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Lattice Input on the Inclusive $\tau$ Decay $V_{us}$ Puzzle

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Recent analyses of flavor-breaking hadronic-$\tau$-decay-based sum rules produce values of $|V_{us}| \sim 3\sigma$ low compared to 3-family unitarity expectations. An unresolved systematic issue is the significant variation in $|V_{us}|$ produced by different prescriptions for treating the slowly converging $D = 2$ OPE series. We investigate the reliability of these prescriptions using lattice data for various flavor-breaking correlators and show the fixed-scale prescription is clearly preferred. Preliminary updates of the conventional $\tau$-based, and related mixed $\tau$-electroproduction-data-based, sum rule analyses incorporating B-factory results for low-multiplicity strange $\tau$ decay mode distributions are then performed. Use of the preferred FOPT $D = 2$ OPE prescription is shown to significantly reduce the discrepancy between 3-family unitarity expectations and the sum rule results.

The conventional inclusive hadronic $\tau$ decay determination of $|V_{us}|$ is obtained by applying the finite energy sum rule (FESR) relation, involving polynomial weight $w(s)$ and kinematic-singularity-free correlator $\Pi(s)$ with spectral function $\rho(s)$,

$$\int_0^{s_0} w(s)\rho(s) \, ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s)\Pi(s) \, ds ,$$

(1)

to the flavor-breaking (FB) difference $\Delta\Pi_\tau \equiv \left[ \Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)} \right]$, where

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\( \Pi_{A;ij}(s) \) are the spin \( J = 0, 1 \) components of the flavor \( ij \), vector (V) or axial vector (A) current-current 2-point functions. The spectral functions, \( \rho^{(0+1)}_{V/A;ij} \), hence also \( \Delta \rho_{\tau} \), are related to the normalized differential decay distributions, \( dR_{V/A;ij}/ds \), of flavor \( ij \) V- or A-current-induced \( \tau \) decay widths, \( R_{V/A;ij} \equiv \Gamma[\tau^+ \rightarrow \nu_\tau \bar{\nu}_\tau \pi^0 \pi^-] / \Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e] \), by

\[
dR_{V/A;ij}/ds = 12\pi^2|V_{ij}|^2 S_{\text{EW}} \left[ w_\tau(y_\tau) \rho^{(0+1)}_{V/A;ij}(s) - w_L(y_\tau) \rho^{(0)}(s) \right]/m_\tau^2, \tag{2}
\]

with \( y_\tau = s/m_\tau^2 \), \( V_{ij} \) the ijk CKM matrix element, \( w_\tau(y) = (1-y)^2(1+2y) \), \( w_L(y) = y(1-y)^2 \), and \( S_{\text{EW}} \) a short-distance electroweak correction factor. The \( J = 0 \) (longitudinal) contributions in (2) are well known phenomenologically and, due to problems with the corresponding \( D = 2 \) OPE series, usually subtracted from \( dR/\text{ds} \) \( \rho \). The subtracted result, \( dR^{(0+1)}_{V/A;ij}/ds \), allows the construction of \( J = 0 + 1 \) reweighted analogues, \( R^{(0+1)}_{V/A;ij}(s_0) = \int_0^{s_0} ds \left[ w(s)/w_\tau(y_\tau) \right] dR^{(0+1)}_{V/A;ij}(s)/ds \), for any \( w(s) \) and \( s_0 < m_\tau^2 \). Defining \( \delta R^{(0+1)}_{V/A;ij}(s_0) = \left[ R^{(0+1)}_{V/A;ud}(s_0)/|V_{ud}|^2 \right] - \left[ R^{(0+1)}_{V/A;us}(s_0)/|V_{us}|^2 \right] \), one has, for \( s_0 \) large enough to allow use of the OPE on the RHS of (1), \( \rho \),

\[
|V_{us}| = \sqrt{R^{w}_{V+A;us}(s_0)/\left[ R^{(0+1)}_{V+A;ud}(s_0)/|V_{ud}|^2 - \delta R^{w,\text{OPE}}_{V+A}(s_0) \right]} \tag{3}
\]

This relation has usually been employed in un-reweighted form, with \( w = w_\tau \), and the single value \( s_0 = m_\tau^2 \). This has the advantage that \( R^{w}_{V+A;ud,us}(m_\tau^2) \) is determinable from branching fraction information alone, but the disadvantage of precluding tests of the \( s_0 \)- and \( w(s) \)-independence of the analysis, which could otherwise be used to investigate potential systematic uncertainties (in particular, those associated with the treatment of OPE contributions). Such self-consistency tests were carried out in Refs. 2, 3, 4 and non-trivial \( w(s) \)- and \( s_0 \)-dependences observed, suggesting shortcomings in the experimental data and/or OPE representation.

The most obvious potential OPE problem lies in the rather slow convergence of the \( D = 2 \) OPE series. In terms of the running \( \mathcal{M}_{\tau S} \) quantities \( m_\tau(Q^2) \) and \( \bar{a} \equiv \alpha_\tau(Q^2)/\pi \), the \( D = 2 \) series, which is known to 4-loops, is given by

\[
[\Delta \Pi_{\tau}(Q^2)]^{\text{OPE}}_{D=2} = \frac{3}{2\pi^2} \frac{m_\tau^2(Q^2)}{Q^2} \sum_{k=0} c_k \bar{a}^k \tag{4}
\]

with \( c_k = 1, 7/3, 19.93, 208.75 \) for \( k = 0 \cdots 3 \). Since \( \bar{a}(m_\tau^2) \gtrapprox 0.10, c_k \bar{a}^3 > c_3 \bar{a}^2 \), at the spacelike point on the contour for all \( s_0 \leq m_\tau^2 \). The problematic convergence complicates the assessment of \( D = 2 \) truncation errors, and manifests itself, e.g., in the \( \sim 0.0020 \) difference in \( |V_{us}| \) values obtained using two alternate (CIPT or FOPT) versions of the 4-loop-truncated, \( w_\tau \)-weighted series.

An alternate determination employs the FB combination \( \Delta \Pi_{\tau-EM} \equiv 9\Pi_{EM} - 5\Pi_{ud}^{(0+1)} + \Pi_{13}^{(0+1)} + \Pi_{us,V+A}^{(0+1)} \) in place of \( \Delta \Pi_{\tau} \). Inclusive electroproduction cross-sections fix the electromagnetic (EM) spectral function. By construction, the \( \Delta \Pi_{\tau-EM} \) \( D = 2 \) series is strongly suppressed, having the form \( \Pi \), with \( c_k \rightarrow \)
$c_k^{\tau-EM} = 0, -1/3, -4.384, -44.943$ for $k = 0 \cdots 3$. The $D = 4$ series is also strongly suppressed. OPE contributions to $\Delta \Pi_{\tau-EM}$ FESRs, hence also estimated OPE errors, are thus very small and the resulting $|V_{us}|$ errors essentially entirely experimental. A check of this predicted suppression is thus of interest.

We investigate the relative merits of the fixed-scale (FOPT-like) and local-scale ($\mu^2 = Q^2$, i.e., CIPT-like) treatments of the $\Delta \Pi_{\tau}$ $D = 2$ series, and the level of $\Delta \Pi_{\tau-EM}$ suppression, by comparing OPE expectations and lattice data for the two correlator combinations over a range of Euclidean $Q^2$. Five RBC/UKQCD domain wall fermion ensembles are employed, three, with $m_{\pi} = 293, 349, 399$ MeV, having $1/a = 2.31$ GeV and two, with $m_{\pi} = 171, 248$ MeV, having $1/a = 1.37$ GeV.

For technical reasons, conserved-local versions of the flavor $us$ 2-point functions are numerically challenging and hence, for $\Delta \Pi_{\tau}$, local-local versions are used. To check that this does not produce residual lattice artifacts which would impact our conclusions, we have also performed the OPE-lattice comparison, using conserved-local data, for the alternate flavor-diagonal FB combination $\Delta \Pi_{\text{diag}} \equiv \Pi_{V;e\ell} - \Pi_{V;ss}$, whose $D = 2$ series is very similar to that of $\Delta \Pi_{\tau}$ ($c_k^{\text{diag}} \rightarrow c_k^{\text{diag}} = 1, 8/3, 24.32, 253.69$ for $k = 0 \cdots 3$ in (4)). The results confirm those of the local-local study.

Representative OPE-lattice data comparisons for $\Delta \Pi_{\tau}$ are shown, for the $1/a = 2.13$ GeV, $m_{\pi} = 293$ MeV ensemble, in Fig. 1. The left (right) panel comparison employs the fixed-scale (local-scale) prescription for the $D = 2$ OPE series. The fixed-scale versions match much better the $Q^2$ dependence of the lattice results, with the 3-loop-truncated version thereof best matching the overall normalization.

![Graph](#)

Fig. 1. OPE and lattice $\Delta \Pi_{\tau}$ data, $1/a = 2.31$ GeV, $m_{\pi} = 293$ MeV ensemble, $O(\bar{a}^{1,2,3})$ $D = 2$ OPE truncation, fixed-scale (left panel) or local-scale (right panel) $D = 2$ prescription.

The comparison of lattice data for $\Delta \Pi_{\tau}$ and $\Delta \Pi_{\tau-EM}$ confirms the very strong suppression of $\Delta \Pi_{\tau-EM}$ (see Ref. for the relevant figure).
We turn to preliminary updates of the $|V_{us}|$ analyses. For the $D = 2$ OPE series, we employ the 3-loop-truncated FOPT prescription favored by lattice data, and for the $ud$ spectral integrals, OPAL data\cite{9} as updated in Ref.\cite{10}. For the $us$ spectral integrals, recent B-factory results are used for the $K\pi$\cite{11}, $K^-\pi^-\pi^+$\cite{12}, and $K_{s}\pi^0$\cite{13} exclusive mode distributions, and ALEPH results\cite{14}, updated for current branching fractions (BFs), for all other modes. Contributions from the latter lie higher in the spectrum, and have much larger errors. The B-factory distributions are unit normalized, and also require current BFs for their overall scales. We work with BFs obtained in a $\pi_\mu2$, $K_{\mu2}$-constrained HFAG fit, supplemented by the update to $B[\tau^- \to K^0\pi^-\pi^0\nu_\tau]$ produced by the recent Belle result\cite{13}. Other non-trivial shifts in the $us$ BFs also remain possible. To illustrate the changes to $|V_{us}|$ that could result, we consider also an alternate set of $us$ BFs with the recent larger, but not yet finalized, BaBar results\cite{15} for $B[\tau^- \to K^{-}n\pi^0\nu_\tau]$, $n \leq 3$, used in place of those of the HFAG fit. The first set of $us$ BFs is labelled “$us$ BF set #1” below, the second, alternate set “$us$ BF set #2”. Changes to the $us$ BFs alter the inclusive $us$ spectral distribution, and hence can affect both the magnitude of $|V_{us}|$ and the $s_0$-dependence of the results. The significantly larger preliminary BaBar $K^-\pi^0$ BF is particularly relevant for the FB FESRs considered here, which weight more strongly the low-$s$ part of the spectrum. We consider FESRs employing the weights $w_\tau$ and $w_2(y) = (1 - y)^2$. $w_2$ weights less strongly the higher-$s$, large-error region of the $us$ spectral distribution. Differences between results obtained using the two different weights can thus point to issues with the $us$ spectral distribution.

$|V_{us}|$ results obtained from the $w_\tau$ and $w_2$ versions of the $\Delta \Pi_\tau$ FESR are shown, as a function of $s_0$, and also the choice of the input $us$ BF set, in the left panel of Fig. 2. Similar results for the $\Delta \Pi_{\tau-EM}$ FESR are shown in the right panel. $w_2$ results, which are less sensitive to the large-error high-$s$ region, show better $s_0$-stability in both cases. For $w_\tau$, $s_0$-stability is also better for the $\Delta \Pi_{\tau-EM}$ case,
where OPE contributions are suppressed. The convergence of $w_\tau$ results to the more stable $w_2$ ones as $s_0 \to m_\tau^2$, seen for both the $\Delta \Pi_\tau$ and $\Delta \Pi_{\tau-EM}$ FESRs, suggests the possibility of residual OPE problems in the $w_\tau$ case, where cancellations on the contour play a larger role. Finally we note that results obtained using the FOPT prescription preferred by the lattice data agree better with 3-family unitarity expectations than do those (not shown here) obtained using CIPT, as do those obtained using $\bar{u} s$ BF set #2 in place of $u s$ BF set #1. More details of these analyses will be presented elsewhere.

We close by stressing the preference for FOPT over CIPT for the $D = 2$ OPE series. The prescription which underlies CIPT (of summing logarithmic terms to all orders while truncating the series of non-logarithmic terms), though plausible, is motivated by heuristic arguments not generally valid for divergent series\textsuperscript{16}, and performs poorly when tested against lattice data for the FB correlators.

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