Homogenization: Preparing Equations for Change of Unknown

HOMOGENIZATION: PREPARING EQUATIONS FOR CHANGE OF UNKNOWN

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ABSTRACT

PRESS is a computer program for solving symbolic, transcendental, non-differential equations, [3]. We describe a new equation solving method, called Homogenization, which we have implemented in PRESS. Homogenization prepares equations, so that they can be solved by the Change of Unknown method. It does this by oausing different occurrences of the unknown to occur within identical subterms. The method has application, outside equation solving, to the problem of generalizing expressions.

I INTRODUCTION

A common method of solving equations is to change the unknown, e.g. if the equation is

$$4/\log_2 x \ ♦ \ \log_2 x \ s \ 5$$

then the equation may be solved by substituting y s logpX and solving the resulting disguised quadratic

$$M/y \ ♦ \ y \ « \ 5$$

Unfortunately, things are seldom as simple as this. Equations, which can be solved by Change of Unknown, are more likely to appear in the form

$$M.\log_x 2 \ ♦ \ \log_2 x \ * \ 5 \qquad (i)$$

in which the occurrences of the unknown, x, appear within dissimilar subterms, namely $\log_x 2$ and $\log_2 x$. Some preparation of the equation is required before the unknown can be changed. In the case of our example, the subterm $\log_x 2$ must first be converted to $1/\log_2 x$, with the aid of the rewrite rule

$$\log_u v \ => \ 1/\log_v u$$

We call this preparation step Homogenization, because it makes the occurrences of x appear in identical subterms. In this paper we describe the implementation of a version of Homogenization.

II SOME TERMINOLOGY

Before we can describe the Homogenization method we must introduce some terminology.

- The original equation, prior to Homogenization, will be called the input equation, and the resulting equation will be called the output equation.

• The output equation belongs to a class of equation which we will call the base olaas. We have restricted the base class to Algebraic equations, l.e those involving only the functions +, -, *, / and exponentiation to a rational number power. This covers all the examples we have found in text books and exam papers. In fact most output equations are (sometimes disguised) quadratics. Any class of equations could be used as the base olass, but the idea is to plck a olass whose members are relatively easy to solve.

- The input equation can always be regarded as an Algebraic equation la AfiBfi. set of nQn-algfiwTIlfi flUbUrM la X, «.g. equation (1) above can be regarded as an Algebraic equation in the set ($\log_x 2$, $\log_2 x$). These subterms are called the offending terms and the set of them Is called the offenders sat. The idea is that this is a set of subterms preventing the equation being Algebraic: a class of equations which PRESS knows a lot about.

• The essence of the Homogenization method is to replace each of the offending terms by some Algebraic funotlon of a single term, called the reduoed term. In the example above the reduced term Is $\log_2 x$.

III THE HOMOGENIZATION METHOD

The Homogenization method is as follows:

(a) The offenders set is found by trying to parse the Input equation as an Algebraic equation. When the parse Is blocked, because the current subterm is x or a non-Algebraic subterm containing x, then this is added to the offenders set and the parse forced to continue.

(b) If the offenders set is a singleton then we must distinguish three cases.

  (1) The singleton is the unknown, x. The equation is already algebraic so exit.

  (2) The singleton Is a funotlon of x, f(x) say, which occurs at least twice in the

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input equation. Exit with success as now the Change of Unknown method can be applied to the output equation, substituting y for f(x).

(3) The singleton is a function of x as above, but occurs only once In the equation. In this case again the Change of Unknown method cannot be used so exit with failure. This case includes examples like

$$_{x}\cos(x) = 2.$$

(c) Classify the input equation acoordlng to its offenders set. For example, if all the offenders are trigonometric terms, then the equation is of type trigonometric. The types currently used are: trigonometric, exponential, hyperbolic, logarithmic and mixed (the miscellaneous type). The classification scheme is explained in more detail in the long version of this paper,

(d) A reduced term is selected. The type of the Input equation determines how this term is chosen, different techniques are needed for trigonometric equations thsn for exponential equations, for example. For mixed type, where the special techniques fall, the reduced term is chosen from the offenders set on the basis of a simple simplicity measure.

(e) Now an attempt is made to rewrite each term in the offenders set as an Algebraic function of the reduced term. In order to avoid the Inefflciences of random search, we have designed the program to make maximum use of its syntactic analysis of the input equation. Thus a special rewrite rule set is used for each of the equation types (see [4] for details). For mixed types each rewrite must be done In single step: with s rule of the form

ot => af(rt)

where ot matches the offending term, rt matches the reduced term and af la an Algebraic function. If some of the offending terms cannot be rewritten, then backtrack to choose a new reduced term if this is possible, otherwise fail.

(f) Substitute the rewrites for the offending terms in the input equation to give the output equation. This equation is now an Algebraic equation of the reduoed term, I.e. it is homogenized, so exit with success. Change of Unknown oan now be successfully applied, substituting y for the reduced term in the output equation.

IV RESULTS AND EXAMPLE EQUATIONS

During our survey of exam papers we discovered 27 questions on which the Homogenlzatlon method could be used, although In some examplea other, better, methods could be used. Of these 27, 25 were successfully processed by our Implementation of Homogenlzatlon, that is the correct output equation was found. The rest of PRESS was then able to solve 11 of these output equations. The 2 examples where Homogenlzatlon failed are discussed in the long version of this paper.

Here are some examples of equations which have been solved using Homogenlzatlon:

$$9.\cosh(x) - 6.\sinh(x) = 7$$
(A level London 1976)

$$e^{3x} - 2.e^{x} - 3.e^{-x} = 0$$
(A level London 1977)

$$4^{x} - 5.2^{x+1} + 16 = 0$$
(A level London 1977)

$$3.\cos^{2}(x) + 5.\sin(x) - 1 = 0$$
(A level A.E.B. 1975)

$$\log_{x}8 + \log_{8}x = 13/6$$
(A level A.E.B. 1975)

$$\cos(x) + \cos(3x) + \cos(5x) = 0$$
(A level A.E.B. 1976)

V GENERALIZATION

Change of Unknown is very siallar to the nethod of Generalization in the program-property theorem-prover of Boyer and Moore, [2]. Before proving a theorem by induction, the Boyer-Moore theorem-prover is able to generalize it by replacing several occurrences of the same subterm by a new akolem constant, e.g. (rev a) in

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(equal (append (rev a) (append b e))
   (append (append (rev a) b) c))
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is replaced by d to produce

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(equal (append d (append b c))
   (append (append d b) c))
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which is then solved by induction.

Work by Boyer, Moore and Aubin, [1], has concentrated on the question of when Generalization is to be done, which occurrences of a subterm are to be replaced and what additional assumptions may need to be introduced to prevent over-generalization. These are not issues in equation solving. There is never a danger of over-generalization and all occurrences of a subterm should always be generalized (i.e. ohanged to a new unknown).

The method of Homogenization described above is complementary to the work done by Boyer, Moore and Aubin, because it suggests how an expression may be prepared for Generalization: subterms which were not previously identical may be made so, in order to allow Generalization to proceed.

VI CONCLUSIONS

We have described the Homogenlzatlon method, which prepares equations for solution by Change of Unknown. This work further illustrates the role of meta-level inference in algebraic manipulation (see

[3]). Meta-level inference consists of a syntactic analysis of the input equation to determine appropriate algebraic solution steps. In this case the syntactic analysis reveals the offenders set, picks a reduoed tern and, hence, suggests what rewrite rules to apply.

We have found that the problem of selecting the reduoed tern is one that requires something more than a general purpose method. Although the simplicity method, used for equations of mixed type is often successful on any equation, there are many problems for which it is inadequate. Thus we have implemented special purpose methods for some types of equations, and these have proved to be of great value. Similar remarks apply to the rewriting of offending terms. We are continuing the development of these methods.

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