Dispersion in particle velocity resulting from random motion through a spatially-varying fluid velocity field in a pipe

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Abstract

A macro-scale probabilistic model of dilute phase pneumatic transport is developed to analyse the dispersion in the velocity of conveyed particles and to predict their velocity statistics. Fluid drag force, proportional to the relative velocity between the particle and fluid, is taken to be the agent that causes particle motion in the axial direction. The basic premise of the approach is that dispersion in axial particle velocity is a result of subsidiary random motion in the radial direction through the fluid velocity field. Two causes are investigated for this radial motion; gravity and inter-particle collisions. As the local fluid velocity experienced by a particle thus continuously varies in an unpredictable fashion, then the associated drag force fluctuates as does the resulting particle velocity. The non-deterministic nature of the fluid velocity acting on the particle is captured by treating fluid velocity as a stochastic process whose description comes from combining knowledge of the flow field with the nature of the radial motion of the particle. This novel approach allows analytical expressions to be obtained for the mean and variance of particle velocity. The accuracy of these predictions was checked by numerical simulation and found to be good. The analysis demonstrates that dispersion in particle velocity is a function of the magnitude of dispersion in fluid velocity (a fluid property), on the inertial rate constant of the particle (a combined particle/fluid property) and the autoregressive parameter (a property reflecting the type of radial motion).

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**Nomenclature**

- $c_D$: Drag coefficient for an individual sphere in an unbounded fluid
- $F_D$: Drag force acting on the conveyed particle
- $g$: Acceleration due to gravity
- $m$: Mass of a particle
- $n$: Parameter in the $1/7$th power law velocity profile which is a function of $Re$
- $p_e$: Effective inertial rate constant of the particle
- $P(u_e)$: Probability density function of effective fluid velocity
- $P(u_f)$: Probability density function of fluid velocity
- $r$: Radial distance from the pipe centreline
- $r_p$: Radius of a particle
- $R$: Internal radius of the pipeline
- $Re_{ac}$: A first-order autoregressive function to describe autocorrelation
- $r_\phi$: Characteristic dimension required in the calculation of $\phi$
- $Re$: Pipe Reynolds number
- $Re_p$: Particle Reynolds number
- $u_e$: Effective fluid velocity, i.e., the average spatial fluid velocity acting on the particle projected area
- $u_f$: Fluid velocity at a radial distance $r$ from the pipe centreline
- $u_m$: Maximum fluid velocity at the pipe centreline
- $u_p$: Axial velocity of the conveyed particle
- $u_\phi$: Characteristic velocity at which conveyed particles move in the radial direction
- $z_t$: A random term which is sampled from the normal distribution
- $\Delta t$: The time step used in the algorithm to calculate $u_e$
- $\mu_{uf}$: Mean fluid velocity
- $\mu_{un}$: Mean normal impact velocity for collisions with the pipe bend
- $\mu_{up}$: Mean particle velocity
- $\nu$: Kinematic viscosity of air
- $\rho$: Density of air
- $\rho_p$: Density of the conveyed particle
\( \rho_{ue} \)  Autocorrelation coefficient in the algorithm used to calculate \( u_e \) from its value at the preceding step

\( \sigma_{uf}^2 \)  Variance in fluid velocity

\( \sigma_{up}^2 \)  Variance in particle velocity

\( \tau \)  Separation time

\( \tau_c \)  Correlation (or decorrelation) time constant

\( \tau_f \)  Representative fluid time constant

\( \tau_p \)  Particle response time constant

\( \varphi \)  Autocorrelation parameter in the first-order autoregressive function \( R_{ue} \)

1  **Introduction**

Pneumatic conveying is described by Fraige and Langston [1] as the use of a gas, which is usually air, to transport solid particles through a pipeline. Pneumatic conveying may be divided into two broad categories: dilute (also known as lean phase) and dense phase. The former is more widely used in industry and is characterised by low mass flow ratios and high velocities [2]. The use of high velocities can cause significant attrition of the conveyed product which can have many adverse consequences for product quality. Coupled CFD-DEM models have the potential to provide great insights into particle attrition during conveying [3,4]. However, pneumatic conveying of powders involves the transport of very large numbers of particles concurrently which makes such simulations difficult at present, even if the system is modelled by representing real particles with ideal spheres. The overall objective of this work is to present an approach to the prediction of particle breakage in pneumatic conveying where the emphasis is on the integration of particle conveying velocity and particle structural response. Furthermore, rather than adopting either an experimental or CFD-DEM approach, some of the uncertainties and complexities in the system are captured by a novel macro-scale probabilistic approach to predict particle breakage. The task is subdivided into two parts: this paper deals with the prediction of the distribution in particle velocity in the pipe while a second, companion paper is concerned with the determination of the distribution in impact velocity at a bend and the attendant breakage [5].

In principle, there are various reasons why particles at any specific cross-section of a pipeline may be transported at different velocities. The presence of a polydispersed particle size means larger, heavier particles require longer times and pipe lengths than fine particles to reach any attainable velocity. Inter-particle collisions and particle-wall collisions add variability to particle velocity profiles. Temporal fluctuations in fluid velocity resulting from turbulence
can also produce dispersion in particle velocity. Finally, the existence of a spatial velocity profile in the gas phase means that those particles which are near the centreline of the pipe are subject to preferentially higher fluid velocities than those close to the pipe wall. The specific aim of this paper is to quantify the dispersion in particle velocity resulting only from the last mentioned phenomenon; the random migration of particles through the carrier gas velocity field. Theoretical predictions will be validated against numerical results and the level of variability in velocity arising from this phenomenon compared to the contributions of the other phenomena. The model pneumatic conveying system under examination consists of a long, straight horizontal pipe section terminated by one 90° bend. The model is predicated on a number of assumptions. The particles considered are monodispersed spheres with diameters which are small compared to the pipe diameter although not so small as to be in the ultra-fine range (< 20 μm). Furthermore, particles are treated as point masses from a kinetic perspective and no rotational effects are included. The system is assumed to be completely dry to avoid complications such as particles clumping together or adhering to the pipe wall.

2 Theory

2.1 Statistics of Fluid Velocity

Several additional assumptions were made regarding the air velocity in the conveying system. The air velocity has a high magnitude to ensure that particles remain in suspension, the radial and tangential components of air velocity are not modelled explicitly, there is no systematic variation in air velocity in the axial direction in the straight length of pipeline, entry length effects are ignored and the effect of the bend on the air flow pattern in the straight section of piping is not taken into account. Fluid velocity is taken as invariant with respect to time and fluctuations in it over turbulent time scales are not analysed. Any modification to the air flow pattern due to the presence of the particles was also neglected. In reality, the presence of a solid phase may affect the spatial and turbulent structure of the flow field; however for a low mass flow ratio the influence of the particles on the carrier gas characteristics can be neglected [6].

The high air velocity implies that the Reynolds number of the air flow is large and the flow is considered to be turbulent. Such flows may be characterised by the empirical 1/7th power law velocity profile, which relates the fluid velocity, \( u_f \), at a radial distance of \( r \) from the centreline to the maximum velocity along the centreline, \( u_{m} \), in a pipe of internal radius \( R \).
This velocity profile is given as eq. 1 [7], in which \( n \) is a function of Reynolds number. Figure 1 schematically illustrates the velocity profile across a pipeline.

\[
    u_f = u_m \left( 1 - \frac{r}{R} \right)^\frac{1}{n} \quad n = 1.77 \log_{10}(Re) - 1.6 \tag{1}
\]

As a spatial variation in fluid velocity exists, the distribution in fluid velocity can be represented by its probability density function (PDF) and its statistics quantified by mean and variance. The PDF of fluid velocity, \( P(u_f) \), is given as eq. 2 by considering the fractional cross-sectional area of the pipe in which the fluid velocity lies between any values \( u_f \) and \( (u_f + du_f) \).

\[
P(u_f) = \frac{2n(u_f^n - u_m^n)}{u_f^n u_m^{2n}} \quad 0 \leq u_f \leq u_m \tag{2}
\]

The first moment of \( P(u_f) \) about zero gives the mean fluid velocity, \( \mu_{uf} \), while the second moment of \( P(u_f) \) about its mean value yields the variance in fluid velocity, \( \sigma_{uf}^2 \).

\[
    \mu_{uf} = \int_{0}^{u_m} P(u_f) u_f \, du_f = \frac{2n^2}{(n+1)(2n+1)} u_m \tag{3}
\]

\[
    \sigma_{uf}^2 = \int_{0}^{u_m} P(u_f)(u_f - \mu_{uf})^2 \, du_f = \frac{n^2(5n+1)}{(n+1)^2(2n+1)^2(n+2)} u_m^2 \tag{4}
\]

Both are defined solely by the fluid velocity parameters \( u_m \) and \( n \). Figure 2 shows the PDFs of fluid velocity for three different values of \( n \). When \( n \) has a realistic value of 6, the mean fluid velocity is just under 80% of \( u_m \) and the standard deviation in fluid velocity is approximately 13% of \( u_m \). The fluid velocity distribution is left-skewed which is in accord with the experimental data presented by Hamed and Mohamed [8]; it becomes more skewed to the left as the magnitude of \( n \) is increased.
2.2 Statistics of Effective (Particle-Averaged) Fluid Velocity

The steady-state velocity of a particle at any radial location within the pipe is determined by the local velocity of the fluid at that point. If the dimensions of the particles being conveyed are infinitesimal compared to the pipe radius, the statistics of particle velocity would correspond exactly to those of fluid velocity. However, real particles occupy a non-infinitesimal fraction of the pipe cross-section and thus experience a variation (in the radial direction) in fluid velocity acting on the cross-section of the particle normal to the axial fluid flow. The effective fluid velocity acting on a particle, \( u_e \), is the spatially-averaged fluid velocity acting on the particle projected area (i.e., the cross-sectional area normal to the axial fluid velocity). For a spherical particle of radius \( r_p \) with its centre a distance \( r \) from the centreline of the pipe, \( u_e \) is approximated by eq. 5:

\[
\begin{align*}
    u_e(r) &= \int_{r-r_p}^{r+r_p} \frac{u_m \left(1 - \frac{r}{R}\right)^{1/2} \pi r dr}{\left[r + r_p\right]^2 - \left(r - r_p\right)^2} \quad 0 \leq r \leq R - r_p \\
    &= \frac{u_m}{2} \left[ \left(1 - \frac{r - r_p}{R}\right)^{1/2} + \left(1 - \frac{r + r_p}{R}\right)^{1/2} \right] \\
    \end{align*}
\]

The validity of eq. 5 depends on the assumption that the particle radius is small compared to the pipe radius. The equation satisfies the required limiting condition that \( \lim_{r_p \rightarrow 0} u_e = u_f \).

Determination of analytical expressions for the statistics of \( u_e \) again requires knowledge of its PDF. A closed-form analytical expression for \( P(u_e) \) could not be obtained in terms of \( u_e \), although an expression can be derived for \( P(u_e) \) in terms of \( u_f \):

\[
P(u_e) = \frac{4r_p \left(u_m^u - u_f^u\right)R^{1/2}}{u_m^u \left[Ru_f^u + r_p u_m^u\right]^{1/2} - \left[Ru_f^u + r_p u_m^u\right]^{1/2}} \quad \{6\}
\]

For physically realistic ranges of \( n \) and \( \frac{r_p}{R} \), the PDFs of fluid velocity and effective fluid velocity are almost identical \( P(u_e) \approx P(u_f) \). This is illustrated by Figure 3 (for \( n = 6 \)) which shows the approximation becomes better as the ratio \( r_p/R \) decreases from 0.1 to 0.01. Hence
for small particles, the mean and variance of $u_e$ can be taken to be the same as those of $u_f$ with a high degree of conformity. The main difference between the two PDFs is in the prediction of extreme values for velocity along the centreline and at the wall boundary; this is a particular issue close to the pipe wall ( $r \to R$) where $P(u_e)$ is not defined for $r > R - r_p$. Considering the fluid velocity $u_f$, its maximum value, $u_{m}$, is attained at the pipe centreline and its minimum value is zero at the wall. For effective fluid velocity, the corresponding statistics are given by eq. 7 and 8:

$$u_{e_{\text{max}}} = \frac{u_m}{2\left[1 + \frac{1}{n}\frac{r_p}{R}\right]} \left[\left(1 + \frac{r_p}{R}\right)^{1+\frac{1}{n}} - \left(1 - \frac{r_p}{R}\right)^{1+\frac{1}{n}}\right] \quad \{7\}$$

$$u_{e_{\text{min}}} = \frac{\frac{1}{n}u_m\left(\frac{r_p}{R}\right)^{1/n}}{1 + \frac{1}{n}\left(\frac{r_p}{R}\right)^{1/n}} \quad \{8\}$$

Both of these equations predict the same extreme values as for fluid velocity as $r_p \to 0$. The effective fluid velocity field has lower radial gradients than the corresponding fluid velocity field.

2.3 Effective Fluid Velocity as a Random Process

Effective fluid velocity is the average fluid velocity experienced by the particle at some radial location and determines particle velocity at that location. If the particles are considered to solely move in straight horizontal lines (in the axial direction) at fixed (but unknown) radial locations within the pipe, then the effective fluid velocity can be represented as a time-invariant random variable with its mean and variance obtained from the theory of Sections 2.1 and 2.2. Since $u_e$ is a random variable, the particle velocity, $u_p$, is likewise a random variable and assuming the particles are small, the mean and variance in particle velocity will correspond to those of $u_f$ (strictly $u_e$).

All conveyed particles have some radial component of motion caused by factors such as temporal variations in fluid velocity (due to turbulence), oblique inter-particle collisions, particle-wall collisions, the real complexity of the interaction between the particle and the fluid (as opposed to a simple drag force) and the effect of gravity on the particles. This motion of the particles in any transverse cross section through the pipe has been
deterministically examined by Lain et al. [9]. In pneumatic conveying, the average velocity of a particle in the radial direction is generally much lower than its average velocity in the axial direction, e.g., [10]. The existence of this radial component of motion means that particles are subjected to time-varying effective fluid velocities (and hence drag forces) in the axial direction as they move through the radially-varying fluid velocity field. The major hypothesis of this paper is that while the actual motion of the population of particles in the radial direction is taken to be unknown (except that it is assumed complex and random), on average the particles move in the radial direction with some characteristic velocity. It is important to note that the actual radial motion of any particular particle is not captured by this approach: only the overall average effect. Figure 4 illustrates the phenomenon under study. Figure 4a shows a realization of hypothetical particle motion in a transverse cross-section through the pipe at each radial position. Qualitatively the trajectories are very similar to those reported by Lain et al. [9]. The corresponding fluid velocity can be found from figure 4b while figure 4c displays the resultant effective fluid velocity acting on the particle. According to this paradigm, the real motion of the particles is equivalent to fixing the radial positions of the particles so that they move only in horizontal, mutually-parallel lines, and then considering each particle to be subject to a time-varying effective fluid velocity.

Thus, \( u_e \) can be described as a time-varying random process experienced by any given particle with a random component of motion in the radial direction. As the origin of the random process results from the motion of the particle, the process must display autocorrelation. For simplicity and considerations of physical reasonableness, the random process is defined as a first-order autoregressive process with the autocorrelation function

\[
R_{u_e}(\tau) = e^{-\phi \tau} \tag{9}
\]

The term \( \phi \) is the autocorrelation parameter of the random process of effective fluid velocity and \( \tau \) is the separation time, i.e., the time interval between successive samples. \( \phi \) determines the rate at which the correlation decays: as \( \phi \) increases, the random process is characterised by more rapidly unpredictable fluctuations. Note the inverse of the autocorrelation parameter is known as the correlation (or decorrelation) time constant, \( \tau_c \).

The magnitude of the autocorrelation parameter, \( \phi \), should be informed by the mechanism and magnitude of the characteristic radial velocity, \( u_\phi \). Where \( u_\phi \) is low, the level of
autocorrelation should be high and vice versa. In the limit as \( u_\phi \) goes to zero (no particle motion in the radial direction), values of \( u_e \) will be perfectly correlated as the case for a time-invariant random variable while as \( u_\phi \) tends to infinity, \( u_e \) should be described as an almost uncorrelated white-noise process. One function that satisfies this condition is

\[
\phi = \frac{u_\phi}{r_\phi}
\]  

\{10\}

where \( r_\phi \) is a characteristic radial distance associated with the mechanism in question. In summary effective fluid velocity, \( u_e \), can be treated as a first-order, autoregressive process with mean \( \mu_{u_e} \) given by eq. 3, variance \( \sigma_{u_e}^2 \) by eq. 4, PDF given by eq. 2 and autocorrelation function by eq. 9. Effective fluid velocity can be written as

\[
u_e(t) = \mu_{u_e} + u_e^*(t)
\]  

\{11\}

where \( u_e^*(t) \) denotes the time-varying random component of effective fluid velocity.

### 2.4 Autocorrelation Parameter of \( u_e \)

It is assumed that the tangential and radial components of particle velocity (i.e., components in the plane of the pipe cross-section) can be separated from the axial velocity of the particle. There are a large number of physical effects that can induce radial motion and each effect will make a contribution to the overall magnitude of \( \phi \). For this work, the aim is solely to estimate reasonable values for its magnitude appropriate for different prevailing circumstances and then to determine the relationship between its magnitude and the resulting particle velocity statistics. Hence, two different possible reasons for radial particle motion will be examined: inter-particle collisions and gravity. The former would be expected to dominate where the number of particles per unit volume is reasonably large and produce a high value for \( \phi \) consistent with rapid fluctuations in \( u_e \). Gravity should yield a lower value for \( \phi \) corresponding to an effective fluid velocity which changes more slowly. This mechanism would seem appropriate for a more dilute system.
Concerning collisions, the particles are assumed to be rigid, monosized elastic spheres. The characteristic radial distance, $r_\phi$, approximates to the inter-collision mean free path which can be found knowing the number of particles per unit volume in the pipe [11]. An estimate for the characteristic radial velocity, $u_\phi$, may be obtained using the concept of granular temperature based on knowledge of the average collision velocity [12]. This is equivalent to estimating the autocorrelation parameter from the inter-particle collision frequency. For the gravitational mechanism, it is assumed that a particle which is initially at rest falls vertically under gravity to collide with the inner wall of the pipe and then rebound elastically. $r_\phi$ can be estimated as the average height through which a particle would fall if placed randomly in the pipe cross-section. The corresponding $u_\phi$ is taken as the average vertical velocity of a particle falling through this distance.

2.5 Particle Velocity

The axial motion of the fluid causes the particles to be transported in the same direction. Interaction between the fluid and solid particles is complex and includes gravity-buoyancy, slip-shear lift force and slip-rotational lift force amongst others. In addition, particle-wall and inter-particle collision forces act on the bodies. However for the selected particle size and the chosen assumptions about particle behaviour, the primary force acting on any particle in the horizontal, straight section of the conveying system is the drag force, $F_D$, arising from its motion through the fluid: the only force considered in this paper. Given that both the acceleration of the particle from its initial velocity up to its asymptotic velocity and particle-wall collisions are neglected, a particle will nominally move in a straight line with a constant velocity assuming the (effective) fluid velocity is time-invariant. The governing differential equation for particle motion in the axial direction is eq. 12, where $m$ is particle mass.

$$F_D = m \frac{du_p}{dt} = -\frac{c_D}{2} \pi r_p^2 \rho \left| u_p - u_e \right| u_p - u_e$$ \hspace{1cm} \{12\}

The assumption of a low mass flow ratio is important to ensure the applicability of this equation which was developed for single, non-interacting spheres. The drag coefficient $c_D$ is a function of the particle Reynolds number. One commonly-used relationship was provided by Clift and Gauvin [13]:

\[ F_D = \frac{1}{2} \rho C_D \pi r_p^2 \left| u_p - u_e \right| u_p - u_e \]
\[ c_D = \frac{24}{\text{Re}_p} \left( 1 + 0.15 \text{Re}_p^{0.687} \right) + \frac{0.42}{1 + 42500 \text{Re}_p^{-1.16}} \]  \{13\}

Eq. 13 is valid for all \( \text{Re}_p < 10^5 \). For the system being examined in this work, \( 1 < \text{Re}_p < 300 \). An approximation for \( c_D \) which is sometimes applied over this range is

\[ c_D \approx \frac{10}{\sqrt{\text{Re}_p}} \text{ where } \text{Re}_p = \frac{2 |u_p - u_e| r_p}{v} \]  \{14\}

The difference between eq. 13 and 14 is less than 20\% for all \( \text{Re}_p \) greater than 15 in the range considered, and rises non-linearly to almost 70\% as \( \text{Re}_p \) approaches 1. Substituting eq. 14 into eq. 12:

\[ \frac{du_p}{dt} = \frac{15 \rho v^{0.5}}{4\sqrt{2} \rho_p r_p^{1.5}} \sqrt{|u_p - u_e|} (u_p - u_e) \]  \{15\}

Subject to certain assumptions, this can be rewritten so that particle acceleration is proportional to the instantaneous difference between \( u_p \) and \( u_e \):

\[ \frac{du_p}{dt} = -p_e (u_p - u_e) \text{ where } p_e = \frac{15 \rho v^{0.5}}{4\sqrt{2} \rho_p r_p^{1.5}} \sqrt{|u_p - u_e|} \]  \{16\}

The parameter \( p_e \) is termed the effective inertial rate constant of the particle. \( |u_p - u_e| \) is a dynamic quantity whose average value is dependent on the magnitude of the autocorrelation parameter for the effective fluid velocity. As \( \phi \) increases from zero, the average absolute relative velocity also increases from zero but is bounded by an upper limit that can be approximated as \( \frac{2 \sigma_{ue}}{\sqrt{\pi}} \). Physically, this reflects the fact that as the high-frequency content of effective fluid velocity increases, the particle is less able to follow these fluctuations and the temporal average of the absolute difference between particle and fluid velocities becomes larger. However over the range of \( \phi \) of interest for this work (broadly between 10 s\(^{-1}\) and 100 s\(^{-1}\)), the average absolute relative velocity is reasonably invariant. Since \( |u_p - u_e| \) takes a value just greater than 1 m/s, applying the square root operator further diminishes its range and a constant median value is adopted for this quantity. As \( |u_p - u_e| \) is a
measure of the average slip velocity, this approach is in accord with the work of Fessler and Eaton [14] who state that all such velocity scales are of the order of 1 m/s.

In the laminar Stokes Law regime, the instantaneous relative velocity between the fluid and particle is low and \( p_e \) is given by the classical equation

\[
p_e = \frac{9 \rho v}{2 \rho_p r_p^2}
\]  \hspace{1cm} \{17\}

and corresponds to the inverse of the classical (Stokes) particle response time.

### 2.6 Statistics of Particle Velocity

Because \( u_e \) contains a time-varying random component (eq. 11) and \( p_e \) is treated as a constant, eq. 16 is a linear, first-order, stochastic differential equation:

\[
d u_p = - p_e (u_p - \mu_{ue}) dt + p_e u_e^* dt
\]  \hspace{1cm} \{18\}

The solution to this equation with initial condition \( u_p \) at \( t=0 \) is \( u_{p0} \) is

\[
u_p(t) = \mu_{ue} + (u_{p0} - \mu_{ue}) e^{-p_e t} + p_e \int_0^t u_e^* dt
\]  \hspace{1cm} \{19\}

As the random component in the differential equation above has zero mean, only the deterministic component influences the solution for mean particle velocity

\[
\mu_{u_p} = \mu_{u_e} + (u_{p0} - \mu_{u_e}) e^{-p_e t}
\]  \hspace{1cm} \{20\}

Hence the asymptotic value of the mean particle velocity in the axial direction is equal to the mean effective fluid velocity. The variance of particle velocity as a function of time will be

\[
\sigma^2_{u_p} = \frac{1}{1 + \frac{\varphi}{p_e}} \sigma^2_{u_e} + \frac{1}{1 - \frac{\varphi}{p_e}} \sigma^2_{u_e} e^{-p_e t} = \frac{2}{\left(1 + \frac{\varphi}{p_e}\right)\left(1 - \frac{\varphi}{p_e}\right)} \sigma^2_{u_e} e^{-(\rho_e + \varphi) t}
\]  \hspace{1cm} \{21\}
For long times, defined as $t >> \frac{1}{2p_e}$ and $t >> \frac{1}{p_e + \varphi}$, eq. 21 reaches the steady-state value given by eq. 22:

$$\sigma^2_{u_p} = \frac{1}{1 + \frac{\varphi}{p_e}} \sigma^2_{u_e}$$  \{22\}

This analysis implies that the particle velocity in the axial direction becomes a stationary random process after some time has elapsed. Variability in $u_p$ is always lower than the dispersion in $u_e$, and the difference depends on the relative magnitudes of $p_e$ and $\varphi$. Given a relatively large and heavy particle with a correspondingly low effective inertial rate constant and with a fluid velocity field characterised by high-frequency fluctuations (i.e., large $\varphi$), then $\sigma_{u_p}^2$ will be much lower than $\sigma_{u_e}^2$ reflecting the fact that the particle is physically unable to follow the rapid disturbances. For the opposite situation, the variance in $u_p$ will almost equal that of $u_e$.

### 3 Materials and Methods

#### 3.1 System Definition

A typical agglomerated infant milk formula powder was selected as the test particle material. Although the particles were assumed *a priori* to be spherical, their other physical properties for use in the model were informed by experimental data for the physical agglomerates. Table 1 summarises the physical infant formula properties and the geometry and operating conditions of the conveying system. The volume mean diameter of a typical commercial infant formula was measured by Hanley et al. [15] as 312 µm; thus, 150 µm was used as the particle radius ($r_p$) in this model. The density of the model particles, $\rho_p$, was approximated as 770 kg/m³. Therefore, the mass of each particle was $1.09 \times 10^{-8}$ kg.

The basic geometry of the conveying system has already been described as a long, straight, horizontal pipe section with a 90° bend at its end. For this paper, only the straight section, allocated a 20 m length, is of relevance. The inner radius of the pipeline, $R$, was chosen as 50
mm so that the ratio \( \frac{r_p}{R} \) is 0.003. The maximum (centreline) fluid velocity, \( u_m \), was chosen as 20 m/s; this velocity is typical for dilute phase conveying and considerably exceeds the saltation velocity for infant formula. The conveying air was assumed to be at 20°C; at this temperature and at standard atmospheric pressure, the kinematic viscosity, \( \nu \), and density of dry air, \( \rho \), were \( 1.506 \times 10^{-5} \) m²/s and 1.206 kg/m³, respectively [16]. Several simulations were conducted to check the length of the initial acceleration zone for a particle with an initial velocity of zero. The particle attained a velocity equal to mean fluid velocity after 0.132 s by which time it had travelled 2.38 m. Since the pipe length is 20 m, the acceleration phase of the particle from rest up to the mean effective fluid velocity was completed within the first 12% of the conveying system. Thus, it was acceptable to neglect this initial phase of the simulations.

There is a large range in mass flow ratios (the mass of solids per unit mass of conveying air) used for dilute phase conveying: Klinzing [17] states that mass flow ratios of around 1 are often used, while Marcus et al. [2] indicate that ratios below 15 would be reasonable for dilute phase transport. Two different values were used in the model: 1 and 10. Two causes of particle motion in the radial direction were discussed in the model development: the force of gravity and inter-particle collisions. Gravity was taken to be the appropriate agent for radial motion where the mass flow ratio was 1 while the inter-particle collision mechanism was applied for a mass flow ratio of 10.

### 3.2 Numerical Simulation

Monte Carlo simulation was selected as the most suitable probabilistic approach for validating the theoretical predictive equations of particle velocity statistics developed for this model. 5,000 simulations were conducted for each mass flow ratio to ensure statistical significance of the model outputs. The algorithm is outlined below and was implemented using MATLAB [18].

For each simulation, the position of the particle in the pipe cross-section, defined by the Cartesian x and y coordinates of its centrepoint, was selected randomly. Two variates were randomly selected from the uniform distribution on the interval [0,1]: \( \gamma_1 \) and \( \gamma_2 \). x, y and the radial distance from the pipe centreline, \( r \), were calculated from eq. 23–25:

\[
x = \gamma_1(R - r_p)\cos(2\pi\gamma_2)
\]

\{23\}
\[ y = \gamma_1 (R - r_p) \sin(2\pi\gamma_2) \]  \[ r = \sqrt{x^2 + y^2} \]  \[ \{24\} \]

The fluid velocity, \( u_f \), could be calculated for this radial distance knowing \( u_m \), \( r \), \( R \) and \( n \). The mean and variance of effective fluid velocity, \( \mu_{ue} \) and \( \sigma^2_{ue} \), were taken to be the same as the equivalent statistics for fluid velocity. The effective fluid velocity, \( u_e \), was calculated at each incremental time step using the recursion scheme of eq. 26 [19] with a time step, \( \Delta t \), of 0.1 ms.

\[ u_e(t + \Delta t) = \mu_{ue} + \rho_{ue} [u_e(t) - \mu_{ue}] + z_t \]  \[ \{26\} \]

The \( \rho_{ue} \) term in this recursion scheme is the autocorrelation coefficient and is a function of the time increment, \( \Delta t \), and the correlation time, \( \tau_c \):

\[ \rho_{ue} = e^{\frac{-\Delta}{\tau_c}} = e^{-\varphi \Delta} \]  \[ \{27\} \]

\( z_t \) is a random term with a mean of zero and a variance of \( (1 - \rho_{ue}^2)\sigma^2_{ue} \) and \( p_e \) is the effective inertial rate constant of the particle. Particle velocity was simulated with the forward difference scheme given by eq. 28 also with a time step of 0.1 ms.

\[ u_p(t + \Delta t) = u_p(t) - p_e [u_p(t) - u_e(t)] \Delta t \]  \[ \{28\} \]

The particle Reynolds number, \( \text{Re}_p \), and effective inertial rate constant were recalculated at each time step since the relative velocity between the fluid and particle changed continuously.

4 Results

4.1 Flow Velocity Statistics

For a \( u_m \) value of 20 m/s, the pipe Reynolds Number is \( 1.09 \times 10^5 \) and the flow parameter, \( n \), evaluates as 7.32. The large magnitude of \( \text{Re} \) confirms that it was acceptable to describe the
radial variation of fluid velocity using the $1/7$ th power law velocity profile. Table 2 summarises the statistics of $u_f$ and $u_e$. The mean and standard deviation of $u_e$ were calculated by numerical integration of the spatial velocity variation in MATLAB [18], while the gradients were found analytically as the first derivatives of eq. 1 or 6 with respect to $r$. The statistics of $u_e$ almost coincide with those of $u_f$ for the selected $r/R$ of 0.003, except the minimum velocities. It should be noted that the maximum $u_e$ is not exactly 20 m/s, but is extremely close to it because of the flattened velocity profile at the pipe centreline.

### 4.2 Effective Fluid Velocity Autoregressive Parameter

The magnitude of the autoregressive parameter depends on the physical mechanism inducing radial motion. For inter-particle collisions, where the mass flow ratio was 10, the number of particles per unit volume in the pipe was calculated as $1.09 \times 10^9$ m$^{-3}$. Hence, the mean free path between collisions, taken as $r_\phi$, was 2.3 mm. Estimation of $u_\phi$ required knowledge of the granular temperature. This was assumed to be 0.01 m$^2$/s$^2$, which is close to the maximum of 0.009 m$^2$/s$^2$ obtained by Rajniak et al. [20] for a fluidised bed granulator. Thus, the average collision velocity was calculated as 0.266 m/s, which was used as $u_\phi$ for the inter-particle collision mechanism. This gave a value of 116 s$^{-1}$ for $\phi$.

At low mass flow ratios, gravity is taken to be the main cause of particle motion in the radial direction. The average height that the particles fall through is 42.3 mm which was taken as $r_\phi$ for this mechanism. $u_\phi$ was calculated by numerical integration as 0.39 m/s. Thus, $\phi$ was equal to 9.24 s$^{-1}$ for the gravitational mechanism. Decorrelation time (the inverse of the autoregressive parameter) is an order of magnitude lower for the collision mechanism than the gravity mechanism indicating that $u_e$ will be characterised as a much higher frequency random signal for this case. Table 3 summarises this data.

### 4.3 Particle Velocity Statistics

The effective inertial rate constant, $p_e$, is composed of both a constant term and a term that depends on the absolute relative velocity between the particle and fluid. The constant term (dependent on fluid and particle properties) has a magnitude of 8.78 m$^{0.5}$ s$^{0.5}$. The average
absolute relative velocity evaluates as 1.21 m/s when the autocorrelation parameter is 116 s\(^{-1}\) and 1.68 m/s when the parameter is 9.24 s\(^{-1}\). The square root of these values are 1.1 m\(^{0.5}\)/s\(^{0.5}\) and 1.3 m\(^{0.5}\)/s\(^{0.5}\) respectively and a median value of 1.2 m\(^{0.5}\)/s\(^{0.5}\) is chosen for the relative velocity term. Hence \(p_c\) has a magnitude of 10.53 s\(^{-1}\).

Figures 5 and 6 each show three simulated realisations of \(u_e\) and \(u_p\) against distance along the pipeline where \(\phi\) was 9.24 s\(^{-1}\) or 116 s\(^{-1}\), respectively. Increasing \(\phi\) has the effect of increasing the frequency of the fluctuations in \(u_e\), and hence in \(u_p\). Table 4 presents basic statistics of \(u_p\) for both \(\phi\) values. These were calculated using two methods: using the theoretical equations developed for \(\mu_{up}\) and \(\sigma_{up}\) and by determining the statistics for 5,000 Monte Carlo simulations conducted at each level of \(\phi\). The agreement between the predicted means is very good; the magnitudes of the differences are less than 1% and are due to sampling effects. Clearly the average velocity at which particles are conveyed is unaffected by the nature of any radial motion that the particles undergo. The standard deviation in \(u_p\) is sensitive to the level of autocorrelation, decreasing from approximately 1.5 m/s down to 0.7 m/s as \(\phi\) increases from 9.24 s\(^{-1}\) to 116 s\(^{-1}\). The standard deviation in effective fluid velocity is 2.17 m/s. As the fluid velocity field fluctuates more quickly with respect to time, the particle is not capable of following these rapid disturbances and the dispersion in velocity is decreased. The agreement between theory and numerical analysis for the standard deviation is not as close as for mean values. The difference is 13% and -17% for \(\phi\) values of 9.24 s\(^{-1}\) and 116 s\(^{-1}\), respectively. However bearing in mind the complexity of the problem, these are regarded as acceptable.

A number of reasons can be advanced to account for the discrepancy. For the case where \(\phi\) is 9.24 s\(^{-1}\), the average residence time in the system of 1.21 s. Although sufficient for steady-state particle velocity statistics to be achieved, this residence time is quite short compared to the fluid velocity decorrelation time of 0.109 s which introduces sampling effects into the simulation. A second reason is treating \(p_c\) as a constant. Studies were undertaken using a very low \(\phi\) value of 1 s\(^{-1}\) and a pipe length of 300 m. In this situation, \(u_e\) changes very slowly with time and \(u_p\) tracks it more closely. The instantaneous difference between particle and effective fluid velocities remains less than 0.3 m/s and laminar conditions prevail. Hence the governing differential equation for \(u_p\) is truly linear with the constant given by eq. 17. For this condition, the difference between the theoretical and simulated value of standard
deviation is less than 10%. This implies that the theory becomes more accurate for lower values of the autocorrelation parameter though the corresponding very long decorrelation times and hence necessary pipe lengths can become problematic. Finally, a remaining source of discrepancy is due to the assumption that $P(u_e)$ is normally distributed which is not the case in reality.

In addition to validating the theoretical predictions of the statistics of particle velocity, the Monte Carlo method provides additional information on the distribution in particle velocity. Figure 7 displays PDFs of $u_p$ for both levels of autocorrelation. The distributions are minimally skewed and close to normal distributions. The lowest particle velocities recorded during an average simulation are 13.94 m/s or 14.58 m/s at $\Phi$ magnitudes of 9.24 s$^{-1}$ or 116 s$^{-1}$, respectively, which are considerably higher than the minimum $u_e$ of 8.75 m/s. The equivalent average maximum particle velocities are 19.2 m/s and 18.5 m/s, respectively, which are just below the maximum effective fluid velocity of 20 m/s. The range in particle velocity is bounded by the range in effective fluid velocity and is lower for the higher level of the autocorrelation parameter reflecting the fact that particles spend less time at any given radial location.

5 Discussion

The prediction of dispersion in particle velocity, as measured by standard deviation, only takes into account the fluctuation in particle velocity arising from its random motion through the non-uniform velocity field; actual dispersion in particle velocity for real systems also includes contributions from inter-particle collisions, particle-wall impacts and fluid turbulence. The relative contributions of some of these other effects can be assessed to demonstrate the significance of the variability analysed in this paper.

For the high mass flow ratio scenario, the concept of granular temperature was used to quantify the radial motion of the particles as a result of inter-particle collisions. Such collisions will give an additional dispersion to particle velocity. Assuming an isotropic random velocity field (and no interaction between this phenomenon and the effect under study in this paper), the standard deviation in particle velocity in the axial direction will be about 0.1 m/s for a quoted granular temperature of 0.009 m$^2$/s$^2$. This is considerably less than the
predicted standard deviation (of the order of 0.7 m/s) resulting from random motion through the spatially-varying fluid velocity field.

Concerning turbulence, this paper assumed that fluid velocity while varying spatially is invariant with respect to time. In reality, fluid turbulence introduces intrinsic temporal fluctuations in fluid velocity at all locations. The effect of this fluid turbulence on particle velocity can be examined using the Stokes number which is defined as a particle dynamic response time divided by a characteristic fluid time constant:

\[ St = \frac{\tau_p}{\tau_f} \]  \hspace{1cm} \{29\}

The larger the Stokes number, the less responsive the particle is to instantaneous fluid motion. For this system, the particle response time can be defined as

\[ \tau_p = \frac{2\rho_p r_p^2}{9\rho V \left(1 + 0.15Re_p^{0.687}\right)} \]  \hspace{1cm} \{30\}

giving a representative value of 100 ms for the solid product. Note \( \tau_p \) can be interpreted as the inverse of the particle effective inertial rate constant, \( p_e \) (10.53 s\(^{-1}\)). For the systems under analysis here, representative turbulent integral time-scales, \( \tau_f \), can range from 0.1 ms up to 10 ms. A heuristic expression provided by Louge et al. [21] yields a value of 0.3 ms for our pipeline. Hence the Stokes number is significantly greater than unity demonstrating that the particle is sufficiently large that it will be almost unaffected by turbulent fluid motion.

This finding can be checked in more detail by assuming that turbulence effects are present in the fluid flow and characterised by a turbulence intensity of 10%, i.e., \( \sigma_{ut} = 0.1u_f \), giving a value of 1.647 m/s where \( \sigma_{ut}^2 \) is the variance in fluid velocity due to turbulence. \( \sigma_{ut}^2 \) can be related to the corresponding variance in particle velocity, \( \sigma_{up}^2 \), using the fluid turbulence energy spectrum and particle energy spectrum, respectively [22]. The variance in particle velocity due to fluid turbulence can be quantified as
\[ \sigma_{up}^2 = \frac{1}{1+St} \sigma_{ai}^2 \]  \hspace{1cm} \{31\}

\( \sigma_{up}^2 \) evaluated as 0.0072 m\(^2\)/s\(^2\). This variance in particle velocity, which is solely attributable to fluid turbulence, is less than 2% of the smallest simulation value of \( \sigma_{up}^2 \) caused by radial motion of the particle in Table 4. In summary, dispersion in particle velocity should be dominated by the primary effect of random motion of the particle through the velocity field for the system under analysis.

Finally, eq. 22 for the asymptotic value of variance in particle velocity can also be expressed in terms of a Stokes number by noting that the autoregressive parameter, \( \varphi \), is the inverse of the decorrelation time constant for particle random motion, \( \tau_c \), and interpreting \( p_e \) as the inverse of \( \tau_p \). For the collision mechanism, the corresponding Stokes number is 11 while for gravity mechanism it is 0.9 demonstrating the relative insensitivity of particle velocity to the high frequency variability associated with inter-particle collisions.

### 6 Conclusions

A macro-scale probabilistic model of dilute phase pneumatic transport has been developed to analyse the dispersion in the velocity of the conveyed particles and to predict the velocity statistics. Analytical expressions have been obtained for the mean and variance of particle velocity. The accuracy of these predictions was checked by numerical simulation (which requires less simplifying assumptions) and found to be good. The analysis demonstrates that dispersion in particle velocity depends on the magnitudes of dispersion in fluid velocity (a fluid property), on the inertial rate constant of the particle (a combined particle/fluid property) and the autoregressive parameter (a property reflecting the type of radial motion). Dispersion in particle velocity will always be less than the dispersion in fluid velocity with the fraction being dependent on the ratio of \( \varphi/p_e \). The more rapidly the particle moves in the radial direction, the lower the resulting dispersion in axial velocity will be. The analysis suggests that an advantage of selecting a high mass flow ratio (assuming it induces higher frequency motion in the radial direction than a low mass flow ratio) may be a more uniform particle velocity whilst maintaining the same average conveying velocity. This result will be sensitive to the particular conveying conditions. Finally it is shown that, for the conditions investigated
in this paper, the variability associated with random radial motion is considerably greater than that produced by other mechanisms.

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**References**


**Figures**

Figure 1: Illustration showing the turbulent velocity profile in a pipeline
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### Tables

**Table 1: Infant formula properties and conveying system parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Particle radius, $r_p$ (µm)</td>
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<td>Particle density, $\rho_p$ (kg/m³)</td>
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<tr>
<td>Particle mass (kg)</td>
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<td>Length of straight section of conveyor (m)</td>
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<tr>
<td>Inner radius of pipeline, $R$ (mm)</td>
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<td>Fluid centreline velocity, $u_m$ (m/s)</td>
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<tr>
<td>Kinematic viscosity of air, $\nu$ (m²/s)</td>
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<td>Density of dry air, $\rho$ (kg/m³)</td>
<td>1.206</td>
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**Table 2: Flow (fluid & effective fluid) velocity statistics**

<table>
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<tr>
<th>Statistic</th>
<th>Fluid velocity, $u_f$ (m/s)</th>
<th>Effective fluid velocity, $u_e$ (m/s)</th>
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<tbody>
<tr>
<td>Mean</td>
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<td>Standard deviation</td>
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<td>Minimum</td>
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<tr>
<td>Maximum</td>
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<td>20</td>
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<td>Maximum radial gradient</td>
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**Table 3: Autoregressive parameter values**

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<th>Mechanism</th>
<th>Collision</th>
<th>Gravity</th>
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<tr>
<td>Characteristic radial distance, $r_\phi$ (mm)</td>
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<td>42.3</td>
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<tr>
<td>Characteristic radial velocity, $u_\phi$ (m/s)</td>
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<td>0.39</td>
</tr>
<tr>
<td>Autoregressive parameter, $\phi$ (s⁻¹)</td>
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<td>9.24</td>
</tr>
<tr>
<td>Decorrelation time, $\tau$ (s)</td>
<td>0.009</td>
<td>0.109</td>
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</table>
### Table 4: Particle velocity statistics

<table>
<thead>
<tr>
<th>φ (s(^{-1}))</th>
<th>Mean (m/s)</th>
<th>Standard deviation (m/s)</th>
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<tbody>
<tr>
<td></td>
<td>Theory</td>
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<tr>
<td>9.24</td>
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<td>116</td>
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