HALPHEN PENCILS ON QUARTIC THREEFOLDS

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Abstract. For any smooth quartic threefold in $\mathbb{P}^4$ we classify pencils on it whose general element is an irreducible surface birational to a surface of Kodaira dimension zero.

1. Introduction

Let $X$ be a smooth quartic threefold in $\mathbb{P}^4$. The following result is proved in [4].

Theorem 1.1. The threefold $X$ does not contain pencils whose general element is an irreducible surface that is birational to a smooth surface of Kodaira dimension $-\infty$.

On the other hand, one can easily see that the threefold $X$ contains infinitely many pencils whose general elements are irreducible surfaces of Kodaira dimension zero.

Definition 1.2. A Halphen pencil is a one-dimensional linear system whose general element is an irreducible subvariety birational to a smooth variety of Kodaira dimension zero.

The following result is proved in [2].

Theorem 1.3. Suppose that $X$ is general. Then every Halphen pencil on $X$ is cut out by

$$\lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}\left(\mathbb{C}[x, y, z, t, w]\right) \cong \mathbb{P}^4,$$

where $l_1$ and $l_2$ are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$.

The assertion of Theorem 1.3 is erroneously proved in [1] without the assumption that the threefold $X$ is general. On the other hand, the following example is constructed in [3].

Example 1.4. Suppose that $X$ is given by the equation

$$w^3 x + w^2 q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}\left(\mathbb{C}[x, y, z, t, w]\right) \cong \mathbb{P}^4,$$

where $q_i$ and $p_i$ are forms of degree $i$. Let $P$ be the pencil on $X$ that is cut out by

$$\lambda x^2 + \mu \left(wx + q_2(x, y, z, t)\right) = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then $P$ is a Halphen pencil if $q_2(0, y, z, t) \neq 0$ by [2, Theorem 2.3].

The purpose of this paper is to prove the following result.

Theorem 1.5. Let $M$ be a Halphen pencil on $X$. Then

- either $M$ is cut out on $X$ by the pencil

$$\lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}\left(\mathbb{C}[x, y, z, t, w]\right) \cong \mathbb{P}^4,$$

where $l_1$ and $l_2$ are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$,

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\[\text{We assume that all varieties are projective, normal and defined over } \mathbb{C}.\]
or the threefold $X$ can be given by the equation
\[ w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj} \left( \mathbb{C}[x, y, z, t, w] \right) \cong \mathbb{P}^4 \]
such that $q_2(0, y, z, t) \neq 0$, and $\mathcal{M}$ is cut out on the threefold $X$ by the pencil
\[ \lambda x^2 + \mu \left( wx + q_2(x, y, z, t) \right) = 0, \]
where $q_i$ and $p_i$ are forms of degree $i$, and $(\lambda : \mu) \in \mathbb{P}^1$.

Let $P$ be an arbitrary point of the quartic hypersurface $X \subset \mathbb{P}^4$.

**Definition 1.6.** The mobility threshold of the threefold $X$ at the point $P$ is the number
\[ \iota(P) = \sup \left\{ \lambda \in \mathbb{Q} \text{ such that } \left| n\left( \pi_*(-K_X) - \lambda E \right) \right| \text{ has no fixed components for } n \gg 0 \right\}, \]
where $\pi : Y \to X$ is the ordinary blow up of $P$, and $E$ is the exceptional divisor of $\pi$.

Arguing as in the proof of Theorem [1.5] we obtain the following result.

**Theorem 1.7.** The following conditions are equivalent:

- the equality $\iota(P) = 2$ holds,
- the threefold $X$ can be given by the equation
\[ w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj} \left( \mathbb{C}[x, y, z, t, w] \right) \cong \mathbb{P}^4, \]
where $q_i$ and $p_i$ are forms of degree $i$ such that
\[ q_2(0, y, z, t) \neq 0, \]
and $P$ is given by the equations $x = y = z = t = 0$.

One can easily check that $2 \geq \iota(P) \geq 1$. Similarly, one can show that

- $\iota(P) = 1 \iff$ the hyperplane section of $X$ that is singular at $P$ is a cone,
- $\iota(P) = 3/2 \iff$ the threefold $X$ contains no lines passing through $P$.

The proof of Theorem 1.7 is completed on board of IL-96-300 Valery Chkalov while flying from Seoul to Moscow. We thank Aeroflot Russian Airlines for good working conditions.

2. Important Lemma

Let $S$ be a surface, let $O$ be a smooth point of $S$, let $R$ be an effective Weil divisor on the surface $S$, and let $\mathcal{D}$ be a linear system on the surface $S$ that has no fixed components.

**Lemma 2.1.** Let $D_1$ and $D_2$ be general curves in $\mathcal{D}$. Then
\[ \text{mult}_O \left( D_1 \cdot R \right) = \text{mult}_O \left( D_2 \cdot R \right) \leq \text{mult}_O (R) \text{mult}_O \left( D_1 \cdot D_1 \right). \]

**Proof.** Put $S_0 = S$ and $O_0 = O$. Let us consider the sequence of blow ups
\[ S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 \]
such that $\pi_1$ is a blow up of the point $O_0$, and $\pi_i$ is a blow up of the point $O_{i-1}$ that is contained in the curve $E_{i-1}$, where $E_{i-1}$ is the exceptional curve of $\pi_{i-1}$, and $i = 2, \ldots, n$.

Let $D^i_j$ be the proper transform of $D_j$ on $S_i$ for $i = 0, \ldots, n$ and $j = 1, 2$. Then
\[ D^i_1 \equiv D^i_2 \equiv \pi^*_i \left( D^{i-1}_1 \right) - \text{mult}_O \left( D^{i-1}_1 \right) E_i \equiv \pi^*_i \left( D^{i-1}_2 \right) - \text{mult}_O \left( D^{i-1}_2 \right) E_i \]
for $i = 1, \ldots, n$. Put $d_i = \text{mult}_O (D^{i-1}_1) = \text{mult}_O (D^{i-1}_2)$ for $i = 1, \ldots, n$. 

Let $R^i$ be the proper transform of $R$ on the surface $S_i$ for $i = 0, \ldots, n$. Then
\[ R^i \equiv \pi^*_i \left( R^{i-1} \right) - \text{mult}_{O_{i-1}}(R^{i-1}) E_i \]
for $i = 1, \ldots, n$. Put $r_i = \text{mult}_{O_{i-1}}(R^{i-1})$ for $i = 1, \ldots, n$. Then $r_1 = \text{mult}_O(R)$.

We may chose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D^n_1 \cap D^n_2$ is empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then
\[ \text{mult}_O(D_1 \cdot D_2) = \sum_{i=1}^n d_i^2. \]

We may chose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D^n_1 \cap R^n$ and $D^n_2 \cap R^n$ are empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then
\[ \text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i, \]

where some numbers among $r_1, \ldots, r_n$ may be zero. Then
\[ \text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i \leq \sum_{i=1}^n d_i r_1 \leq \sum_{i=1}^n d_i^2 r_1 = \text{mult}_O(R) \text{mult}_O(D_1 \cdot D_2), \]
because $d_i \leq d_i^2$ and $r_i \leq r_1 = \text{mult}_O(R)$ for every $i = 1, \ldots, n$. □

The assertion of Lemma 2.1 is a cornerstone of the proof of Theorem 1.5.

3. Curves

Let $X$ be a smooth quartic threefold in $\mathbb{P}^4$, let $\mathcal{M}$ be a Halphen pencil on $X$. Then
\[ \mathcal{M} \sim -nK_X, \]

since $\text{Pic}(X) = \mathbb{Z}K_X$. Put $\mu = 1/n$. Then
- the log pair $(X, \mu\mathcal{M})$ is canonical by [3, Theorem A],
- the log pair $(X, \mu\mathcal{M})$ is not terminal by [2, Theorem 2.1].

Let $\mathcal{CS}(X, \mu\mathcal{M})$ be the set of non-terminal centers of $(X, \mu\mathcal{M})$ (see [2]). Then
\[ \mathcal{CS}(X, \mu\mathcal{M}) \neq \emptyset, \]
because $(X, \mu\mathcal{M})$ is not terminal. Let $M_1$ and $M_2$ be two general surfaces in $\mathcal{M}$.

**Lemma 3.1.** Suppose that $\mathcal{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then
\[ \text{mult}_P(M) = n\text{mult}_P(T) = 2n, \]

where $M$ is any surface in $\mathcal{M}$, and $T$ is the surface in $| - K_X|$ that is singular at $P$.

**Proof.** It follows from [6, Proposition 1] that the inequality
\[ \text{mult}_P(M_1 \cdot M_2) \geq 4n^2 \]
holds. Let $H$ be a general surface in $| - K_X|$ such that $P \in H$. Then
\[ 4n^2 = H \cdot M_1 \cdot M_2 \geq \text{mult}_P(M_1 \cdot M_2) \geq 4n^2, \]
which gives $(M_1 \cdot M_2)_P = 4n^2$. Arguing as in the proof of [6, Proposition 1], we see that
\[ \text{mult}_P(M_1) = \text{mult}_P(M_2) = 2n, \]
Lemma 3.2. Suppose that $\text{CS}(X, \mu M)$ contains a point $P \in X$. Then
\[ M_1 \cap M_2 = \bigcup_{i=1}^{r} L_i, \]
where $L_1, \ldots, L_r$ are lines on the threefold $X$ that pass through the point $P$.

Proof. Let $H$ be a general surface in $|-K_X|$ such that $P \in H$. Then
\[ 4n^2 = H \cdot M_1 \cdot M_2 = \mu_P(M_1 \cdot M_2) = 4n^2 \]
by Lemma 3.1. Then $\text{Supp}(M_1 \cdot M_2)$ consists of lines on $X$ that pass through $P$. □

Lemma 3.3. Suppose that $\text{CS}(X, \mu M)$ contains a point $P \in X$. Then
\[ n/3 \leq \mu_L(M) \leq n/2 \]
for every line $L \subset X$ that passes through the point $P$.

Proof. Let $D$ be a general hyperplane section of $X$ through $L$. Then we have
\[ M|_D = \mu_L(M)L + \Delta, \]
where $M$ is a general surface in $M$ and $\Delta$ is an effective divisor such that
\[ \mu_P(\Delta) \geq 2n - \mu_L(M). \]

On the surface $D$ we have $L \cdot L = -2$. Then
\[ n = (\mu_L(M)L + \Delta) \cdot L = -2\mu_L(M) + \Delta \cdot L \]
on the surface $D$. But $\Delta \cdot L \geq \mu_P(\Delta) \geq 2n - \mu_L(M)$. Thus, we get
\[ n \geq -2\mu_L(M) + \mu_P(\Delta) \geq 2n - 3\mu_L(M), \]
which implies that $n/3 \leq \mu_L(M)$.

Let $T$ be the surface in $|-K_X|$ that is singular at $P$. Then $T \cdot D$ is reduced and
\[ T \cdot D = L + Z, \]
where $Z$ is an irreducible plane cubic curve such that $P \in Z$. Then
\[ 3n = (\mu_L(M)L + \Delta) \cdot Z = 3\mu_L(M) + \Delta \cdot Z \]
on the surface $D$. The set $\Delta \cap Z$ is finite by Lemma 3.2. In particular, we have
\[ \Delta \cdot Z \geq \mu_P(\Delta) \geq 2n - \mu_L(M), \]
because $\text{Supp}(\Delta)$ does not contain the curve $Z$. Thus, we get
\[ 3n \geq 3\mu_L(M) + \mu_P(\Delta) \geq 2n + 2\mu_L(M), \]
which implies that $\mu_L(M) \leq n/2$. □

In the rest of this section we prove the following result.
Proposition 3.4. Suppose that $\text{CS}(X, \mu M)$ contains a curve. Then $n = 1$.

Suppose that $\text{CS}(X, \mu M)$ contains a curve $Z$. Then it follows Lemmas 3.2 and 3.3 that the set $\text{CS}(X, \mu M)$ does not contain points of the threefold $X$ and

\[
\text{mult}_Z(M) = n,
\]

because $(X, \mu M)$ is canonical but not terminal. Then $\deg(Z) \leq 4$ by [2, Lemma 2.1].

Lemma 3.6. Suppose that $\deg(Z) = 1$. Then $n = 1$.

Proof. Let $\pi : V \rightarrow X$ be the blow up of $X$ along the line $Z$. Let $B$ be the proper transform of the pencil $M$ on the threefold $V$, and let $B$ be a general surface in $B$. Then

\[
B \sim -nK_V
\]

by (3.5). There is a commutative diagram

\[
\begin{array}{c}
V \\
\downarrow \pi \\
X \\
\downarrow \psi \\
\eta \\
\downarrow \\
\mathbb{P}^2
\end{array}
\]

where $\psi$ is the projection from the line $Z$ and $\eta$ is the morphism induced by the linear system $| - K_V|$. Thus, it follows from (3.7) that $B$ is the pull-back of a pencil $P$ on $\mathbb{P}^2$ by $\eta$.

We see that the base locus of $B$ is contained in the union of fibers of $\eta$.

The set $\text{CS}(V, \mu B)$ is not empty by [2, Theorem 2.1]. It easily follows from (3.5) that the set $\text{CS}(V, \mu B)$ does not contain points because $\text{CS}(X, \mu M)$ contains no points.

We see that there is an irreducible curve $L \subset V$ such that

\[
\text{mult}_L(B) = n
\]

and $\eta(L)$ is a point $Q \in \mathbb{P}^2$. Let $C$ be a general curve in $P$. Then $\text{mult}_Q(C) = n$. But

\[
C \sim \mathcal{O}_{\mathbb{P}^2}(n)
\]

by (3.7). Thus, we see that $n = 1$, because general surface in $M$ is irreducible. \(\square\)

Thus, we may assume that the set $\text{CS}(X, \mu M)$ does not contain lines.

Lemma 3.8. The curve $Z \subset \mathbb{P}^4$ is contained in a plane.

Proof. Suppose that $Z$ is not contained in any plane in $\mathbb{P}^4$. Let us show that this assumption leads to a contradiction. Since $\deg(Z) \leq 4$, we have

\[
\deg(Z) \in \{3, 4\},
\]

and $Z$ is smooth if $\deg(Z) = 3$. If $\deg(Z) = 4$, then $Z$ may have at most one double point.

Suppose that $Z$ is smooth. Let $\alpha : U \rightarrow X$ be the blow up at $Z$, and let $F$ be the exceptional divisor of the morphism $\alpha$. Then the base locus of the linear system

\[
\left| \alpha^* \left( - \deg(Z) K_X \right) - F \right|
\]

does not contain any curve. Let $D_1$ and $D_2$ be the proper transforms on $U$ of two sufficiently general surfaces in the linear system $M$. Then it follows from (3.5) that

\[
\left( \alpha^* \left( - \deg(Z) K_X \right) - F \right) \cdot D_1 \cdot D_2 = n^2 \left( \alpha^* \left( - \deg(Z) K_X \right) - F \right) \cdot \left( \alpha^* \left( - K_X \right) - F \right)^2 \geq 0,
\]
because the cycle $D_1 \cdot D_2$ is effective. On the other hand, we have

$$\left(\alpha^* \left( - \deg(Z) K_X \right) - F \right) \cdot \left( \alpha^* \left( - K_X \right) - F \right)^2 = \left( 3 \deg(Z) - \left( \deg(Z) \right)^2 - 2 \right) < 0,$$

which is a contradiction. Thus, the curve $Z$ is not smooth.

Thus, we see that $Z$ is a quartic curve with a double point $O$.

Let $\beta: W \to X$ be the composition of the blow up of the point $O$ with the blow up of the proper transform of the curve $Z$. Let $G$ and $E$ be the exceptional surfaces of the morphism $\beta$ such that $\beta(E) = Z$ and $\beta(G) = O$. Then the base locus of the linear system

$$\left| \beta^* \left( - 4K_X \right) - E - 2G \right|$$

does not contain any curve. Let $R_1$ and $R_2$ be the proper transforms on $W$ of two sufficiently general surfaces in $M$. Put $m = \text{mult}_O(M)$. Then it follows from \((3.5)\) that

$$\left( \beta^* \left( - 4K_X \right) - E - 2G \right) \cdot R_1 \cdot R_2 = \left( \beta^* \left( - 4K_X \right) - E - 2G \right) \cdot \left( \beta^* \left( - nK_X \right) - nE - mG \right)^2 \geq 0,$$

and $m < 2n$, because the set $\mathcal{CS}(X, \mu M)$ does not contain points. Then

$$\left( \beta^* \left( - 4K_X \right) - E - 2G \right) \cdot \left( \beta^* \left( - nK_X \right) - nE - mG \right)^2 = -8n^2 + 6mn - m^2 < 0,$$

which is a contradiction. \qed

If $\deg(Z) = 4$, then $n = 1$ by Lemma 3.8 and [2, Theorem 2.2].

**Lemma 3.9.** Suppose that $\deg(Z) = 3$. Then $n = 1$.

**Proof.** Let $\mathcal{P}$ be the pencil in $|-K_X|$ that contains all hyperplane sections of $X$ that pass through the curve $Z$. Then the base locus of $\mathcal{P}$ consists of the curve $Z$ and a line $L \subset X$.

Let $D$ be a sufficiently general surface in the pencil $\mathcal{P}$, and let $M$ be a sufficiently general surface in the pencil $\mathcal{M}$. Then $D$ is a smooth surface, and

\[(3.10) \quad M|_D = nZ + \text{mult}_L(\mathcal{M})L + B \equiv nZ + nL,\]

where $B$ is a curve whose support does not contain neither $Z$ nor $L$.

On the surface $D$, we have $Z \cdot L = 3$ and $L \cdot L = -2$. Intersecting \((3.10)\) with $L$, we get

$$n = (nZ + nL) \cdot L = 3n - 2 \text{mult}_L(\mathcal{M}) + B \cdot L \geq 3n - 2 \text{mult}_L(\mathcal{M}),$$

which easily implies that $\text{mult}_L(\mathcal{M}) \geq n$. But the inequality $\text{mult}_L(\mathcal{M}) \geq n$ is impossible, because we assumed that $\mathcal{CS}(X, \mu M)$ contains no lines. \qed

**Lemma 3.11.** Suppose that $\deg(Z) = 2$. Then $n = 1$.

**Proof.** Let $\alpha: U \to X$ be the blow up of the curve $Z$. Then $|-K_U|$ is a pencil, whose base locus consists of a smooth irreducible curve $L \subset U$.

Let $D$ be a general surface in $|-K_U|$. Then $D$ is a smooth surface.

Let $\mathcal{B}$ be the proper transform of the pencil $\mathcal{M}$ on the threefold $U$. Then

$$-nK_U|_D \equiv B|_D \equiv nL,$$
where $B$ is a general surface in $B$. But $L^2 = -2$ on the surface $D$. Then

$$L \in \mathcal{CS}(U, \mu B)$$

which implies that $B = |-K_U|$ by [2] Theorem 2.2. Then $n = 1$. □

The assertion of Proposition 3.4 is proved.

4. Points

Let $X$ be a smooth quartic threefold in $\mathbb{P}^4$, let $\mathcal{M}$ be a Halphen pencil on $X$. Then

$$\mathcal{M} \sim -nK_X,$$

since $\text{Pic}(X) = \mathbb{Z}K_X$. Put $\mu = 1/n$. Then

- the log pair $(X, \mu \mathcal{M})$ is canonical by [3] Theorem A,
- the log pair $(X, \mu \mathcal{M})$ is not terminal by [2] Theorem 2.1.

Remark 4.1. To prove Theorem 1.5 it is enough to show that $X$ can be given by

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where $q_i$ and $p_i$ are homogeneous polynomials of degree $i \geq 2$ such that $q_2(0, y, z, t) \neq 0$.

Let $\mathcal{CS}(X, \mu \mathcal{M})$ be the set of non-terminal centers of $(X, \mu \mathcal{M})$ (see [2]). Then

$$\mathcal{CS}(X, \mu \mathcal{M}) \neq \emptyset,$$

because $(X, \mu \mathcal{M})$ is not terminal. Suppose that $n \neq 1$. There is a point $P \in X$ such that

$$P \in \mathcal{CS}(X, \mu \mathcal{M})$$

by Proposition 3.4. It follows from Lemmas 3.1, 3.2 and 3.3 that

- there are finitely many distinct lines $L_1, \ldots, L_r \subset X$ containing $P \in X$,
- the equality $\text{mult}_P(\mathcal{M}) = 2n$ holds, and
  $$n/3 \leq \text{mult}_{L_i}(M) \leq n/2,$$

  where $M$ is a general surface in the pencil $\mathcal{M}$,
- the equality $\text{mult}_P(T) = 2$ holds, where $T \in |-K_X|$ such that $\text{mult}_P(T) \geq 2$,
- the base locus of the pencil $\mathcal{M}$ consists of the lines $L_1, \ldots, L_r$, and
  $$\text{mult}_P(M_1 \cdot M_2) = 4n^2,$$

  where $M_1$ and $M_2$ are sufficiently general surfaces in $\mathcal{M}$.

Lemma 4.2. The equality $\mathcal{CS}(X, \mu \mathcal{M}) = \{P\}$ holds.

Proof. The set $\mathcal{CS}(X, \mu \mathcal{M})$ does not contain curves by Proposition 3.4.

Suppose that $\mathcal{CS}(X, \mu \mathcal{M})$ contains a point $Q \in X$ such that $Q \neq P$. Then $r = 1$.

Let $D$ be a general hyperplane section of $X$ that passes through $L_1$. Then

$$M|_D = \text{mult}_{L_1}(\mathcal{M})L_1 + \Delta,$$

where $M$ is a general surface in $\mathcal{M}$ and $\Delta$ is an effective divisor such that

$$\text{mult}_P(\Delta) \geq 2n - \text{mult}_{L_1}(\mathcal{M}) \leq \text{mult}_Q(\Delta).$$
On the surface $D$, we have $L_1^2 = -2$. Then
\[ n = \left( \mult_{L_1}(\mathcal{M})L_1 + \Delta \right) \cdot L_1 = -2\mult_{L_1}(\mathcal{M}) + \Delta \cdot L \geq -2\mult_{L_1}(\mathcal{M}) + 2\left( 2n - \mult_{L_1}(\mathcal{M}) \right), \]
which gives $\mult_{L_1}(\mathcal{M}) \geq 3n/4$. But $\mult_{L_1}(\mathcal{M}) \leq n/2$ by Lemma 3.3. \qed

The quartic threefold $X$ can be given by an equation
\[ w^3x + w^2q_2(x, y, z, t) + wq_3(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}\left( \mathbb{C}[x, y, z, t, w] \right) \cong \mathbb{P}^4, \]
where $q_i$ is a homogeneous polynomial of degree $i \geq 2$.

Remark 4.3. The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations
\[ x = q_2(x, y, z, t) = q_3(x, y, z, t) = q_4(x, y, z, t) = 0, \]
the surface $T$ is cut out on $X$ by $x = 0$, and $\mult_P(T) = 2 \iff q_2(0, y, z, t) \neq 0$.

Let $\pi: V \to X$ be the blow up of the point $P$, let $E$ be the $\pi$-exceptional divisor. Then
\[ B \equiv \pi^*\left( -nK_X \right) - 2nE \equiv -nK_V, \]
where $B$ is the proper transform of the pencil $\mathcal{M}$ on the threefold $V$.

Remark 4.4. The pencil $B$ has no base curves in $E$, because
\[ \mult_P(M_1 \cdot M_2) = \mult_P(M_1)\mult_P(M_2). \]

Let $\tilde{L}_i$ be the proper transform of the line $L_i$ on the threefold $V$ for $i = 1, \ldots, r$. Then
\[ B_1 \cdot B_2 = \sum_{i=1}^{r} \mult_{\tilde{L}_i}(B_1 \cdot B_2)\tilde{L}_i, \]
where $B_1$ and $B_2$ are proper transforms of $M_1$ and $M_2$ on the threefold $V$, respectively.

Lemma 4.5. Let $Z$ be an irreducible curve on $X$ such that $Z \notin \{L_1, \ldots, Z_r\}$. Then
\[ \deg(Z) \geq 2\mult_P(Z), \]
and the equality $\deg(Z) = 2\mult_P(Z)$ implies that
\[ \tilde{Z} \cap \left( \bigcup_{i=1}^{r} \tilde{L}_i \right) = \emptyset, \]
where $\tilde{Z}$ is a proper transform of the curve $Z$ on the threefold $V$.

Proof. The curve $\tilde{Z}$ is not contained in the base locus of the pencil $\mathcal{B}$. Then
\[ 0 \leq B_1 \cdot \tilde{Z} \leq n\left( \deg(Z) - 2\mult_P(Z) \right), \]
which implies the required assertions. \qed

To conclude the proof of Theorem 1.5, it is enough to show that
\[ q_3(x, y, z, t) = xp_2(x, y, z, t) + q_2(x, y, z, t)p_1(x, y, z, t), \]
where $p_1$ and $p_2$ are some homogeneous polynomials of degree 1 and 2, respectively.
5. Good points

Let us use the assumptions and notation of Section 4. Suppose that the conic
\[ q_2(0, y, z, t) = 0 \subset \text{Proj} \left( \mathbb{C}[y, z, t] \right) \cong \mathbb{P}^2 \]
is reduced and irreducible. In this section we prove the following result.

**Proposition 5.1.** The polynomial \( q_3(0, y, z, t) \) is divisible by \( q_2(0, y, z, t) \).

Let us prove Proposition 5.1. Suppose that \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \).

Let \( R \) be the linear system on the threefold \( X \) that is cut out by quadrics
\[ xh_1(x, y, z, t) + \lambda (wx + q_2(x, y, z, t)) = 0, \]
where \( h_1 \) is an arbitrary linear form and \( \lambda \in \mathbb{C} \). Then \( R \) does not have fixed components.

**Lemma 5.2.** Let \( R_1 \) and \( R_2 \) be general surfaces in the linear system \( R \). Then
\[ \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \leq 6. \]

**Proof.** We may assume that \( R_1 \) is cut out by the equation
\[ wx + q_2(x, y, z, t) = 0, \]
and \( R_2 \) is cut out by \( xh_1(x, y, z, t) = 0 \), where \( h_1 \) is sufficiently general. Then
\[ \text{mult}_{L_i}(R_1 \cdot R_2) = \text{mult}_{L_i}(R_1 \cdot T). \]

Put \( m_i = \text{mult}_{L_i}(R_1 \cdot T). \) Then
\[ R_1 \cdot T = \sum_{i=1}^{r} m_i L_i + \Delta, \]
where \( m_i \in \mathbb{N} \), and \( \Delta \) is a cycle, whose support contains no lines passing through \( P \).

Let \( \tilde{R}_1 \) and \( \tilde{T} \) be the proper transforms of \( R_1 \) and \( T \) on \( V \), respectively. Then
\[ \tilde{R}_1 \cdot \tilde{T} = \sum_{i=1}^{r} m_i \tilde{L}_i + \Omega, \]
where \( \Omega \) is an effective cycle, whose support contains no lines passing through \( P \).

The support of the cycle \( \Omega \) does not contain curves that are contained in the exceptional divisor \( E \), because \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \) by our assumption. Then
\[ 6 = E \cdot \tilde{R}_1 \cdot \tilde{T} = \sum_{i=1}^{r} m_i (E \cdot \tilde{L}_i) + E \cdot \Omega \geq \sum_{i=1}^{r} m_i (E \cdot \tilde{L}_i) = \sum_{i=1}^{r} m_i, \]
which is exactly what we want. \( \square \)

Let \( M \) and \( R \) be general surfaces in \( M \) and \( R \), respectively. Put
\[ M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta, \]
where \( m_i \in \mathbb{N} \), and \( \Delta \) is a cycle, whose support contains no lines passing through \( P \).

**Lemma 5.3.** The cycle \( \Delta \) is not trivial.
Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil. □

We have $\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i$. On the other hand, the inequality

$$\text{mult}_P(\Delta) \geq 6n - \sum_{i=1}^{r} m_i$$

holds, because $\text{mult}_P(M) = 2n$ and $\text{mult}_P(R) \geq 3$. It follows from Lemma 4.5 that

$$\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i \geq 2\text{mult}_P(\Delta) \geq 2\left(6n - \sum_{i=1}^{r} m_i\right),$$

which implies that $\sum_{i=1}^{r} m_i \geq 4n$. But it follows from Lemmas 2.1 and 3.3 that

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every $i = 1, \ldots, r$, where $R_1$ and $R_2$ are general surfaces in $\mathcal{R}$. Then

$$\sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2)n/2 \leq 3n$$

by Lemma 5.2, which is a contradiction.

The assertion of Proposition 5.1 is proved.

6. Bad points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$$

is reduced and reducible. Therefore, we have

$$q_2(x, y, z, t) = (\alpha_1 y + \beta_1 z + \gamma_1 t)(\alpha_2 y + \beta_2 z + \gamma_2 t) + xp_1(x, y, z, t)$$

where $p_1(x, y, z, t)$ is a linear form, and $(\alpha_1 : \beta_1 : \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2 : \beta_2 : \gamma_2)$.

Proposition 6.1. The polynomial $q_3(0, y, z, t)$ is divisible by $q_2(0, y, z, t)$.

Suppose that $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$. Then without loss of generality, we may assume that $q_3(0, y, z, t)$ is not divisible by $\alpha_1 y + \beta_1 z + \gamma_1 t$.

Let $Z$ be the curve in $X$ that is cut out by the equations

$$x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0.$$  

Remark 6.2. The equality $\text{mult}_P(Z) = 3$ holds, but $Z$ is not necessary reduced.

Hence, it follows from Lemma 4.5 that $\text{Supp}(Z)$ contains a line among $L_1, \ldots, L_r$.

Lemma 6.3. The support of the curve $Z$ does not contain an irreducible conic.

Proof. Suppose that $\text{Supp}(Z)$ contains an irreducible conic $C$. Then

$$Z = C + L_i + L_j$$

for some $i \in \{1, \ldots, r\} \ni j$. Then $i = j$, because otherwise the set

$$\left(C \cap L_i\right) \cup \left(C \cap L_j\right)$$

contains a point that is different from $P$, which is impossible by Lemma 4.5. We see that

$$Z = C + 2L_i,$$
and it follows from Lemma 4.5 that $C \cap L_i = P$. Then $C$ is tangent to $L_i$ at the point $P$.

Let $\bar{C}$ be a proper transform of the curve $C$ on the threefold $V$. Then

$$\bar{C} \cap \bar{L}_i \neq \emptyset,$$

which is impossible by Lemma 4.5. The assertion is proved. □

**Lemma 6.4.** The support of the curve $Z$ consists of lines.

*Proof.* Suppose that $\text{Supp}(Z)$ does not consist of lines. It follows from Lemma 6.3 that

$$Z = L_i + C,$$

where $C$ is an irreducible cubic curve. But $\text{mult}_P(Z) = 3$. Then

$$\text{mult}_P(C) = 2,$$

which is impossible by Lemma 4.5 □

We may assume that there is a line $L \subset X$ such that $P \notin P$ and

$$Z = a_1L_1 + \cdots + a_kL_k + L,$$

where $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 \geq a_2 \geq a_3$ and $\sum_{i=1}^k a_i = 3$.

**Remark 6.5.** We have $L_i \neq L_j$ whenever $i \neq j$.

Let $H$ be a sufficiently general surface of $X$ that is cut out by the equation

$$\lambda x + \mu (\alpha_1 y + \beta_1 z + \gamma_1 t) = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then $H$ has at most isolated singularities.

**Remark 6.6.** The surface $H$ is smooth at the points $P$ and $L \cap L_i$, where $i = 1, \ldots, k$.

Let $\bar{H}$ and $\bar{L}$ be the proper transforms of $H$ and $L$ on the threefold $V$, respectively.

**Lemma 6.7.** The inequality $k \neq 3$ holds.

*Proof.* Suppose that the equality $k = 3$ holds. Then $H$ is smooth. Put

$$B \bigg|_{\bar{H}} = m_1\bar{L}_1 + m_2\bar{L}_2 + m_3\bar{L}_3 + \Omega,$$

where $B$ is a general surface in $\mathcal{B}$, and $\Omega$ is an effective divisor on $\bar{H}$ whose support does not contain any of the curves $\bar{L}_1, \bar{L}_2$ and $\bar{L}_3$. Then

$$\bar{L} \notin \text{Supp}(\Omega) \nsubseteq \bar{H} \cap E,$$

because the base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = L \cdot \left( m_1L_1 + m_2L_2 + m_3L_3 + \Omega \right) = \sum_{i=1}^3 m_i + L \cdot \Omega \geq \sum_{i=1}^3 m_i,$$

which implies that $\sum_{i=1}^3 m_i \leq n$. On the other hand, we have

$$-n = \bar{L}_i \cdot \left( m_1\bar{L}_1 + m_2\bar{L}_2 + m_3\bar{L}_3 + \Omega \right) = -3m_i + \bar{L}_i \cdot \Omega \geq -3m_i,$$

which implies that $m_i \geq n/3$. Thus, we have $m_1 = m_2 = m_3 = n/3$ and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,$$

which implies that $\text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \text{Supp}(\Omega) \cap \bar{L}_3 = \emptyset$. 

Lemma 6.8. The inequality \( k \neq 2 \) holds.

Proof. Suppose that the equality \( k = 2 \) holds. Then \( Z = 2L_1 + L_2 + L \). Put

\[
B' \bigg|_\bar{H} = m_1\bar{L}_1 + m_2\bar{L}_2 + \bar{\Omega},
\]

where \( B \) is a general surface in \( \mathcal{B} \), and \( \Omega \) is an effective divisor on \( \bar{H} \) whose support does not contain the curves \( \bar{L}_1 \) and \( \bar{L}_2 \). Then \( \bar{L} \notin \text{Supp}(\Omega) \not\supset H \cap E \) and

\[
n = L \cdot \left( m_1L_1 + m_2L_2 + \Omega \right) = m_1 + m_2 + L \cdot \Omega \geq m_1 + m_2,
\]

which implies that \( m_1 + m_2 \leq n \). On the other hand, we have

\[
\bar{T} \big|_\bar{H} = 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E \bigg|_{\bar{H}} \equiv \left( \pi^* \left( -K_X \right) - 2E \right) \bigg|_{\bar{H}},
\]

where \( T \) is the proper transform of the surface \( T \) on the threefold \( V \). Then

\[
-1 = \bar{L}_1 \cdot \left( 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E \big|_{\bar{H}} \right) = 2 \left( \bar{L}_1 \cdot \bar{L}_1 \right) + 2,
\]

which implies that \( \bar{L}_1 \cdot \bar{L}_1 = -3/2 \) on the surface \( \bar{H} \). Then

\[
-n = \bar{L}_1 \cdot \left( m_1L_1 + m_2L_2 + \Omega \right) = -3m_1/2 + L_1 \cdot \Omega \geq -3m_1/2,
\]

which gives \( m_1 \geq 2n/3 \). Similarly, we see that \( \bar{L}_2 \cdot \bar{L}_2 = -3 \) on the surface \( \bar{H} \). Then

\[
-n = \bar{L}_2 \cdot \left( m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega \right) = -3m_2 + L_2 \cdot \Omega \geq -3m_2,
\]

which implies that \( m_2 \leq n/3 \). Thus, we have \( m_1 = 2m_2 = 2n/3 \) and

\[
\Omega \cdot L = \Omega \cdot L_1 = \Omega \cdot L_2 = 0,
\]

which implies that \( \text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \emptyset \).
Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that
\[
B'|_H = \frac{2n}{3}L_1 + \frac{n}{3}L_2 + \Omega',
\]
where $\Omega'$ is an effective divisor on $H$ whose support does not contain $L_1$ and $L_2$ such that
\[
\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \emptyset,
\]
which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that
\[
\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,
\]
and arguing as in the proof of Lemma 6.7 we obtain a contradiction. $\square$

It follows from Lemmas 6.7 and 6.8 that $Z = 3L_1 + L$. Put
\[
B'|_H = m_1 \bar{L}_1 + \Omega,
\]
where $B$ is a general surface in $\mathcal{B}$, and $\Omega$ is a curve such that $\bar{L}_1 \not\subseteq \text{Supp}(\Omega)$. Then
\[
\bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E,
\]
because the base locus of $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then
\[
n = \bar{L} \cdot (m_1 \bar{L}_1 + \Omega) = m_1 + \bar{L} \cdot \Omega \geq m_1,
\]
which implies that $m_1 \leq n$. On the other hand, we have

\[
T|_H = 3L_1 + L + E|_H \equiv \left(\pi^*\left(-K_X\right) - 2E\right)|_H,
\]
where $\bar{T}$ is the proper transform of the surface $T$ on the threefold $V$. Then
\[
-1 = \bar{L}_1 \cdot (3\bar{L}_1 + \bar{L} + E|_H) = 3\bar{L}_1 \cdot \bar{L}_1 + 2,
\]
which implies that $\bar{L}_1 \cdot \bar{L}_1 = -1$ on the surface $\bar{H}$. Then
\[
-n = \bar{L}_1 \cdot (m_1 \bar{L}_1 + \Omega) = -m_1 + L_1 \cdot \Omega \geq -m_1,
\]
which gives $m_1 \geq n$. Thus, we have $m_1 = n$ and $\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = 0$. Then $\text{Supp}(\Omega) \cap \bar{L}_1 = \emptyset$.

Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that
\[
B'|_H = n\bar{L}_1 + \Omega',
\]
where $\Omega'$ is an effective divisor on $H$ whose support does not contain $\bar{L}_1$ such that
\[
\text{Supp}(\Omega') \cap \bar{L}_1 = \emptyset,
\]
which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that $\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset$.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Hence, we have
\[
\text{Supp}(\Omega') \cap \bar{L}_1 = \emptyset,
\]
but $\bar{L}_1 \cap H = \emptyset$ whenever $\bar{L}_1 \neq \bar{L}_1$. Then $\text{Supp}(\Omega) \cap \bar{L}_1 \neq \emptyset$, because
\[
\bar{L}_1 \cup \left(\text{Supp}(\Omega) \cap \text{Supp}(\Omega')\right) = \text{Supp}(\Omega') \cap \bar{L}_1 = \bar{L}_1,
\]
which is a contradiction. The assertion of Proposition 6.1 is proved.
7. Very bad points

Let us use the assumptions and notation of Section 4. Suppose that \( q_2 = y^2 \).

The proof of Proposition 6.1 implies that \( q_3(0, y, z, t) \) is divisible by \( y \). Then
\[
q_3 = yf_2(z, t) + xh_2(z, t) + x^2a_1(x, y, z, t) + xyb_1(x, y, z, t) + y^2c_1(y, z, t)
\]
where \( a_1, b_1, c_1 \) are linear forms, \( f_2 \) and \( h_2 \) are are homogeneous polynomials of degree two.

**Proposition 7.1.** The equality \( f_2(z, t) = 0 \) holds.

Let us prove Proposition 7.1 by reductio ad absurdum. Suppose that \( f_2(z, t) \neq 0 \).

**Remark 7.2.** By choosing suitable coordinates, we may assume that \( f_2 = zt \) or \( f_2 = z^2 \).

We must use smoothness of the threefold \( X \) by analyzing the shape of \( q_4 \). We have
\[
q_4 = f_4(z, t) + xu_3(z, t) + yv_3(z, t) + x^2a_2(x, y, z, t) + xyb_2(x, y, z, t) + y^2c_2(y, z, t),
\]
where \( a_2, b_2, c_2 \) are homogeneous polynomials of degree two, \( u_3 \) and \( v_3 \) are homogeneous polynomials of degree three, and \( f_4 \) is a homogeneous polynomial of degree four.

**Lemma 7.3.** Suppose that \( f_2(z, t) = zt \) and
\[
f_4(z, t) = t^2g_2(z, t)
\]
for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(z, 0) \neq 0 \).

**Proof.** Suppose that \( v_3(z, 0) = 0 \). The surface \( T \) is given by the equation
\[
w^2y^2 + yzt + y^2c_1(x, y, z, t) + t^2g_2(z, t) + yv_3(z, t) + y^2c_2(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, w]) \cong \mathbb{P}^3
\]
because \( T \) is cut out on \( X \) by the equation \( x = 0 \). Then \( T \) has non-isolated singularity along the line \( x = y = t = 0 \), which is impossible because \( X \) is smooth.

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

**Corollary 7.4.** Suppose that \( f_2(z, t) = zt \) and
\[
f_4(z, t) = z^2g_2(z, t)
\]
for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(0, t) \neq 0 \).

**Lemma 7.5.** Suppose that \( f_2(z, t) = zt \). Then \( f_4(0, t) = f_4(z, 0) = 0 \).

**Proof.** We may assume that \( f_4(0, 0) \neq 0 \). Let \( \mathcal{H} \) be the linear system on \( X \) that is cut out by
\[
\lambda x + \mu y + \nu t = 0,
\]
where \((\lambda : \mu : \nu) \in \mathbb{P}^2\). Then the base locus of \( \mathcal{H} \) consists of the point \( P \).

Let \( \mathcal{R} \) be a proper transform of \( \mathcal{H} \) on the threefold \( V \). Then the base locus of \( \mathcal{R} \) consists of a single point that is not contained in any of the curves \( L_1, \ldots, L_r \).

The linear system \( \mathcal{R}|_B \) has not base points, where \( B \) is a general surface in \( \mathcal{B} \). But
\[
R \cdot R \cdot B = 2n > 0,
\]
where \( R \) is a general surface in \( \mathcal{R} \). Then \( \mathcal{R}|_B \) is not composed from a pencil, which implies that the curve \( R \cdot B \) is irreducible and reduced by the Bertini theorem.

Let \( H \) and \( M \) be general surfaces in \( \mathcal{H} \) and \( \mathcal{M} \), respectively. Then \( M \cdot H \) is irreducible and reduced. Thus, the linear system \( \mathcal{M}|_H \) is a pencil.
The surface $H$ contains no lines passing through $P$, and $H$ can be given by
\[ w^3 x + w^2 y^2 + w \left( y^2 l_1(x, y, z) + x l_2(x, y, z) \right) + l_4(x, y, z) = 0 \subset \text{Proj} \left( \mathbb{C}[x, y, z, w] \right) \cong \mathbb{P}^3, \]
where $l_i(x, y, z)$ is a homogeneous polynomials of degree $i$.

Arguing as in Example 1.4, we see that there is a pencil $Q$ on the surface $H$ such that
\[ Q \sim O_{\mathbb{P}^3}(2) \bigg|_H, \]
general curve in $Q$ is irreducible, and $\text{mult}_P(Q) = 4$. Arguing as in the proof of Lemma 3.1, we see that $\mathcal{M}|_H = Q$ by [2, Theorem 2.2]. Let $M$ be a general surface in $\mathcal{M}$. Then
\[ M \equiv -2K_X, \]
and $\text{mult}_P(M) = 4$. The surface $M$ is cut out on $X$ by an equation
\[ \lambda x^2 + x \left( A_0 + A_1(y, z, t) \right) + B_2(y, z, t) + B_1(y, z, t) + B_0 = 0, \]
where $A_i$ and $B_i$ are homogeneous polynomials of degree $i$, and $\lambda \in \mathbb{C}$.

It follows from $\text{mult}_P(M) = 4$ that $B_1(y, z, t) = B_0 = 0$.

The coordinated $(y, z, t)$ are also local coordinates on $X$ near the point $P$. Then
\[ x = -y^2 - y \left( z t + yp_1(y, z, t) \right) + \text{higher order terms}, \]
which is a Taylor power series for $x = x(y, z, t)$, where $p_1(y, z, t)$ is a linear form.

The surface $M$ is locally given by the analytic equation
\[ \lambda y^4 + \left( -y^2 - yzt - y^2 p_1(y, z, t) \right) \left( A_0 + A_1(y, z, t) \right) + B_2(y, z, t) + \text{higher order terms} = 0, \]
and $\text{mult}_P(M) = 4$. Hence, we see that $B_2(y, z, t) = A_0 y^2$ and
\[ A_1(y, z, t) y^2 + A_0 y \left( z t + yp_1(y, z, t) \right) = 0, \]
which implies that $A_0 = A_1(y, z, t) = B_2(y, z, t) = 0$. Hence, we see that a general surface in the pencil $\mathcal{M}$ is cut out on $X$ by the equation $x^2 = 0$, which is an absurd. □

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

**Corollary 7.6.** Suppose that $f_2(z, t) = z^2$. Then $f_4(0, t) = 0$.

Let $\mathcal{R}$ be the linear system on the threefold $X$ that is cut out by cubics
\[ x h_2(x, y, z, t) + \lambda \left( w^2 x + wy^2 + q_3(x, y, z, t) \right) = 0, \]
where $h_2$ is a form of degree 2, and $\lambda \in \mathbb{C}$. Then $\mathcal{R}$ has no fixed components.

Let $M$ and $R$ be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put
\[ M \cdot R = \sum_{i=1}^r m_i L_i + \Delta, \]
where $m_i \in \mathbb{N}$, and $\Delta$ is a cycle, whose support contains no lines among $L_1, \ldots, L_r$.

**Lemma 7.7.** The cycle $\Delta$ is not trivial.

*Proof.* Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil. □
We have \( \text{mult}_P(\Delta) \geq 8n - \sum_{i=1}^{r} m_i \), because \( \text{mult}_P(M) = 2n \) and \( \text{mult}_P(R) \geq 4 \). Then
\[
    \deg(\Delta) = 12n - \sum_{i=1}^{r} m_i \geq 2\text{mult}_P(\Delta) \geq 2\left(8n - \sum_{i=1}^{r} m_i\right)
\]
by Lemma 4.3, because \( \text{Supp}(\Delta) \) does not contain any of the lines \( L_1, \ldots, L_r \).

**Corollary 7.8.** The inequality \( \sum_{i=1}^{r} m_i \geq 4n \) holds.

Let \( R_1 \) and \( R_2 \) be general surfaces in the linear system \( R \). Then
\[
    m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2)\text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2
\]
for every \( 1 \leq i \leq 4 \) by Lemmas 2.1 and 3.3. Then
\[
    4n \leq \sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2)n/2.
\]

**Corollary 7.9.** The inequality \( \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8 \) holds.

Now we suppose that \( R_1 \) is cut out on the quartic \( X \) by the equation
\[
    w^2x + wy^2 + q_3(x, y, z, t) = 0,
\]
and \( R_2 \) is cut out by \( xh_2(x, y, z, t) = 0 \), where \( h_2 \) is sufficiently general. Then
\[
    \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot T) = \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8,
\]
where \( T \) is the hyperplane section of the hypersurface \( X \) that is cut out by \( x = 0 \). But
\[
    R_1 \cdot T = Z_1 + Z_2,
\]
where \( Z_1 \) and \( Z_2 \) are cycles on \( X \) such that \( Z_1 \) is cut out by \( x = y = 0 \), and \( Z_2 \) is cut out by
\[
    x = wy + f_2(z, t) + yc_1(x, y, z, t) = 0.
\]

**Lemma 7.10.** The equality \( \sum_{i=1}^{r} \text{mult}_{L_i}(Z_1) = 4 \) holds.

*Proof.* The lines \( L_1, \ldots, L_r \subset \mathbb{P}^4 \) are given by the equations
\[
    x = y = q_4(x, y, z, t) = 0,
\]
which implies that \( \sum_{i=1}^{r} \text{mult}_{L_i}(Z_1) = 4 \). \( \square \)

Hence, we see that \( \sum_{i=1}^{r} \text{mult}_{L_i}(Z_2) \geq 4 \). But \( Z_2 \) can be considered as a cycle
\[
    wy + f_2(z, t) + yc_1(y, z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}\left(\mathbb{C}[y, z, t, w]\right) \cong \mathbb{P}^3,
\]
and, putting \( u = w + c_1(y, z, t) \), we see that \( Z_2 \) can be considered as a cycle
\[
    uy + f_2(z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}\left(\mathbb{C}[y, z, t, u]\right) \cong \mathbb{P}^3,
\]
and we can consider the set of lines \( L_1, \ldots, L_r \) as the set in \( \mathbb{P}^3 \) given by \( y = f_4(z, t) = 0 \).

**Lemma 7.11.** The inequality \( f_2(z, t) \neq zt \) holds.
Proof. Suppose that \( f_2(z, t) = zt \). Then it follows from Lemma \ref{lem:corollary7.3} that
\[
f_4(z, t) = zt(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)
\]
for some \((\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)\). Then \( Z_2 \) can be given by
\[
uy + zt = yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,
\]
which implies \( Z_2 = Z_2^1 + Z_2^2 \), where \( Z_2^1 \) and \( Z_2^2 \) are cycles in \( \mathbb{P}^3 \) such that \( Z_2^1 \) is given by
\[
y = uy + zt = 0,
\]
and \( Z_2^2 \) is given by \( uy + zt = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0 \).

We may assume that \( L_1 \) is given by \( y = z = 0 \), and \( L_2 \) is given by \( y = t = 0 \). Then
\[
Z_2^1 = L_1 + L_2,
\]
which implies that \( \sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) \geq 2 \).

Suppose that \( r = 4 \). Then \( \alpha_1 \neq 0, \beta_1 \neq 0, \alpha_2 \neq 0, \beta_2 \neq 0 \). Hence, we see that
\[
L_1 \not\subset \text{Supp}(Z_2^2) \not\supset L_2,
\]
because \( v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) \) does not vanish on \( L_1 \) and \( L_2 \). But
\[
L_3 \not\subset \text{Supp}(Z_2^2) \not\supset L_4,
\]
because \( zt \) does not vanish on \( L_3 \) and \( L_4 \). Then \( \sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0 \), which is impossible.

Suppose that \( r = 3 \). We may assume that \( (\alpha_1, \beta_1) = (1, 0) \), but \( \alpha_2 \neq 0 \neq \beta_2 \). Then
\[
L_2 \not\subset \text{Supp}(Z_2^2),
\]
because \( v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t) \) does not vanish on \( L_2 \). We have
\[
f_4(z, t) = z^2t(\alpha_2 z + \beta_2 t),
\]
which implies that \( v_3(0, t) \neq 0 \) by Corollary \ref{cor:corollary7.4}. Hence, we see that
\[
L_1 \not\subset \text{Supp}(Z_2^2) \not\supset L_3,
\]
because \( v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t) \) and \( zt \) do not vanish on \( L_1 \) and \( L_3 \), respectively, which implies that \( \sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0 \). The latter is a contradiction.

We see that \( r = 2 \). We may assume that \( (\alpha_1, \beta_1) = (1, 0) \), and either \( \alpha_2 = 0 \) or \( \beta_2 = 0 \).

Suppose that \( \alpha_2 = 0 \). Then \( f_4(z, t) = \beta_2 z^2t^2 \). By Lemma \ref{lem:lemma7.3} and Corollary \ref{cor:corollary7.3} we get
\[
v_3(0, t) \neq 0 \neq v_3(z, 0),
\]
which implies that \( v_3(z, t) + yc_2(y, z, t) - \beta_2 z^2t \) does not vanish on neither \( L_1 \) nor \( L_2 \). Then
\[
L_1 \not\subset \text{Supp}(Z_2^2) \not\supset L_2,
\]
which implies that \( \sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0 \), which is a contradiction.

We see that \( \alpha_2 \neq 0 \) and \( \beta_2 = 0 \). We have \( f_4(z, t) = \alpha_2 z^3t \). Then
\[
v_3(0, t) \neq 0
\]
by Corollary \ref{cor:corollary7.4} Then \( L_1 \not\subset \text{Supp}(Z_2^2) \) because the polynomial
\[
v_3(z, t) + yc_2(y, z, t) - \alpha_2 z^2
\]
does not vanish on \( L_1 \).

The line \( L_2 \) is given by the equations \( y = t = 0 \). But \( Z_2 \) is given by the equations
\[
uy + zt = v_3(z, t) + yc_2(y, z, t) - \alpha_2 uz^2 = 0,
\]
which implies that $L_2 \not\subseteq \text{Supp}(Z_2^3)$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^3) = 0$, which is a contradiction. 

Therefore, we see that $f_2(z, t) = z^2$. It follows from Corollary 7.6 that

$$f_4(z, t) = zg_3(z, t)$$

for some $g_3(z, t) \in \mathbb{C}[z, t]$. We may assume that $L_1$ is given by $y = z = 0$.

**Lemma 7.12.** The equality $g_3(0, t) = 0$ holds.

**Proof.** Suppose that $g_3(0, t) \neq 0$. Then $\text{Supp}(Z_2) = L_1$, because $Z_2$ is given by

$$uy + z^2 = zg_3(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0,$$

and the lines $L_2, \ldots, L_r$ are given by the equations $y = g_3(z, t) = 0$.

The cycle $Z_2 + L_1$ is given by the equations

$$uy + z^2 = z^2g_3(z, t) + zyv_3(z, t) + zy^2c_2(y, z, t) = 0,$$

which implies that the cycle $Z_2 + L_1$ can be given by the equations

$$uy + z^2 = zyv_3(z, t) + zy^2c_2(y, z, t) - uyg_3(z, t) = 0.$$

We have $Z_2 + L_1 = C_1 + C_2$, where $C_1$ and $C_2$ are cycles in $\mathbb{P}^3$ such that $C_1$ is given by $y = uy + z^2 = 0$,

and the cycle $C_2$ is given by the equations

$$uy + z^2 = zv_3(z, t) + zyc_2(y, z, t) - uyg_3(z, t) = 0.$$

We have $C_1 = 2L_2$. But $L_1 \not\subseteq \text{Supp}(C_2)$ because the polynomial

$$zv_3(z, t) + zyc_2(y, z, t) - uyg_3(z, t)$$

does not vanish on $L_1$, because $g_3(0, t) \neq 0$. Then

$$Z_2 + L_1 = 2L_2,$$

which implies that $Z_2 = L_1$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2) = 1$, which is a contradiction. 

Thus, we see that $r \leq 3$ and

$$f_4(z, t) = z^2(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$$

for some $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \not\supseteq (\alpha_2 : \beta_2)$. Then

$$v_3(0, t) \neq 0$$

by Corollary 7.4. But $Z_2$ can be given by the equations

$$uy + z^2 = yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,$$

which implies $Z_2 = Z_2^1 + Z_2^2$, where $Z_2^1$ and $Z_2^2$ are cycles on $\mathbb{P}^3$ such that $Z_2^1$ is given by $y = uy + z^2 = 0$,

and the cycle $Z_2^2$ is given by the equations

$$uy + z^2 = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0,$$

which implies that $Z_2^1 = 2L_1$. Thus, we see that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^1) \geq 2$.

**Lemma 7.13.** The inequality $r \neq 3$ holds.
Proof. Suppose that $r = 3$. Then $\beta_1 \neq 0 \neq \beta_2$, which implies that

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on $L_1$. But

$$L_2 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_3,$$

because $\beta_1 \neq 0 \neq \beta_2$. Then $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. \qed

Thus, we see that either $r = 1$ or $r = 2$.

**Lemma 7.14.** The inequality $r \neq 2$ holds.

Proof. Suppose that $r = 2$. We may assume that

- either $\beta_1 \neq 0 = \beta_2$,
- or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$.

Suppose that $\beta_2 = 0$. Then $f_4(z, t) = \alpha_2 z^3(\alpha_1 z + \beta_1 t)$ and

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - \alpha_2 uz(\alpha_1 z + \beta_2 t)$ does not vanish on $L_1$. But $L_2$ is given by

$$y = \alpha_1 z + \beta_1 t = 0,$$

which implies that $z^2$ does not vanish on $L_2$, because $\beta_1 \neq 0$. Then

$$L_2 \not\subseteq \text{Supp}(Z_2^2),$$

which implies that $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Hence, we see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$. Then $L_1 \not\subseteq \text{Supp}(Z_2^2)$, because

$$v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)^2$$

does not vanish on $L_1$. But $L_2 \not\subseteq \text{Supp}(Z_2^2)$, because $z^2$ does not vanish on $L_2$. Then

$$\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0,$$

which is a contradiction. \qed

We see that $f_4(z, t) = z^2$ and $f_4(z, t) = \mu z^4$ for some $0 \neq \mu \in \mathbb{C}$. Then $Z_2^2$ is given by

$$uy + z^2 = v_3(z, t) + yc_2(y, z, t) - \mu z^2 = 0,$$

where $v_3(0, t) \neq 0$ by Corollary 7.4. Thus, we see that $L_1 \not\subseteq \text{Supp}(Z_2^2)$, because

$$v_3(z, t) + yc_2(y, z, t) - \mu z^2$$

does not vanish on $L_1$. Then $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

The assertion of Proposition 7.1 is proved.

The assertion of Theorem 1.5 follows from Propositions 3.4, 5.1, 6.1, 7.1.
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