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Probabilistic analysis of particle impact at a pipe bend in pneumatic conveying

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Abstract
A macro-scale model of the breakage of particles at a 90° bend during dilute phase pneumatic transport is presented. Breakage results if the impact force between the particle and pipe bend exceeds the intrinsic strength of the particle. The latter is taken to be distributed according to the Weibull distribution. Impact force depends on impact velocity and this relationship is obtained by a two-phase structural model of the particle, based on the widely-used Kelvin-Voigt model. The impact velocity is distributed as a result of a distribution in particle velocity and in impact angle though the variability in the latter is shown to be the significant component. The results of the theoretical analysis are confirmed by Monte Carlo simulations. For infant formula agglomerates with typical dimensions, slightly less than 2% of the infant formula agglomerates are predicted to fail when conveyed through a simple system containing one 90° bend (radius of 0.8 m) at a maximum superficial velocity of 20 m/s.

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Pneumatic conveying, Probabilistic modelling, Granular materials, Impact, Breakage

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**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>b</td>
<td>Impact model parameter</td>
</tr>
<tr>
<td>c</td>
<td>Contact viscous damping coefficient</td>
</tr>
<tr>
<td>E</td>
<td>Young’s Modulus</td>
</tr>
<tr>
<td>$F_{\text{max}}$</td>
<td>Maximum impact force</td>
</tr>
<tr>
<td>$F_0$</td>
<td>Characteristic particle strength</td>
</tr>
<tr>
<td>$k_c$</td>
<td>Contact stiffness parameter</td>
</tr>
<tr>
<td>$k_v$</td>
<td>Contact stiffness parameter</td>
</tr>
<tr>
<td>L</td>
<td>Half-length of a narrow strip of width dy</td>
</tr>
<tr>
<td>m</td>
<td>Mass of particle</td>
</tr>
<tr>
<td>$m_w$</td>
<td>Weibull modulus</td>
</tr>
<tr>
<td>$P_S$</td>
<td>Survival probability</td>
</tr>
<tr>
<td>$p_e$</td>
<td>Particle inertial rate constant</td>
</tr>
<tr>
<td>r</td>
<td>Radial distance from the pipe centreline</td>
</tr>
<tr>
<td>$r_p$</td>
<td>Particle radius</td>
</tr>
<tr>
<td>R</td>
<td>Internal radius of the pipeline</td>
</tr>
<tr>
<td>$R_b$</td>
<td>Mean radius of the pipe bend</td>
</tr>
<tr>
<td>$t_c$</td>
<td>Total impact duration</td>
</tr>
<tr>
<td>$t_1$</td>
<td>Duration of phase 1 of loading</td>
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<td>Inter-particle collision velocity</td>
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<td>Effective fluid velocity</td>
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<tr>
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<td>Axial velocity of the conveyed particle</td>
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<tr>
<td>x</td>
<td>Axial displacement/deformation</td>
</tr>
<tr>
<td>$x_e$</td>
<td>Particle deformation at the end of phase 1 of loading</td>
</tr>
<tr>
<td>y</td>
<td>Vertical displacement from the centre of the pipe cross-section</td>
</tr>
<tr>
<td>$z_t$</td>
<td>A random term which is sampled from the normal distribution</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>The time step used in the algorithm to calculate $u_e$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Impact model parameter</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Variate drawn from the uniform distribution on the interval [0,1]</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Dimensionless damping factor</td>
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<tr>
<td>$\theta$</td>
<td>Impact angle for a particle collision with the pipe bend</td>
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<tr>
<td>$\theta_c$</td>
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<td>$\theta_{\text{max}}$</td>
<td>Maximum impact angle</td>
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</tbody>
</table>
1. Introduction

Pneumatic conveying is widely used in industrial plants for transporting granular materials. Most installed systems operate in dilute, or lean, phase, which is characterised by high gas velocities and low mass flow ratios. These high air speeds can cause attrition of pneumatically-conveyed materials which is generally undesirable as it can result in significant changes in bulk properties of the material such as the particle size distribution and bulk density [1]. Most attrition in pneumatic conveying systems occurs at the bends where pipe geometry changes abruptly and impact velocity will be related to particle absolute velocity [2], which makes the selection of bend type critical [3-6]. For any pipe bend, the impact angle is an extremely significant determinant of agglomerate impact breakage [4]. Particle breakage can also occur due to inter-particle collisions and tangential contact with a pipe wall in a straight section. However it can be expected that the effect of inter-particle collisions on breakage should be small because relative velocities between particles are small compared to their absolute velocities. Therefore, conveyed particles will experience an impact meaning at each bend where the collision velocity will be considerably greater than the relative velocity between agglomerates. A complicating factor is that agglomerates which are suspended in the air tend to be swept around the bend by the motion of the air; hence, the impact angle is often relatively low, and for some fine particles, impact may not occur at all. It is also possible for multiple impacts with the pipe wall to occur at a bend. Both of these effects are disregarded in this paper.

The model pneumatic conveying system under examination consists of a long, straight horizontal pipe section terminated by one 90° bend. In a companion paper, expressions for the mean and variance in fluid and particle velocity in the straight section of the pipe were obtained. The source of the variability in particle velocity was the radial gradient in fluid
velocity in the pipe. In this paper, this will be extended to determine the statistics of normal impact velocity at the pipe bend and impact angle with the bend. When this information is combined with a suitable structural model to relate the particle impact velocity to the corresponding impact force and a statistical criterion to determine the resulting breakage probability, it becomes possible to predict the fraction of particles which fail during the conveying process. The theory will be applied to model the pneumatic conveying of a real material, an agglomerated infant formula powder, and the results will be compared with Monte Carlo simulations.

2. Theory

2.1 Particle Velocity Statistics

The dispersion in the axial velocity of the particles as they are blown along the straight section of piping is estimated using a method that is detailed in the companion paper [7]. The fluid velocity profile in the radial direction is characterised by a power law variation associated with turbulent flow. The underlying approach is that any random motion that the particles have in the radial direction (that is not explicitly modelled) causes random fluctuations in the local fluid velocity experienced by the particles and hence in the drag force acting on the particles. The asymptotic values of mean and variance in particle velocity in the axial direction were shown by Hanley et al. [7] to be:

\[ \mu_{up} = \mu_{ue} \]  \hspace{1cm} (1)

\[ \sigma_{up}^2 = \frac{1}{1 + \frac{\phi}{P_e}} \sigma_{ue}^2 \]  \hspace{1cm} (2)

These are taken to be the statistics of particle velocity just prior to impact with the wall. As seen from equation 2, dispersion in particle velocity is sensitive to the magnitude of the autocorrelation parameter, \( \phi \). This is the parameter that quantifies the frequency of the fluctuations in the fluid velocity acting on the particle: larger values of \( \phi \) signify more rapid particle motion in the radial direction.
2.2 Normal Impact Velocity Statistics at Pipe Bend

At the end of the straight section of pipe, the particles collide with the inner wall of the single 90° bend in the system. The normal component of impact velocity is of interest here as it determines to a large extent the breakage force. A number of simplifying assumptions are made in the development of this sub-model: all particles are assumed to travel in straight lines at the bend with no curvilinear motion caused by the changed fluid flow, particles collide only once with the bend wall and the particles are assumed to be uniformly distributed over the pipe cross-sectional area. Since the radial component of air velocity is disregarded, its effect on the impact velocity of the particle was also not considered. The assumption that particles travel in straight lines in the bend requires a minimum particle size to ensure validity. Furthermore, the pipe wall at the bend has a double curvature in reality, i.e., is curved in two orthogonal directions. Therefore, the impact angle for any real particle collision depends on its vertical and horizontal position in the pipe cross-section at the moment of impact. To simplify the analysis, it was assumed that the bend geometry is defined by only a single, rather than a double, curvature. Thus, the impact angle depends solely on the vertical location of a particle in the pipe at the point of impact. As illustrated in figure 1, the normal impact velocity at the bend, $u_n$, is given by equation 3, in which $u_p$ is the axial velocity of the particle at impact and $\theta$ is the impact angle.

$$u_n = u_p \sin \theta \quad \{3\}$$

The velocity and vertical location in the pipe of any given particle at the point of impact is unknown. Hence probability distributions must be used to represent both the particle velocity, $u_p$, and the possible impact angle, $\theta$. Both can be considered to be represented by random variables (since $u_p$ is being examined at a particular moment in time). This complicates the calculation of the statistics of normal impact velocity: equation 3 is non-linear for $u_n$ due to the presence of a product and a sine function. However, the mean and variance of $u_n$ can be estimated by the method of statistical differentials, having knowledge of the mean and variance of both input random variables. The mean normal impact velocity is given by equation 4:

$$\mu_{u_n} \approx \mu_{u_p} \sin \mu_{\theta} \quad \{4\}$$
In contrast to inter-particle collisions where the mean collision velocity is proportional to the standard deviation in particle velocity [7], the mean normal collision velocity between the particles and wall of the pipe bend is proportional to the mean particle velocity, \( \mu_{up} \). As the mean particle velocity is typically an order of magnitude greater than the standard deviation in particle velocity, it is clear that collisions with the bend wall will be the dominant cause of particle breakage. The variance of normal impact velocity is given by equation 5, in which the partial derivatives are evaluated at the mean values of the input variables and \( \rho_{up,\theta} \) is the cross-correlation coefficient of particle velocity and impact angle.

\[
\sigma_{un}^2 \approx \left( \frac{\partial u_p}{\partial u_p} \right)_{\mu}^2 \sigma_{up}^2 + 2\rho_{u_p,\theta} \left( \frac{\partial u_p}{\partial u_p} \right)_{\mu} \left( \frac{\partial u_p}{\partial \theta} \right) \sigma_{up} \sigma_{\theta} + \left( \frac{\partial u_p}{\partial \theta} \right)_{\mu}^2 \sigma_{\theta}^2 \tag{5}
\]

To quantify \( \rho_{up,\theta} \) it is necessary to take into consideration the dependence of particle velocity, \( u_p \), and impact angle, \( \theta \). For the situation where there is negligible correlation between \( u_p \) and \( \theta \), \( \rho_{up,\theta} \approx 0 \) and the variance in normal impact velocity is given by equation 6:

\[
\sigma_{un}^2 \approx \sin^2 \mu_{\theta} \sigma_{up}^2 + \mu_{up}^2 \cos^2 \mu_{\theta} \sigma_{\theta}^2 \tag{6}
\]

Equations 4 and 6 allow estimates of the mean and variance of normal impact velocity to be calculated given the corresponding statistics for particle velocity and impact angle. Expressions for the mean and variance of particle velocity are available, while expressions for the corresponding statistics for the impact angle are developed in Section 2.3.

### 2.3 Impact Angle Statistics

The 90° bend at the end of the modelled conveying system is defined by the mean bend radius, \( R_b \). The geometry of the bend is shown in figure 2. The impact angle, \( \theta \), is related to the vertical displacement from the centre of the cross-section, \( y \), by equation 7.
\[
\theta(y) = \arccos \left( \frac{1 - \frac{y}{R_b}}{1 + \frac{R - r_p}{R_b}} \right) \quad \{7\}
\]

At the bottom wall as \( y \to -(R - r_p) \), impact angle \( \theta \to 0 \) while the maximum impact angle, \( \theta_{\text{max}} \), is attained at the top of the pipe, i.e., where \( y = (R - r_p) \). The impact angle corresponding to motion along the pipe centreline (where \( y = 0 \)) is designated as the centreline impact angle, \( \theta_c \). The magnitudes of \( \theta_c \) and \( \theta_{\text{max}} \) are given by equation 8:

\[
\begin{align*}
\theta_c &= \arccos \left( \frac{1}{1 + \frac{R - r_p}{R_b}} \right) \\
\theta_{\text{max}} &= \arccos \left( \frac{1 - \frac{R - r_p}{R_b}}{1 + \frac{R - r_p}{R_b}} \right)
\end{align*}
\{8\}
\]

For a tight bend angle with an \( R_b/R \) ratio of unity, centreline and maximum impact angles can approach the theoretical limits of 60° and 90° respectively. As the bend ratio becomes more generous, both angles will fall. The probability density function of impact angle, \( P(\theta) \), is given by equation 9, and the derivation is given in Appendix A:

\[
P(\theta) = \frac{2 \sin \theta}{\pi (R - r_p)^2} \sqrt{(R - r_p - R_b + (R_b + R - r_p) \cos \theta)(\cos \theta - (R_b + R - r_p) \cos \theta)^2} \quad 0 \leq \theta \leq \theta_{\text{max}}
\{9\}
\]

Figure 3 illustrates the nature of this PDF for a range of \( R_b/R \) values. The lower the \( R_b/R \) ratio, the more left-skewed is the distribution. The mean and variance in impact angle are defined by equation 10:

\[
\begin{align*}
\mu_\theta &= \int_0^{\theta_{\text{max}}} \theta P(\theta) \, d\theta \\
\sigma^2_\theta &= \int_0^{\theta_{\text{max}}} [\theta - \mu_\theta]^2 P(\theta) \, d\theta
\end{align*}
\{10\}
\]

It was not possible to obtain analytical expressions for the mean and variance of impact angle from the probability density function. However, it can be shown numerically that the mean
impact angle, $\mu_\theta$, is fractionally less than the centreline impact angle, $\theta_c$, and the standard deviation of impact angle, $\sigma_\theta$, is 0.29 times this mean, i.e.:

$$\mu_\theta \approx 0.965 \theta_c \quad \sigma_\theta \approx 0.29 \mu_\theta \quad \text{for } 1 \leq \frac{R_b}{R} < \infty \quad \{11\}$$

These results are almost independent of $\frac{R_b}{R}$. Figure 4a compares the mean impact angle obtained using equation 11 with the values obtained by numerically integrating equation 10 between the limits 0 and $\theta_{\text{max}}$ using Mathematica [8]. The agreement between both plots is excellent across a wide range of $\frac{R_b}{R}$ values. A similar comparison was made for the standard deviations in figure 4b, where the variance was obtained by numerically integrating equation 10, taking $\mu_\theta$ from equation 11. The correspondence is not as close as in figure 4a, but the maximum error remains small: 4.4% when $\frac{R_b}{R}$ is 2. Since bend radii are generally much greater than the pipe radius in pneumatic conveying systems, the error for a realistic $\frac{R_b}{R}$ ratio would be $< 1\%$.

### 2.4 Dependence between Particle Velocity and Impact Angle

To quantify $\rho_{u_p,\theta}$ it is necessary to take into consideration the dependence between particle velocity, $u_p$, and impact angle, $\theta$, just prior to impact and in turn their joint dependence on vertical position. For a single curvature bend, impact angle has a solely deterministic dependence on vertical position. Measuring the vertical ordinate, $y$, from the centre of the pipe cross section, the impact angle increases from $\theta_c$ with increasingly positive values of $y$ and decreases from $\theta_c$ with negative values of $y$. If the particles are initially considered to have no radial motion in the pipe (i.e., moving axially at fixed radial positions), the particle velocity will have a deterministic relationship on radial position. As there is no unique correspondence between radial and vertical positions, particle velocity will exhibit a random dependence on vertical position, though on average being larger at $y$ values close to zero along the pipe centreline and approaching zero as $y$ reaches its extreme values of $\pm R$. Hence there will be a positive correlation between particle velocity and impact angle in the lower
half of the pipe cross section and a negative correlation in the upper half. Owing to the non-linear relationship between impact angle and vertical position, as shown in figure 5, these effects will not be self-cancelling and there will be a net (small) positive correlation between particle velocity and impact angle. In summary for this situation, the variables $u_p$ and $\theta$ will be dependent random variables with a small positive correlation between them.

However, the degree of dependence between these variables also depends on the level of autocorrelation in the effective fluid velocity, $u_e$, which was quantified by the autocorrelation parameter, $\phi$. As $\phi$ approaches a value of 0, the particles move in almost straight lines at an almost constant radial location in the straight section of pipe and consequently there will be dependence between $u_p$ and $\theta$ as shown above. As $\phi$ increases, the particle velocity becomes more random with respect to time, corresponding to more rapid fluctuations in the radial position of the particle in the pipe. Thus, the connection between particle velocity and impact angle becomes much weaker. This paper assumes that the magnitude of $\phi$ is sufficiently high to ensure a negligible correlation between $u_p$ and $\theta$, i.e., $\rho_{u_p,\theta} \approx 0$, and hence the variance in normal impact velocity is given by equation 6.

### 2.5 Particle Structural Model

In order to quantify particle breakage, it is necessary to select a suitable structural model of the particle to relate the impact velocity to the corresponding impact force. It is known from the work of Zumaeta et al. [9], that maximum impact force would be the most appropriate criterion to predict whether breakage would result or not. There are many appropriate continuous contact dynamic models in literature; these are comprehensively surveyed by Gilardi and Sharf [10]. One possibility is the standard Kelvin-Voigt model where the impact is simulated with a spring in parallel with a damper. However major deficiencies of this model include the existence of a discontinuous contact force at the beginning of the impact and the presence of an attractive force at the end of contact, both of which are physically unrealistic [11]. In addition as the only mechanism for energy loss is through the damper, for particles with a low coefficient of restitution (of the order of 0.3 for IMF particles), very high damper forces result with this model causing an overestimate of the maximum impact force.

Since the impact forces were of particular interest in this study, the standard Kelvin-Voigt model was modified to give an impact model that would produce more reliable estimates of impact force; particularly to enable an analytical expression for the maximum impact force to
be formulated. This was achieved by introducing an additional spring which is active up to a certain threshold deflection level, $X_e$ i.e. only during the initial phase (phase 1) of the impact. Once particle impact deflection exceeds $X_e$, this spring takes no further part in the contact and conceptually can be considered to be replaced by a frictionless slider element. For deflections beyond this threshold level, the impact force is transmitted through a second spring in parallel with a damper (phase 2 which encompasses the remainder of the loading stage and the unloading stage). A schematic diagram of the impact model is shown in figure 6. Hence the contact elements of the impact model can be taken to be an ‘elastic’ spring with stiffness, $K_e$, a ‘viscoelastic’ spring with stiffness $K_v$ and a damper with viscous damping coefficient, $C$.

The maximum impact force occurs at a time before maximum particle compression is reached i.e. during the loading part of the impact and is the output parameter of interest for this study. During the unloading or rebound part of the impact, energy is removed from the system to account for any plastic deformation of the particle that may occur. For this implementation of the model, the removed energy is arbitrarily set equal to the stored energy of spring $K_e$; this is about half of the energy absorbed by the damper. So the ultimate effect of the initial spring is to decelerate the body with an irreversible loss in energy. Also, it was decided to deem contact between the particle and pipe wall to be terminated when the impact force in the unloading stage returns to zero, since force is the most important consideration in the model. If contact were instead terminated when the deflection returns to zero, an attractive force would be developed between the particle and impact surface which would not have a physical meaning.

Phase 1 commences at the initiation of impact and continues until the threshold deflection $X_e$ is reached after a time $t_1$. This time is given as

$$
t_1 = \frac{\sin^{-1} \left( \frac{\omega_{ne} X_e}{u_n} \right)}{\omega_{ne}} \quad \omega_{ne} = \sqrt{\frac{K_e}{m}} \quad \{12\}
$$

Note $u_n$ is the normal impact velocity of the particle at $t = 0$ and $\omega_{ne}$ is system natural frequency pertaining for phase 1. The model is intended to represent impacts where particle initial velocity is sufficient to ensure maximum deflection exceeds $X_e$ which is the case for
the vast majority of impacts analysed in this paper. For this situation, the force on the particle, \( F(t) \), is given by equation 13, where \( \zeta \) is the dimensionless damping factor (assumed to be less than 1), \( \omega_d \) is the damped natural frequency and \( \omega_{nv} \) is the natural frequency pertaining to phase 2.

\[
F(t) = mu_n e^{-\zeta \omega_{nv} (t - t_e)} \left( \frac{\zeta^2 \omega_{nv}^2 - \omega_d^2}{\omega_d^2} \right) \sin(\omega_d (t - t_e)) - 2\zeta \omega_{nv} \cos(\omega_d (t - t_e)) \tag{13}
\]

where \( \omega_{nv} = \sqrt{\frac{k}{m}} \), \( \zeta = \frac{c}{2 \sqrt{k_n m}} \), \( \omega_d = \omega_{nv} \left(1 - \zeta^2\right) \tag{14} \)

The duration of the viscoelastic phase of the impact is denoted as \( t_2 \) and can be determined from

\[
t_2 = \frac{\tan^{-1} \left( \frac{2}{\zeta \omega_{nv} - \omega_d} \right) + \pi}{\omega_d} \tag{15}
\]

Subject to certain restrictions, for each impact the maximum impact force can be analytically expressed in terms of the normal impact velocity, \( u_n \) as

\[
F_{\text{max}} = \beta u_n \tag{16}
\]

where the parameter \( \beta \) is a constant that is quantified by the magnitudes of the contact elements and is defined as

\[
\beta = \frac{2me^{-\frac{2\zeta}{\sqrt{1-\zeta^2}}(\pi + \arctan b))}}{1 + b^2} \left[ \frac{\zeta^2 \omega_{nv}^2 - \omega_d^2}{\omega_d^2} \right] \left[ b - \zeta \omega_{nv} \left(1 - b^2\right) \right] \quad b = \frac{(2\zeta + 1)\sqrt{1 - \zeta^2}}{2\zeta^2 + \zeta - 1} \tag{17}
\]

In addition to particle mass and impact velocity, four parameters are required to implement this contact model: \( x_e \), \( k_e \), \( k_v \) and \( c \). Physically the deflection parameter, \( x_c \) should represent the deflection of the body at which yielding occurs for the material and the force-deflection
characteristic loses its elasticity. For Hertzian contact of a sphere (powder particle) against a very stiff surface (pipe wall) it can be estimated as

\[ x_e = \left( \frac{\pi^2}{4 \sqrt{3}} \right) \left( 1 - v^2 \right) \frac{\sigma_y}{E} r_p \]  

\{18\}

The threshold deflection is proportional to the particle radius, and has a quadratic dependence on the ratio of yield strength to Young’s modulus. The three remaining parameters can be found from experimental measurement of the coefficient of restitution, \( \varepsilon \) and contact time, \( t_c \) that prevail for impact (at the mean normal impact velocity) and from the requirement that there must be continuity of impact force at time, \( t_1 \) when force transmission switches instantaneously from the initial spring to the second spring and damper in parallel. Using this data, the three inter-related criteria listed below can be used to calculate the required parameters.

\[ c = \frac{k_e x_e}{\sqrt{u_n^2 - \omega_{nv}^2 x_e^2}} \]  

\{19\}  

(for continuity of forces at \( t = t_1 \))

\[ \varepsilon = -\frac{x'(t_2)}{u_n} \frac{\cos(\omega_{nv} t_1)}{\omega_d} e^{-\omega_{nv} (t(t_2 - t_1))} \left[ \frac{2}{\omega_{nv}} \sin(\omega_d (t_2 - t_1)) - \omega_d \cos(\omega_d (t_2 - t_1)) \right] \]  

\{20\}

\[ t_c = t_1 + t_2 \]  

\{21\}

as the total contact time is the sum of the times for each successive phase of the impact. Figure 7 illustrates a typical force versus time history as predicted by the impact model. Impact force rises steeply from zero over phase 1 of the contact and then more slowly in phase 2, reaches a maximum value and falls back to zero. Ultimately the particle structural model allows the determination of the maximum impact force that can be expected to occur using parameter values for the contact elements that agree with known kinematic information (contact duration and coefficient of restitution) of the impact.

2.6 Particle Breakage Probability

Using the structural model, the statistics of mean and standard deviation in maximum impact force can be estimated from the statistics of normal impact velocity as
\[
\mu_{F_{\text{max}}} = \beta \mu_{\text{un}} \quad \sigma_{F_{\text{max}}} = \beta \sigma_{\text{un}} \quad \{22\}
\]

where \(\beta\) is evaluated for the particle having a normal impact velocity equal to the mean value. Once the maximum force has been calculated for any particle impact, a criterion is necessary to determine whether or not this force is sufficient to cause breakage. Generally, there are major differences between the strengths of individual infant formula agglomerates. Weibull statistics are often used to model the variability in particle strength, [12, 13]. Equation 23 is the fundamental Weibull equation, in which \(P_s\) is the probability of survival of a particle when exposed to a force \(F_{\text{max}}\). The equation is defined by two parameters: \(F_o\) is the characteristic force at which 37\% of such samples survive and the exponent \(m_w\) is the Weibull modulus, a measure of the level of dispersion in strength between granules (note the lower the modulus, the greater the dispersion).

\[
P_s = e^{-\left(\frac{F_{\text{max}}}{F_o}\right)^{m_w}} \quad \{23\}
\]

Inputting the value for \(F_{\text{max}}\) associated with an impact velocity prevailing for any particular impact, allows the probability of survival to be determined.

### 3. Materials and Methods

#### 3.1 System Definition

Although the particles were assumed \textit{a priori} to be spherical, their other physical properties for use in the model were informed by experimental data obtained for infant formula agglomerates. For reasons discussed in the companion paper, 150 \(\mu\)m was used as the particle radius in the model, the density of the model particles was approximated as 770 kg/m\(^3\) and the mass of each particle was \(1.09 \times 10^{-8}\) kg [7]. For many powder systems, the ratio of yield strength to Young’s Modulus lies within the range from 0.001 to 0.1 [14]. A value of 0.075 was selected for this ratio giving the magnitude of \(x_c\) as 2.4 \(\mu\)m. The remaining parameters for the model of particle dynamic load response, \(k_c\), \(k_v\) and \(c\) were found from knowledge of impact coefficient of restitution, \(\varepsilon\), contact duration \(t_c\) (and the requirement for force continuity) based on the results of drop tests of individual infant formula agglomerates that were described by Hanley et al. [15]. For similar agglomerates, the mean coefficient of
restitution was measured as 0.29. A discrete element model of an individual infant formula agglomerate was used to obtain a suitable estimate for the contact time of 3 µs [16]. Thus, $c$, $k_e$, $x_e$, $k_v$, and consequently $\zeta$, $\omega_d$, $\omega_{nv}$, $\omega_{ne}$ and $\beta$ were calculated and the values presented in Table 1. The behaviour is highly underdamped ($\zeta << 1$), and so equations in Section 2.5 are applicable. Both of the required Weibull parameters for particle strength were found using a discrete element model of an individual infant formula agglomerate which was calibrated using experimental data [16].

The basic geometry of the conveying system was described in the companion paper as a long, straight, horizontal pipe section, of length 20 m and with an inner radius, $R$, of 50 mm, terminated by a 90º bend [7]. The radius of this bend, $R_b$, was selected as 800 mm, as industrial systems often use long, sweeping bends to minimise product attrition and pipeline erosion. Therefore, the impact angle along the pipe centreline, $\theta_c$, and the maximum impact angle, $\theta_{max}$, were 0.345 rad (19.75º) and 0.490 rad (28.07º), respectively. From the first paper, the maximum (centreline) fluid velocity was chosen as 20 m/s, resulting in a particle velocity with a mean, $\mu_{up}$, of 16.47 m/s and standard deviation, $\sigma_{up}$, of 1.58 m/s (for $\phi = 9.24$ s$^{-1}$) and 0.66 m/s (for $\phi = 116$ s$^{-1}$) [7].

### 3.2 Numerical Simulation

The Monte Carlo simulation approach was selected in the first paper to compare the theoretical predictive equations developed, and the application of this method to the simulation of particle velocity is described in detail therein [7]. For each simulation, the starting position of the particle in the pipe cross-section was selected randomly (assumed uniformly-distributed over the cross-section) and is invariant with respect to time. Hence, the impact angle is defined at this point by equation 7. The particle velocity was simulated with the forward difference scheme given by equation 24, selecting a time step of 0.1 ms.

$$u_p(t + \Delta t) = u_p(t) - p \left[ u_p(t) - u_e(t) \right] \Delta t$$

(24)

The axial displacement along the pipeline, $x_p$, was also found at each time step using equation 25:
\[ x_p(t + \Delta t) = x_p(t) + \frac{1}{2} [u_p(t) + u_p(t + \Delta t)] \Delta t \]  \hfill (25)

\( x_p \) and \( t \) were initialised at 0 at the start of each simulation run. \( u_p \) was initialised at the local fluid velocity at the position the particle initially occupied in the pipe cross-section. \( x_p \) was used as a termination condition for this time-stepping algorithm, i.e., the algorithm was completed when \( x_p(t + \Delta t) \) coincided with the position of the pipe wall at the bend. The impact angle, \( \theta \), and the normal impact velocity, \( u_n \), were found for each simulation, noting that the particle velocity required to calculate the latter was taken as the final value recorded for \( u_p \). The maximum impact force was found using equation 16. Equation 23 was then used to determine the probability that the particle survives the impact, \( P_s \). Whether or not the particle failed was determined by randomly selecting a variate from the uniform distribution on the interval \([0,1]\). If \( \gamma \geq P_s \), the particle was deemed to have failed, otherwise, it survived the loading. At this point, all of the relevant data were appended to a text file for subsequent analysis, the variables were cleared from memory and the algorithm recommenced.

4. Results

4.1 Impact Angles and Normal Impact Velocities

The statistics of the angle of impact, \( \theta \), from the theoretical equations (equation 11) and the Monte Carlo simulation data are given in Table 2. Since the autocorrelation parameter of particle velocity, \( \phi \) had no influence on these statistics, the results for both values of it are reported together. The means compared extremely well, and the mean impact angle is quite close to the centreline impact angle, \( \theta_c \) of 19.75°. The standard deviations obtained from the simulations and predictive equations differed by 23% (relative to the Monte Carlo simulation results). The Monte Carlo analysis gave a minimum impact angle of 1.35° and a maximum impact angle of 27.96°, which almost reached the theoretical \( \theta_{\text{max}} \) of 28.07°. Thus, impact angle ranged almost over the entire possible domain.
The means and standard deviations in the normal impact velocity, $u_n$, as predicted by theory (equations 4 and 6) and by the simulations, for both values of particle velocity autocorrelation parameter, are given in Table 3. The means and standard deviations in the normal impact velocity showed excellent agreement, particularly considering that the theoretical equations were approximations obtained by the method of statistical differentials, and thus would be expected to contain some error. The mean normal impact velocity is close to 5.4 m/s which is less than one-third of the mean particle velocity of 16.47 m/s. This observation is partly caused by the selection of a long bend radius of 0.8 m; choosing progressively smaller bend radii would cause the ratio of $u_n$:$u_p$ to increase asymptotically towards a value of 1.

The standard deviation in normal impact velocity, $\sigma_{un}$ is 1.31 m/s for a magnitude of $\Phi$ of 9.24 s$^{-1}$ and 1.26 m/s for the higher $\Phi$ value of 116 s$^{-1}$, (from the theoretical equations). These compare to a standard deviation in particle velocity, $\sigma_{up}$ of 1.58 m/s and 0.66 m/s (from the theoretical equations) for the same two values of $\Phi$ [7]. Hence dispersion in particle normal impact velocity can be greater or less than dispersion in particle velocity depending on the prevailing conditions of impact. Examining equation 6, for variance in normal impact velocity, it can be seen to be the sum of two distinct terms, one proportional to the variance in particle velocity and the other proportional to the variance in impact angle. Their relative importance is also quantified in Table 3. The term containing dispersion in impact angle (as measured by $\sigma_{\theta}^2$) makes the dominant contribution to dispersion in the normal component of impact velocity $\sigma_{un}^2$ for both values of $\Phi$ tested, and its relative importance becomes more marked as $\Phi$ increases. In fact for high levels of the autocorrelation parameter, dispersion in impact velocity is almost entirely due to the variability in impact angle. Even for no dispersion in particle velocity, $\sigma_{up} = 0$, the standard deviation in impact velocity will exceed 1 m/s for these conditions. Thus, if it is desirable to minimise the variance in normal impact velocity, it may be more effective to optimise the design of the pipe bend rather than to minimise the variance in particle velocity, e.g., by adjusting the solids loading ratio.

The cross-correlation coefficient of particle velocity and impact angle, $\rho_{up,\theta}$, was calculated to assess the validity of omitting the partial derivative terms from the equation for $\sigma_{un}$. $\rho_{up,\theta}$ was found to be equal to -0.005 or 0.021 when $\Phi$ was 9.24 s$^{-1}$ or 116 s$^{-1}$, respectively. The magnitudes of both are negligible, confirming that it was acceptable to omit the partial derivative terms. In the limit, the maximum value for the cross correlation coefficient, as $\Phi$
approaches 0, was found to be 0.12. Even at the lower level of $\Phi$ of 9.24 s$^{-1}$, it can be shown that $\rho_{up,\theta}$ must exceed 0.1 to have an effect greater than 5% on the total variance in normal impact velocity.

4.2 Particle Strength, Impact Forces and Agglomerate Breakage

Statistics

Concerning the statistics of particle strength, (as measured by the force required to break it), the Weibull modulus, $m_w$, was found as 1.718 and the 37% characteristic force, $F_o$, was 1.336 N. Equation 23 gives the additive inverse of the cumulative distribution function of the force needed to cause particle failure in impact; the corresponding probability density function of this failure force is shown in figure 8. This displays the distribution in the force required to produce failure of the particles in impact; as can be seen for the data of this paper, it broadly ranges from 0 N (very weak particles) up to 4 N (for the strongest ones). The characteristic force, $F_o$, is shown on the graph by means of a dashed line, and the fractional area under the curve to the right of this is equal to 37%. The relatively low magnitude of the modulus of 1.718 is indicative of a large dispersion in particle strength. Mean particle strength will be 1.19 N while standard deviation will be 0.714 N for these values of $m_w$ and $F_o$ and are tabulated in Table 4.

Statistics of the maximum impact force, $F_{max}$ as predicted by theory and simulation are also provided in Table 4. The mean and standard deviation of the maximum impact forces can be predicted analytically using equation 22 where mean normal impact velocity, $\mu_{un}$, was taken as 5.4 m/s while standard deviation in impact force $\sigma_{un}$ was averaged as 1.3 m/s. The output from the simulations is the average of 10,000 individual runs which were combined for both values of the auto-regressive parameter $\Phi$ as it has little effect on prediction of maximum impact force. MATLAB [17] was used to fit a Weibull distribution to the simulation data for $F_{max}$. The scale and shape parameters of the probability density function (PDF) were 0.1181 N and 4.859 respectively, and the PDF is shown in figure 9. The mean of this Weibull distribution is 0.108 N and its standard deviation is 0.026 N, both of which show excellent agreement with the theoretical method. The lowest simulated impact force was 0.006 N and the maximum 0.185 N.
As mean maximum impact force is 0.108 N compared to mean particle strength of 1.19 N, this suggests that the actual forces developed at impact are less than one-tenth of the force required to break a specimen and thus breakage rates should be modest. Equation 23 can be used to estimate the proportion of particles that will survive impact at the bend without breaking. On average, the probability of survival for all agglomerates conveyed is 98.6 %, i.e., less than 2 % of particles would be expected to break under the conveying conditions described above. It should be noted that the approach only calculates breakage rates and makes no prediction about the size distribution of any daughter particles. The probability density functions of particle strength (figure 8) and maximum impact force (figure 9) overlap each other and the curves intersect at two points. The force corresponding to the higher intersection point is termed \( F^* \) as depicted in figure 10. The magnitude of \( F^* \) is 0.18 N for these simulations and is a realistic measure of the largest impact force any particle will experience. The survival probability at this impact force is 96.9 %, which may be treated as a lower limit for any individual particle.

5. Discussion

The probability of a particle failing in impact at a pipe bend depends on the balance between the resulting impact force and particle strength. Impact force will be proportional to normal impact velocity which in turn is the product of particle velocity and the sine of impact angle. The average particle velocity will be close to the average fluid velocity which, as shown in the first paper [7], will be about 80% of the fluid centreline velocity, \( \bar{u}_m \), for air flows used in pneumatic conveying. The average impact angle will have a value slightly less than the centreline impact angle, which is primarily a function of the severity of the bend as quantified by \( R_b/R \). Thus, the proportion of the agglomerates which fail would be expected to increase as \( \bar{u}_m \) increases and \( R_b/R \) decreases. It should be noted that this simple model has not considered breakage due to shear as normal impact at the pipe wall is often the most significant cause of particle breakage and cannot sensibly be disregarded. Dispersion in breakage rates is sensitive to many factors, though the large inherent variability in particle strength will generally outweigh any process contributions to the dispersion. For the case where particle strength is tightly controlled, generously radiused bends will limit the dispersion in breakage (in addition to lowering the mean level of breakage) by minimising the dispersion in impact angle and hence the impact velocity.
Inter-particle collisions are not considered explicitly as a cause of particle breakage in this model. The severity of these collision forces will depend on the mean inter-particle collision velocity. A theoretical estimate for the mean inter-particle collision velocity can be obtained only if the distribution in particle velocity is known. It is also necessary to have data regarding the homogeneity and isotropy of the velocity field. However, estimates of the relative magnitude of the mean collision velocity can still be calculated in the absence of such information. The mean collision velocity can be obtained by considering the granular temperature of the mixture, in which case the result is given by equation 26:

$$\mu_{uc} = \frac{\sqrt{3\pi}}{2} \sigma_{up} = 1.535\sigma_{up}$$  \{26\}

The mean inter-particle collision velocity was calculated to be 2.147 m/s, taking \(\sigma_{up}\) as 1.399 m/s [7]. These estimates of \(\mu_{uc}\) were much less than half of the mean normal impact velocities at the pipe bend in Table 2, and since fatigue of the particles was not considered, the assumption to neglect inter-particle collisions as a cause of particle breakage was acceptable. Crucially, the average inter-particle collision velocities are proportional to the standard deviation in particle velocity while the average bend wall impact velocities are proportional to mean particle velocity. In turn, standard deviation in particle velocity cannot exceed the standard deviation in fluid velocity and mean particle velocity equals mean fluid velocity. As shown in paper 1, the ratio of standard deviation in fluid velocity to mean fluid velocity is close to 0.13 for air flows of interest. Thus it can be said that mean values of particle velocity will be significantly higher than the standard deviation and hence wall impact rather than inter-particle contact will be the dominant contributor to breakage.

6. Conclusions

An approach to quantify the breakage in particles during pneumatic conveying based on a stochastic representation of the process has been outlined. This model is straightforward compared to some alternative approaches and has the benefit of a modular structure: any of the sub-models comprising the overall model can be substituted by a reasonable alternative without necessarily compromising the functionality of the model as a whole. Key outputs of the model (impact angle with the pipe bend, normal impact velocity and maximum absolute impact force) have been compared with results obtained from Monte Carlo simulations. The statistics generally compared well, particularly the means, although there was a disparity of
around 20% between the standard deviations of impact angle. For infant formula agglomerates with typical dimensions, slightly less than 2% of the infant formula agglomerates are predicted to fail when conveyed through a simple system containing one 90° bend (radius of 0.8 m) at a maximum superficial velocity of 20 m/s. The speed at which the agglomerates move in the radial direction, and hence the autocorrelation parameter, do not influence the probability of survival of the agglomerates conveyed. The cross-correlation coefficients between particle velocity and impact angle have been shown to be negligible. Mean inter-particle collision velocity is generally much lower than the normal impact velocity at the pipe bend, thus justifying the assumption to neglect inter-particle collisions as a mechanism for particle breakage.

**Acknowledgements**

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**References**


Appendix A: Derivation of the PDF of Impact Angle

The cross-sectional area of the pipe of inner radius \( R \) within which the centrepoint of any particle of radius \( r_p \) may be located is \( \pi (R - r_p)^2 \). The cross-section was divided into horizontal strips of length \( 2L \) and width \( dy \), as shown in figure 11. The area of each such strip is \( 2Ldy \). Pythagoras’s theorem was applied to relate \( L \) to \( y \) (the vertical displacement from the centrepoint of the cross-section):

\[
L = \sqrt{(R - r_p)^2 - y^2}
\]  \{A1\}

The proportion of the particles in any horizontal strip at a distance \( y \) from the centrepoint of the pipeline, and hence the proportion of the particles subjected to the commensurate impact angle, \( \theta \), is given by equation A2:

\[
\text{Proportion of particles in strip} = \frac{2\sqrt{(R - r_p)^2 - y^2} \, dy}{\pi (R - r_p)^2}
\]  \{A2\}

Equation 7 was solved for \( y \) and differentiated to find \( dy \):

\[
y = R_b - (R_b + R - r_p) \cos \theta
\]  \{A3\}

\[
dy = (R_b + R - r_p) \sin \theta \, d\theta
\]  \{A4\}

Thus, equation A2 could be rewritten as equation A5 after substituting equations A3 and A4. Therefore, the PDF is given by equation 9.

\[
\text{Proportion} = \frac{2 \sin \theta \, d\theta}{\pi (R - r_p)^2} \sqrt{(R - r_p - R_b + (R_b + R - r_p) \cos \theta)(1 - \cos \theta)(R_b + R - r_p)^3}
\]  \{A5\}
Figures

Figure 1: Geometry for impact of a particle with the bend

Figure 2: Illustration of the geometric relationship between the impact angle, θ, and the vertical ordinate of particle position, y
Figure 3: Probability density functions of impact angle for a constant $\frac{R_p}{R}$ of 0.01 and $\frac{R_b}{R}$ values of 1, 5 and 10

Figure 4: Variation of the mean (a) and standard deviation (b) in impact angle (rad) obtained both by numerical integration and using Eq. 11, with the ratio of bend radius to pipe radius, $\frac{R_b}{R}$
Figure 5: Illustration showing the variation in impact angle against the vertical ordinate, y

Figure 6: Schematic of the two-phase particle impact model
Figure 7: Impact force versus time behaviour of the two-phase model described in this paper for one particle impact

Figure 8: Probability density function of the Weibull distribution in particle strength
Figure 9: Probability density function of the Weibull distribution fitted to the simulation data for maximum impact force

Figure 10: Identification of $F^*$: the point of intersection of the Weibull PDFs for force at failure and for the maximum impact forces in the conveying system
Figure 11: Cross-section through the pipe bend showing the geometry used to derive the PDF of impact angle

Tables

Table 1: Parameters calculated for the two-phase model at a normal impact velocity of 5.4 m/s

<table>
<thead>
<tr>
<th>Phase 1</th>
<th></th>
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<tbody>
<tr>
<td>$k_e$</td>
<td>25,921 N/m</td>
<td></td>
</tr>
<tr>
<td>$x_e$</td>
<td>$2.384 \times 10^{-6}$ m</td>
<td></td>
</tr>
<tr>
<td>$\omega_{ne}$</td>
<td>$1.542 \times 10^6$ rad/s</td>
<td></td>
</tr>
<tr>
<td>Phase 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>0.0156 N s/m</td>
<td></td>
</tr>
<tr>
<td>$k_v$</td>
<td>171,871 N/m</td>
<td></td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.1805</td>
<td></td>
</tr>
<tr>
<td>$\omega_d$</td>
<td>$3.842 \times 10^6$ rad/s</td>
<td></td>
</tr>
<tr>
<td>$\omega_{nv}$</td>
<td>$3.971 \times 10^6$ rad/s</td>
<td></td>
</tr>
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</table>

Table 2: Statistics of impact angle (°)

<table>
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<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory</td>
<td>19.08</td>
<td>5.53</td>
</tr>
<tr>
<td>Simulation</td>
<td>19.14</td>
<td>4.51</td>
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</table>
Table 3: Mean and standard deviation of normal impact velocity (m/s), with the percentage contribution to the variance in normal impact velocity made by the particle velocity and impact angle terms

<table>
<thead>
<tr>
<th>φ (s⁻¹)</th>
<th>Mean (m/s)</th>
<th>Standard deviation (m/s)</th>
<th>σ_υ² Effect</th>
<th>σ_θ² Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Simulation</td>
<td>Theory</td>
<td>Simulation</td>
</tr>
<tr>
<td>9.24</td>
<td>5.46</td>
<td>5.45</td>
<td>1.31</td>
<td>1.37</td>
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<tr>
<td>116</td>
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<td>5.38</td>
<td>1.26</td>
<td>1.27</td>
</tr>
</tbody>
</table>

Table 4: Statistics of the maximum impact force (N) and particle strength (N)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Simulation</td>
</tr>
<tr>
<td>Maximum impact force (N)</td>
<td>0.108</td>
<td>0.108</td>
</tr>
<tr>
<td>Particle strength (N)</td>
<td>1.191</td>
<td>-</td>
</tr>
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