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Citation for published version:

Digital Object Identifier (DOI):
10.1016/0370-2693(96)00137-2

Link:
Link to publication record in Edinburgh Research Explorer

Published In:
Physics Letters B
MONTE CARLO SIMULATION OF THE THREE DIMENSIONAL THIRRING MODEL

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Abstract

We study the Thirring model in three spacetime dimensions, by means of Monte Carlo simulation on lattice sizes $8^3$ and $12^3$, for numbers of fermion flavors $N_f = 2, 4, 6$. For sufficiently strong interaction strength, we find that spontaneous chiral symmetry breaking occurs for $N_f = 2, 4$, in accordance with the predictions of the Schwinger-Dyson approach. The phase transitions which occur are continuous and with critical scaling behaviour depending on $N_f$. For $N_f = 6$ our results are preliminary, and no firm conclusions about the existence or otherwise of chiral symmetry breaking are possible.

PACS numbers: 11.10.Kk, 11.15.Ha, 11.30.Rd

Keywords: Model field theory, $1/N$ expansion, Schwinger-Dyson equation, chiral symmetry breaking, lattice simulation, dynamical fermions.
The three dimensional Thirring model is a field theory of relativistic fermions interacting via a contact term between conserved vector currents. Its Lagrangian is written

\[ \mathcal{L} = \bar{\psi}_i (\partial \psi + m) \psi_i + \frac{g^2}{2N_f} (\bar{\psi}_i \gamma_{\mu} \psi_i)^2, \]  

(1)

where $\psi_i, \bar{\psi}_i$ are four-component spinors, $m$ is a bare, parity-conserving mass, and the index $i$ runs over $N_f$ distinct fermion species. Since the coupling $g^2$ has mass dimension $-1$, naive power-counting suggests that the model (1) is non-renormalisable. However, as has been suspected for many years [1,2], an expansion in powers of $1/N_f$, rather than $g^2$, is exactly renormalisable. At leading order in $1/N_f$, interaction between vector currents is dominated by exchange of a fermion - anti-fermion bound state described in terms of a chain of vacuum-polarisation “bubbles”. In the ultra-violet limit the interaction is thus transformed from a momentum-independent contact term to a softer $A/(N_f k)$ behaviour, where $A$ is a numerical constant independent of $g$: this asymptotic behaviour can be used to evaluate divergent graphs at higher orders in $1/N_f$, eg. in [3], where renormalisability of the massless model is explicitly demonstrated to $O(1/N_f)$.

The property of renormalisability signals that the model’s $1/N_f$ expansion exhibits a UV-stable fixed point of the renormalisation group, the continuum limit being taken in the limit $g^2 \Lambda \to \infty$, where $\Lambda$ is a UV cutoff. RG fixed points have also been observed in other three-dimensional four-fermi models [4,5]. The distinctive feature of the Thirring model is that for $d < 4$ the vacuum polarisation is UV-finite so long as the regularisation respects current conservation (eg. Pauli-Villars). This means that there is no need to fine-tune $g^2$ to a critical value: a continuum limit may be taken for any value of the dimensionless parameter $mg^2$ (at least to leading order in $1/N_f$ [3]), the theory thus obtained having a variable ratio of, say, physical fermion mass to vector bound state mass. In the RG sense the interaction $(\bar{\psi}_i \gamma_{\mu} \psi_i)^2$ is a marginal operator for $2 < d < 4$, whereas, say, the interaction $(\bar{\psi}_i \psi_i)^2$ in the Gross-Neveu model is relevant [6].

Another possibility raised by the $1/N_f$ expansion is the equivalence of the Thirring
model in the strong-coupling limit $mg^2 \to \infty$, in which the vector particle becomes massless, with the infra-red limiting behaviour of QED in three spacetime dimensions. In massless large-$N_f$ QED$_3$ vacuum polarisation screens one-photon exchange to the extent that the $1/k^2$ interaction is again transformed to $1/k$ [7]. The $O(1/N_f)$ corrections to the models evaluated in respectively UV (Thirring) or IR (QED) limits appear to coincide [3,8].

The $1/N_f$ expansion may not, however, describe the true behaviour of the model, particularly for small $N_f$. For instance, spontaneous chiral symmetry breaking, signalled by a vacuum condensate $\langle \bar{\psi} \psi \rangle$, is forbidden to all orders in $1/N_f$, and yet may be predicted by a self-consistent approach such as solution of the Schwinger-Dyson equations [2,9,10]. In this approach a non-trivial solution for the dressed fermion propagator $S(p) = (A(p)i\gamma^\mu + \Sigma(p))^{-1}$ is sought, ie. one in which the self-energy $\Sigma(p)$ and hence $\langle \bar{\psi} \psi \rangle$ are non-vanishing in the chiral limit $m \to 0$. Unfortunately, the SD equations can only be solved by truncating them in a somewhat arbitrary fashion. The usual approximation [2,10] is to assume that the vector propagator is given by its leading-order form for $m = 0$ in the $1/N_f$ expansion, viz.

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(1 + \frac{g^2(k^2)^{1\over 2}}{8}\right)^{-1} + \frac{k_\mu k_\nu}{k^2}, \quad (2)$$

and that the fermion-vector vertex function is well-approximated by the bare vertex (the so-called “planar” or “ladder” approximation):

$$\Gamma_\mu(p,q) = -\frac{ig}{\sqrt{N_f}} \gamma_\mu. \quad (3)$$

The longitudinal part of $D_{\mu\nu}$ raises potential ambiguities: the most systematic treatment has been given by Itoh et al [10], who note the equivalence of the Thirring model with a gauge-fixed form of a fermion-scalar model possessing a local gauge symmetry and then use a non-local gauge-fixing condition to find a gauge in which the “wavefunction renormalisation” $A(p) \equiv 1$. This simplification enables the SD equations to be exactly
solved in the limit $g^2 \to \infty$, with the result that a non-trivial solution for $\Sigma(p)$ exists for

$$N_f < N_{fc} = \frac{128}{3\pi^2} \simeq 4.32. \quad (4)$$

Moreover since the integral equations require the introduction of a UV cutoff $\Lambda$, a feature of this solution is that the induced physical mass scale $\mu$ depends on $N_f$ in an essentially singular way:

$$\frac{\mu}{\Lambda} \propto \exp \left(-\frac{2\pi}{\sqrt{N_{fc}/N_f - 1}}\right); \quad \langle \bar{\psi}\psi \rangle \propto \Lambda^{\frac{1}{2}}\mu^{\frac{3}{2}} \propto \exp \left(-\frac{3\pi}{\sqrt{N_{fc}/N_f - 1}}\right). \quad (5)$$

This implies that a continuum limit only exists as $N_f \to N_{fc}$, the scenario being very similar to that proposed by Miranskii and collaborators for strongly-coupled QED$_4$ [11]. Unfortunately no analytic solution exists for $g^2 < \infty$; however using different techniques Kondo [12] has argued that a critical line $N_{fc}(g^2)$ exists in the $(g^2, N_f)$ plane, which is a smooth invertible function. Therefore for integer $N_f < N_{fc}$ one might expect a critical scaling behaviour

$$\langle \bar{\psi}\psi \rangle \propto \exp \left(-\frac{a}{\sqrt{g^2 - g_{ec}^2}}\right), \quad (6)$$

corresponding to a symmetry restoring transition at some critical point $g^2 = g_{ec}^2$. Presumably in this scenario the Thirring interaction has become relevant: there may exist a novel strongly-coupled continuum limit at the critical point not described by the $1/N_f$ expansion.

There are good reasons to be cautious of this picture, however. Using a different sequence of truncations Hong and Park have found chiral symmetry breaking for all $N_f$ [9], with

$$\frac{1}{g_{ec}^2} \propto \exp \left(-\frac{N_f \pi^2}{16}\right), \quad (7)$$

a result which is non-analytic in $1/N_f$. Moreover in the limit $g^2 \to \infty$ the system of SD equations obtained are very similar to those of large-$N_f$ QED$_3$, in which case studies
beyond the planar approximation, using improved ansätze for the vertex $\Gamma_\mu$, suggest that the condition $A(p) \equiv 1$ is unphysical, and that chiral symmetry is spontaneously broken for all $N_f$ \cite{13}. In the current context this would imply $g^2_c < \infty$ for all $N_f$.

For these reasons we consider a numerical study of the lattice-regularised model to be timely. If, as suggested above, the Thirring model lies in the same universality class as QED$_3$, then a numerical study may shed light on the value of $N_f c$ for this model \cite{7}; previous lattice studies \cite{14} have been plagued by large finite volume effects due to the slow fall-off of the photon propagator $\propto 1/x$. The corresponding propagator in the Thirring model falls as $1/x^2$, so sensible results may emerge on smaller systems. A second motivation is the possible existence of a novel continuum limit: since vacuum polarisation corrections to the vector propagator are finite to all orders in $1/N_f$ there should be no competing effects of charge screening, which obscures the issue in QED$_4$.

The lattice action we have used is as follows:

$$S = \frac{1}{2} \sum_{x \mu i} \bar{\chi}_i(x) \eta_\mu(x) (\chi_i(x + \hat{\mu}) - \chi_i(x - \hat{\mu})) + m \sum_{xi} \bar{\chi}_i(x) \chi_i(x)$$

$$+ \frac{g^2}{2N} \sum_{x \mu ij} \bar{\chi}_i(x) \chi_i(x + \hat{\mu}) \bar{\chi}_j(x + \hat{\mu}) \chi_j(x)$$

$$= \frac{1}{2} \sum_{x \mu i} \bar{\chi}_i(x) \eta_\mu(x) (1 + i A_\mu(x)) \chi_i(x + \hat{\mu}) + \text{h.c.} + m \sum_{xi} \bar{\chi}_i(x) \chi_i(x) + \frac{N}{4g^2} \sum_{x \mu} A^2_\mu(x),$$

$$= \frac{1}{2} \sum_{xy \mu i} \bar{\chi}_i(x) M_{(A,m)}(x,y) \chi_i(y) + \frac{N}{4g^2} \sum_{x \mu} A^2_\mu(x),$$

where $\chi, \bar{\chi}$ are staggered fermion fields, $\eta_\mu$ the Kawamoto-Smit phases, $m$ is the bare mass, the flavor index $i$ runs from 1 to $N$, and we have introduced $M_{(A,m)}$ for the fermionic bilinear, which depends on both the auxiliary field and the mass. The second form of the action is the one actually simulated: the equivalence of the two forms follows from Gaussian integration over the real-valued auxiliary field $A_\mu$ defined on the lattice links (for $N = 1$ there is an alternative formulation in terms of a compact complex-valued auxiliary
[15,16]). The vector-like interaction of the action allows the introduction of a checkerboard, which in turn enables simulation of the system for any $N$. In three Euclidean dimensions staggered fermions describe two continuum species of four-component fermions, with a parity-conserving mass term [17]. Hence the number of physical flavors $N_f = 2N$. An interesting feature of the lattice formulation (8) is that the interaction current is not exactly conserved. The conserved current in lattice gauge theory incorporates the gauge connection $\exp(iA_\mu)$. This means that at leading order in $1/N$ the vector propagator receives an extra contribution from vacuum polarisation, essentially due to the absence of the diagram of Fig. 1. The effect can be absorbed into a redefinition of the coupling:

$$ g^2_R = \frac{g^2}{1 - g^2 J(m)}, $$

where $J(m)$ is the value of the integral depicted in Fig. 1. The physics described by continuum $1/N_f$ perturbation theory occurs for the range of couplings $g^2_R \in [0, \infty)$, ie. for $g^2 \in [0, g^2_{lim})$; to leading order in $1/N$

$$ \frac{1}{g^2_{lim}} = J(m); \quad \text{with} \quad J(0) = \frac{2}{3}. $$

We therefore expect to see some kind of discontinuous behaviour in our simulations for small values of $1/g^2$.

In this letter, we aim to clarify the chiral symmetry breaking pattern by studying the chiral condensate:

$$ \langle \bar{\psi} \psi \rangle = \frac{1}{V} \text{Tr} \left( M^{-1}_{(A,m)} \right) $$

which, in the limit $m \to 0$, is an order parameter for the spontaneous symmetry breaking. We performed simulations on $8^3$ and $12^3$ lattices for $N_f = 2, 4, 6$, using the hybrid Monte Carlo algorithm. Bare mass values $m$ ranged from 0.4 down to 0.02, with most attention paid to the range 0.05 – 0.02. For each mass and coupling we performed roughly 500 HMC trajectories, the trajectory length being drawn from a Poisson distribution with mean 0.9.
The condensate $\langle \bar{\psi}\psi \rangle$ was measured with a stochastic estimator every few trajectories. To maintain reasonable acceptance rates the timestep varied from 0.15 on $8^3$ at $m = 0.4$ down to 0.022 on $12^3$ at $m = 0.02$. We found that considerably more work was needed to perform matrix inversion in this model than for the Gross-Neveu simulations described in [5]. Another difference is that in this case since the critical region of interest occurs at successively stronger couplings as $N_f$ is raised, the CPU required also grows with $N_f$, despite the $1/N_f$ suppression of quantum corrections.

In Fig. 2, we plot $\sigma \equiv \langle \bar{\psi}\psi \rangle$ vs. $1/g^2$ for the three values of $N_f$ studied, for $m = 0.10$ on a $8^3$ lattice. The models with different $N_f$ have apparently coincident condensates in the strong–coupling region $1/g^2 \leq 0.3$, but thereafter the $\langle \bar{\psi}\psi \rangle$ signals peak to maxima at distinct values of $1/g^2$ before falling away. It is tempting to associate the strong–coupling region with $g_R^2 < 0$ from the discussion following Eq. (9), although the correspondence with the value of $g_{\lim}^2$ predicted in Eq. (10) is not good. It may well be that the value of the diagram of Fig. 1 is considerably altered in a chirally broken vacuum.

As stated above, in order to study spontaneous chiral symmetry breaking, one has to monitor the value of $\langle \bar{\psi}\psi \rangle$ as $m \to 0$. Our results for the chiral condensate for $N_f = 2$ are reported in Fig. 3 for different values of the bare mass. A naive extrapolation to the chiral limit from the lattice data at fixed $1/g^2$ is probably unreliable in the range of bare masses we have explored. In order to determine the critical point, we need to perform a global fit of our data incorporating many values of $m$ and $1/g^2$. Therefore, we have to use an equation of state (EOS) relating the external symmetry breaking parameter $m$ to the response of the system $\langle \bar{\psi}\psi \rangle$ and the coupling $1/g^2$ [16,18].

A generic EOS, inspired by the critical behaviour of spin systems, can be written in terms of the scaled variables:

$$m\langle \bar{\psi}\psi \rangle^{-\delta} = \mathcal{F}\left(\Delta(1/g^2)\langle \bar{\psi}\psi \rangle^{1/\beta}\right),$$  

(11)

where $\Delta(1/g^2) = 1/g_c^2 - 1/g^2$ is the reduced coupling. At $g = g_c$, Eq. (11) is the usual...
scaling relation:
\[ \langle \bar{\psi} \psi \rangle \sim m^{1/\delta}, \]
while a Taylor expansion for small \( \Delta(1/g^2) \) yields:
\[
m = B \langle \bar{\psi} \psi \rangle^\delta + A \Delta(1/g^2) \langle \bar{\psi} \psi \rangle^{\delta-1/\beta} + \ldots \tag{12}
\]
where \( A, B = \mathcal{F}'(0), \mathcal{F}(0) \) respectively. At this stage, one sees that, for vanishing \( m \), Eq. (12) is simply the definition of the critical exponent \( \beta \):
\[
\langle \bar{\psi} \psi \rangle \sim (1/g_c^2 - 1/g^2)^\beta,
\]
and that there are no logarithmic corrections, since these should only appear in 4-d [18]. If the critical behaviour is described by mean-field theory, we expect \( \delta = 3 \) and \( \beta = 1/2 \), yielding:
\[
\langle \bar{\psi} \psi \rangle^2 = \kappa_1 \frac{m}{\langle \bar{\psi} \psi \rangle} + \kappa_2 \Delta(1/g^2), \tag{13}
\]
which shows that \( \langle \bar{\psi} \psi \rangle^2 \) is a linear function of the ratio \( m/\langle \bar{\psi} \psi \rangle \). Such a plot is known as Fisher plot. From Eq. (13) we see that a positive value of the intercept corresponds to a non-vanishing value of the chiral condensate for \( m = 0 \), while the intercept will be exactly zero at the critical coupling. In Fig. 4 and 5 we show the Fisher plots for \( N_f = 2, 4 \) respectively, where we can see at first glance an indication of chiral symmetry breaking, according to the criterion stated above. In order to get a more quantitative evidence, we have fitted our data using Eq. (12) and a simpler version of it based on the hypothesis that \( \delta - 1/\beta = 1 \) [19], which we will call respectively fit I and II in what follows. We should stress that in the absence of a systematic critical theory the forms I and II are used simply as effective descriptions of the data. The values of the fit parameters, their errors and the \( \chi^2 \) are listed in Table 1. The number of values of the chiral condensate included in the fit is chosen in order to minimize the value of the reduced \( \chi^2 \). The results of fit II are shown in Figs. 4 and 5 and seem to describe the data quite well. The dashed line in Fig. 3 is the
curve one obtains using the results of fit II with $m = 0$. It shows clearly that, within the range of $m$ we have explored, the chiral condensate is still far from its chiral limit value, thus providing a justification \textit{a posteriori} for the impossibility of extrapolating the data naively.

There are a few conclusions one can draw from the numerical analysis that we would like to emphasize. First, for both values of $N_f$, we find clear evidence of chiral symmetry breaking at finite values of the coupling, as predicted by the Schwinger-Dyson approach. From Eq. (7) we get:

$$
\frac{g_c^2(N_f = 2)}{g_c^2(N_f = 4)} = \exp(-\pi^2/8) = 0.291
$$

which is not too far from the fitted value $0.342 \pm 0.015$ (using the data from fit II).

Secondly, although any claims to understand the details of the critical scaling must be premature, the fits strongly suggest that the models with $N_f = 2$ and 4 are described by distinct critical theories, in a sense on “either side” of the mean field theory. If we relax the requirement $\delta - 1/\beta = 1$ (which means using fit I instead of fit II) then the difference in the fitted values of $\delta$ becomes even more apparent. This is significant because similar EOS fits in QED$_4$ reveal no such differences between $N_f = 2$ and $N_f = 4$ [20].

Finally, we report some preliminary results for $N_f = 6$. Figure 6 shows $\langle \bar{\psi}\psi \rangle$ vs. $m$ for two values of $g$. The $1/g^2 = 0.5$ data suggest a linear extrapolation to the chiral limit, yielding a small condensate equal to 0.013(4). For $1/g^2 = 0.4$ it is less clear how to make the extrapolation. Clearly in either case reliable data at much smaller mass values would be needed for confirmation or otherwise of chiral symmetry breaking: comparing data from different lattice sizes, we have found that finite size effects become more important as we go to larger $N_f$, which means that larger lattice sizes will probably be needed before we can proceed to a more quantitative study.

We conclude by briefly summarizing our results. We have shown by numerical simulations that spontaneous chiral symmetry breaking does occur in the Thirring model for
finite $N_f$, in contradiction with the $1/N_f$ perturbative expansion, but in agreement with the Schwinger-Dyson approach at least for $N_f = 2, 4$. In these two cases we were able to determine the critical coupling and the critical exponent $\delta$ by fitting to a plausible EOS, and the fits suggest that the two models have different critical behaviour. For $N_f = 6$, we were unable to find clear evidence in favour of symmetry breaking, but cannot yet exclude a non-vanishing condensate in the chiral limit. In the future we plan to investigate in more detail the theory for $N_f = 2, 4, 6$ at the critical point, focussing on critical exponents, the renormalized charge, and spectroscopy.

Acknowledgments

LDD is supported by an EC HMC Institutional Fellowship under contract No. ERBCH-BGCT930470, and SJH by a PPARC Advanced Fellowship. Some of the numerical work was performed on the Cray Y-MP at Rutherford–Appleton Laboratory under PPARC grant GR/J675.5. We have enjoyed discussing this project with Jiří Jersák, Kei-Ichi Kondo, Mike Pennington, Craig Roberts and Paolo Rossi.
References

S. Hikami and T. Muta, Prog. Theor. Phys. 57 (1977) 785;


D. Atkinson, P.W. Johnson and M.R. Pennington, Brookhaven preprint BNL-41615 (1988);


**B371** (1992) 713.


Phys. **B413** (1994) 503;
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<tr>
<td>$B$</td>
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**Table 1**

Results from the fits
Figure Captions

Figure 1: Diagram contributing to coupling constant renormalisation at leading order in $1/N_f$.

Figure 2: chiral condensate $\sigma$ vs. $1/g^2$ for $N = 1, 2, 3$, corresponding respectively to $N_f = 2, 4, 6$.

Figure 3: chiral condensate $\sigma$ vs. $1/g^2$ for $N_f = 2$ and different values of the bare mass $m$.

Figure 4: Fisher plot for $N_f = 2$, from data at $\beta = 1.6(\triangle), 1.8(\triangleleft), 2.0(\nabla), 2.2(\triangleright), 2.4(\rhd)$.

Figure 5: Fisher plot for $N_f = 4$, from data at $\beta = 0.5(\circ), 0.6(\blacksquare), 0.7(\triangledown), 0.8(\triangle), 0.9(\triangleleft), 1.0(\nabla), 1.1(\triangleright), 1.2(\rhd), 1.3(\times), 1.4(\ast)$.

Figure 6: chiral condensate $\sigma$ vs. $m$ for different values of the coupling (here $\beta \equiv 1/g^2$ and is not related to the critical exponent).
Figure 1
Fig. 3

![Graph showing data points for different values of m: m=0.05, m=0.04, m=0.03, m=0.02, m=0, from fit. The x-axis represents 1/g^2, and the y-axis represents σ.](image-url)
Fig. 4
Fig. 6