance (Syn a, Syn b) ⇒ Syn (a, b) where
    type Internal (a, b) = (Internal a, Internal b)
    loop p = Pair (loop (fst p)) (loop (snd p))

This permits us to manipulate pairs as normal, with (a, b), fst a, and snd a. Argument p is duplicated in the definition of fromDp, which may require common subexpression elimination as discussed in Section 5.1.

We have now developed sufficient machinery to define a for loop in terms of a while loop.

for :: Syn a ⇒ Dp Int → a → (Dp Int → a → a) → a
for n s0 b = snd (while (λ(i, s) → i < n)
(λ(i, s) → (i + 1, b i s))
(0, s0))

The state of the while loop is a pair consisting of a counter and the state of the for loop. The body b of the for loop is a function that expects both the counter and the state of the for loop. The counter is discarded when the loop is complete, and the final state of the for loop returned.

Thanks to our machinery, the above definition uses only ordinary Haskell pairs. The condition and body of the while loop are functions that expect both the counter and the state of the for loop. The counter is discarded when the loop is complete, and the final state of the for loop returned.

The next obvious step is to define a suitable monad over the type Opt'. The natural definitions to use are as follows:

data Opt' a = Opt' { def :: Dp Bool, val :: a }
instance Syn a ⇒ Syn (Opt' a) where
    type Internal (Opt' a) = (Bool, Internal a)
    loop (Opt' b x) = Opt' (loop (pair b (loop x))
    fromDp p = Opt' (loop (pair (fst p)) (loop (snd p))

some' :: a ⇒ Opt' a
some' x = Opt' true x
none' :: Undef a ⇒ Opt' a
none' = Opt' false undef
option' :: Syn b ⇒ b → (a → b) → Opt' a → b
option' d f o = def o ? (f (val o), d)

The next obvious step is to define a suitable monad over the type Opt'. The natural definitions to use are as follows:

newtype Opt a =
    O { unO :: ∀b. Undef b ⇒ ((a → Opt' b) → Opt' b)

instance Monad Opt where
    return x = O (λg → g x)
    lift o fromDp = lift o fromDp
These definitions support the EDSL code presented in Section 5.2.

5.8 Embedding vector
Recall that values of type `Array` are created by construct `MkArr`, while `LnArr` extracts the length and `IxArr` fetches the element at the given index. Corresponding to the deep embedding `Array` is a shallow embedding `Vec`.

This is easy to define operations on vectors, including combining two primitives the intermediate vector is always eliminated.

Invoking `toVec` enables conversion between types.

6. Related work

QDSL forces all conversions to be written out, while EDSL silently converts between representations; following the pattern that QDSL is more explicit, while EDSL is more compact. For QDSL it is the subformula property which guarantees that all intermediate uses of `Vec` are eliminated, while for EDSL this is established by operational reasoning on the behaviour of the type `Vec`.

6. Conclusion

A good idea can be much better than a new one.

– Gerard Berry

We have compared QDSLs and EDSLs, arguing that QDSLs offer competing expressiveness and efficiency.

The subformula property may have applications in DSLs other that QDSLs. For instance, after Section 6 of this paper was drafted, it occurred to us that a different approach would be to extend type `Dp` with constructs for type `Maybe`. So long as type `Maybe` does not appear in the input or output of the program, a normaliser that ensures the subformula property could guarantee that C code for such constructs need never be generated.

Rather than building a special-purpose tool for each QDSL, it should be possible to design a single tool for each host language. Our next step is to design a QDSL library for Haskell that restores
the type information for quasi-quotations currently discarded by GHC and uses this to support type classes and overloading in full, and to supply a more general normaliser. Such a tool would subsume the special-purpose translator from Ql to Dp described at the beginning of Section 3 and lift most of its restrictions.

These forty years now I've been speaking in prose without knowing it! — Molière

Like Molière’s Monsieur Jourdain, many of us have used QDSLs for years, if not by that name. DSL via quotation lies at the heart of Lisp macros, Microsoft LINQ, and Scala LMS, to name but three. By naming the concept and drawing attention to the benefits of normalisation and the subformula property, we hope to help the concept to prosper for years to come.

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References
A. Gill. Type-safe observable sharing in Haskell. In Haskell, 2009.

J. Maraist, M. Odersky, and P. Wadler. The call-by-need lambda calculus. JFP, 8(03), 1998.
Limsoon Wong. Normal forms and conservative extension properties for query languages over collection types. JCSS, 52(3), 1996.