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POINTS IN PROJECTIVE SPACES AND APPLICATIONS

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Abstract. We prove the factoriality of a nodal hypersurface in \( \mathbb{P}^4 \) of degree \( d \) that has at most \( 2(d - 1)^2/3 \) singular points, and factoriality of a double cover of \( \mathbb{P}^3 \) branched over a nodal surface of degree \( 2r \) having less than \( (2r - 1)r \) singular points.

1. Introduction.

Let \( \Sigma \) be a finite subset in \( \mathbb{P}^n \) and \( \xi \in \mathbb{N} \), where \( n \geq 2 \). The points of the set \( \Sigma \) impose independent linear conditions on homogeneous forms of degree \( \xi \) if and only if for every point \( P \in \Sigma \) there is a homogeneous form of degree \( \xi \) that vanishes at \( \Sigma \setminus P \) and does not vanish at \( P \), which is equivalent to \( h^1(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}(\xi)) = 0 \), where \( \mathcal{I}_\Sigma \) is the ideal sheaf of \( \Sigma \).

In this paper we prove the following result (see Section 2).

Theorem 1.1. Suppose that at most \( \lambda k \) points of the set \( \Sigma \) lie on a curve of degree \( k \), where \( \lambda \in \mathbb{N} \) and \( \lambda \geq 2 \). Then \( h^1(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}(\xi)) = 0 \) if one of the following conditions holds:

- \( \xi = \lfloor 3\lambda/2 - 3 \rfloor \) and \( |\Sigma| < \lambda \lfloor \lambda/2 \rfloor \);
- \( \xi = \lfloor 3\mu - 3 \rfloor \) and \( |\Sigma| \leq \lambda \mu \), where \( \mu \in \mathbb{Q} \) such that \( \lfloor 3\mu \rfloor - \mu - 2 \geq \lambda \geq \mu \);
- \( \xi = \lfloor n\mu \rfloor \) and \( |\Sigma| \leq \lambda \mu \), where \( \mu \in \mathbb{Q} \) such that \( (n - 1)\mu \geq \lambda \).

Let us consider applications of Theorem 1.1.

Definition 1.2. An algebraic variety is called factorial if its divisor class group is \( \mathbb{Z} \).

Let \( \pi : X \to \mathbb{P}^3 \) be a double cover branched over a surface \( S \) of degree \( 2r \geq 4 \) such that the only singularities of \( S \) are isolated ordinary double points. Then \( X \) is a hypersurface

\[
\mathbb{w}^2 = f_{2r}(x, y, z, t) \subset \mathbb{P}(1^4, r) \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),
\]

where \( \text{wt}(x) = \text{wt}(y) = \text{wt}(z) = \text{wt}(t) = 1 \), \( \text{wt}(w) = r \), and \( f_{2r} \) is a homogeneous polynomial of degree \( 2r \) such that \( f_{2r} = 0 \) defines the surface \( S \subset \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, t]) \).

It follows from [10] and [8] that the following conditions are equivalent:

- the threefold \( X \) is factorial;
- the singularities of the threefold \( X \) are \( \mathbb{Q} \)-factorial;
- the equality \( \text{rk} H_4(X, \mathbb{Z}) = 1 \) holds;
- the ring \( \mathbb{C}[x, y, z, t, w]/I \) is a UFD, where \( I = \langle \mathbb{w}^2 - f_{2r}(x, y, z, t) \rangle \);
- the points of the set \( \text{Sing}(S) \) impose independent linear conditions on homogeneous forms on \( \mathbb{P}^3 \) of degree \( 3r - 4 \).

In the case \( r = 3 \), the threefold \( X \) is known to be non-rational if it is factorial (see [4]), but the threefold \( X \) is rational if the surface \( S \) is the Barth sextic (see [1]).

Theorem 1.3. Suppose that \( |\text{Sing}(S)| < (2r - 1)r \). Then \( X \) is factorial.

Proof. The subset \( \text{Sing}(S) \subset \mathbb{P}^3 \) is a set-theoretic intersection of surfaces of degree \( 2r - 1 \), which implies that \( X \) is factorial by Theorem 1.1. \( \square \)

We assume that all varieties are projective, normal, and defined over \( \mathbb{C} \).
Theorem 1.9. Suppose that $g$ where $g_i$ is a general homogeneous polynomial of degree $i$. Then $X$ is not factorial, singular points of the surface $S$ are isolated ordinary double points, and $|\text{Sing}(S)| = (2r - 1)r$.

Example 1.8. Suppose that the hypersurface $V$ is a general determinantal quartic threefold is rational and has isolated ordinary double points. Then $V$ can be given by the equation
\begin{equation}
\left(g^2(x, y, z, w) = g_1(x, y, z, w)g_{2r-1}(x, y, z, w) \subset \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, w])\right),
\end{equation}
where $g_i$ is a general homogeneous polynomial of degree $i$. Then $X$ is not factorial, singular points of the surface $S$ are isolated ordinary double points, and $|\text{Sing}(S)| = (2r - 1)r$.

We prove the following result in Section 3.

Theorem 1.6. Suppose that $|\text{Sing}(S)| \leq (2r - 1)r + 1$. Then the threefold $X$ is not factorial if and only if the surface $S \subset \mathbb{P}^3$ can be defined by the equation $(1.5)$

\begin{equation}
|g^2(x, y, z, w) = g_1(x, y, z, w)g_{2r-1}(x, y, z, w)| \subset \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, w]),
\end{equation}

where $f_n$ is a homogeneous polynomial of degree $n$. It follows from [10] and [8] that the hypersurface $V$ is factorial if and only if one of the following conditions holds:

- the hypersurface $V$ has $\mathbb{Q}$-factorial singularities;
- the equality $\text{rk}H_d(V, \mathbb{Z}) = 1$ holds;
- the ring $\mathbb{C}[x, y, z, t, u]/I$ is a UFD, where $I = \langle f_n(x, y, z, u) \rangle$;
- the points of the set $\text{Sing}(V)$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^4$ of degree $2d - 5$.

In the case $d = 4$, the hypersurface $V$ is not rational if it is factorial (see [12]), but a general determinantal quartic threefold is rational and has isolated ordinary double points.

Conjecture 1.7. Suppose that $|\text{Sing}(V)| < (d - 1)^2$. Then $V$ is factorial.

The claim of Conjecture 1.7 is proved in [3] and [5] in the case when $d \leq 7$.

Example 1.8. Suppose that the hypersurface $V$ is given by the equation
\begin{equation}
xg(x, y, z, w) + yf(x, y, z, w, t) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, w, t]),
\end{equation}
where $g$ and $f$ are general homogeneous polynomials of degree $d - 1$. Then $V$ is not factorial, singular points of $V$ are isolated ordinary double points, and $|\text{Sing}(V)| = (d - 1)^2$.

The factoriality of $V$ is proved in [2] in the case when $|\text{Sing}(V)| \leq (d - 1)^2/4$.

Theorem 1.9. Suppose that $|\text{Sing}(V)| \leq 2(d - 1)^2$ then $V$ is factorial.

Proof. The set $\text{Sing}(V)$ is a set-theoretic intersection of hypersurfaces of degree $d - 1$, which implies the claim for $d \geq 7$ by Theorem 1.1. In the case $d \leq 6$, the factoriality of the hypersurface $V$ follows from Theorem 2 in [9].

Let $Y$ be a complete intersection of hypersurfaces $F$ and $G$ in $\mathbb{P}^5$ of degree $m$ and $k$, respectively, such that $m \geq k$ and $Y$ has at most isolated ordinary double points.

Example 1.10. Let $F$ and $G$ be general hypersurfaces that contain a two-dimensional linear subspace in $\mathbb{P}^5$. Then $F$ and $G$ are smooth, the threefold $Y$ has isolated ordinary double points, and $|\text{Sing}(Y)| = (m + k - 2)^2 - (m - 1)(k - 1)$, but $Y$ is not factorial.

\begin{footnote}{The claim of Theorem 1.6 is conjectured in [11], and it is proved in [11] the case $r = 3$.}
\end{footnote}
It follows from \(8\) that the threefold \(Y\) is factorial if its singular points impose independent linear conditions on homogeneous forms on \(\mathbb{P}^5\) of degree \(2m + k - 6\).

**Theorem 1.11.** Suppose that \(G\) is smooth, and \(Y\) has at most \((m + k - 2)(2m + k - 6)/5\) ordinary double points. Then the complete intersection \(Y\) is factorial for \(m \geq 7\).

**Proof.** The set \(\text{Sing}(Y)\) is a set-theoretic intersection of hypersurfaces of degree \(m + k - 2\), which concludes the proof by Theorem 1.1. \(\square\)

Arguing as in the proof of Theorem 1.11 we obtain the following result.

**Theorem 1.12.** Suppose that \(G\) is smooth, and \(Y\) has at most \((2m + k - 3)(m + k - 2)/3\) ordinary double points. Then the complete intersection \(Y\) is factorial for \(m \geq k + 6\).

Let \(H\) be a smooth hypersurface in \(\mathbb{P}^4\) of degree \(d\), and \(\eta: U \rightarrow H\) be a double cover ramified in a surface \(R \subset H\) that is cut out by a hypersurface of degree \(2r \geq d\) that has isolated ordinary double points. Then \(U\) is factorial if the points of \(\text{Sing}(R)\) impose independent linear conditions on homogeneous forms of degree \(3r + d - 5\) (see \(8\)).

**Theorem 1.13.** The threefold \(U\) is factorial if \(|\text{Sing}(R)| \leq (2r + d - 2)/r2\) and \(r \geq d + 7\).

**Proof.** The set \(\text{Sing}(R)\) is a set-theoretic intersection of hypersurfaces of degree \(2r + d - 2\), which implies the claim by Theorem 1.11. \(\square\)


2. Main result.

Let \(\Sigma\) be a finite subset in \(\mathbb{P}^n\), where \(n \geq 2\). In this section we prove the following special case of Theorem 1.1 leaving other cases to the reader, because their proofs are similar.

**Proposition 2.1.** Suppose that at most \((2r - 1)k\) points of the set \(\Sigma\) lie on a curve of degree \(k\), and \(|\Sigma| < (2r - 1)r\), where \(r \in \mathbb{N}\) and \(r \geq 2\). Then the points of the set \(\Sigma\) impose independent linear conditions on homogeneous forms of degree \(3r - 4\).

We may assume that \(n \geq 3\) due to the following result, which is Corollary 4.3 in \(7\).

**Theorem 2.2.** Let \(\pi: Y \rightarrow \mathbb{P}^2\) be a blow up of points \(P_1, \ldots, P_5\), and \(E_i\) be the \(\pi\)-exceptional divisor such that \(\pi(E_i) = P_i\). Then the linear system \(|\pi^*(\mathcal{O}_\mathbb{P}(\xi)) - \sum_{i=1}^{5} E_i|\) does not have base points if at most \(k(\xi + 3 - k) - 2\) points of the set \(\{P_1, \ldots, P_5\}\) lie on a curve of degree \(k\) for every natural number \(k \leq (\xi + 3)/2\), and the inequality

\[
\delta \leq \max \left\{ \left[ \frac{\xi + 3}{2} \right] \left( \xi + 3 - \left[ \frac{\xi + 3}{2} \right] \right) - 1, \left[ \frac{\xi + 3}{2} \right]^2 \right\},
\]

holds, where \(\xi\) is a natural number such that \(\xi \geq 3\).

Hence, to prove Proposition 2.1 we may assume that \(n = 3\) due to the following result.

**Lemma 2.3.** Let \(\Pi\) be an \(m\)-dimensional linear subspace in \(\mathbb{P}^n\) such that \(n > m \geq 2\), and

\[
\psi: \mathbb{P}^n \dashrightarrow \Pi \cong \mathbb{P}^m
\]

be a projection from a general \((n - m - 1)\)-dimensional linear subspace \(\Omega \subset \mathbb{P}^n\) such that there is a subset \(\Lambda \subset \Sigma\) such that \(|\Lambda| \geq \lambda k + 1\), but the set \(\psi(\Lambda)\) is contained in an irreducible curve of degree \(k\), and \(\mathcal{M}\) be the linear system of hypersurfaces in \(\mathbb{P}^n\) of degree \(k\) that contain \(\Lambda\). Then the base locus of \(\mathcal{M}\) is zero-dimensional, and either \(m = 2\), or \(k > \lambda\).
Proposition 2.4. The points of the set $\Sigma$ lie on a curve of degree $k$ that contains all points of $\Lambda$, and $\Psi$ lie on a curve in $\Pi$ of degree $k$. Hence, at most $\delta k$ points of the set $\Sigma$. Hence, the base locus of the linear system $M$ is zero-dimensional.

Suppose that $m > 2$ and $k \leq \lambda$. Let us show that the latter assumption leads to a contradiction. We may assume that $m = 3$ and $n = 4$, because we may consider $\psi$ as a composition of $n - m$ projections from points. Thus, the projection $\psi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is a projection from the point $\Omega \in \mathbb{P}^4$.

Let $\mathcal{Y}$ be the set of all irreducible reduced surfaces in $\mathbb{P}^4$ of degree $k$ that contains all points of the set $\Lambda$, and $\mathcal{Y}$ be a subset of $\mathbb{P}^4$ consisting of points that are contained in every surface of $\mathcal{Y}$. Then $\Lambda \subseteq \mathcal{Y}$, but the previous arguments imply that $\mathcal{Y}$ is a finite set.

Let $\mathcal{S}$ be the set of all surfaces in $\mathbb{P}^3$ of degree $k$ such that $S \in \mathcal{S}$ if and only if there is a surface $Y \in \mathcal{Y}$ such that $\psi(Y) = S$ and $\psi|_Y$ is a birational morphism. Then $\mathcal{S}$ is not empty, because the projection $\psi$ is general enough and the construction of the set $\mathcal{Y}$ does not depend on the choice of the projection $\psi$. Let $\Psi$ be a subset of $\mathbb{P}^3$ consisting of points that are contained in every surface of the set $\mathcal{S}$. Then $\psi(\Lambda) \subseteq \psi(\mathcal{Y}) \subseteq \Psi$ by construction.

The generality of $\Omega$ implies that $\psi(\mathcal{Y}) = \Psi$. Indeed, for every point $O \in \Pi \setminus \Psi$ and any general surface $Y \in \mathcal{Y}$, we may assume that the line passing through $O$ and $\Omega$ does not intersect $Y$, but the restriction $\psi|_Y$ is a birational morphism.

Thus, the set $\Psi$ is a set-theoretic intersection of surfaces in $\Pi$ of degree $k$, which implies that at most $\delta k$ points in $\Psi$ lie on a curve in $\Pi$ of degree $\delta$. Hence, at most $k^2$ points of the set $\Psi$ lie on a curve in $\Pi$ of degree $k$, but $\psi(\Lambda)$ contains at least $\lambda k + 1$ points contained in an irreducible curve in $\Pi$ of degree $k$, which is a contradiction.

Thus, we have a finite subset $\Sigma \subset \mathbb{P}^3$ such that $|\Sigma| < (2r - 1)r$, and at most $(2r - 1)k$ points of $\Sigma$ lie on a curve of degree $k$, where $r \in \mathbb{N}$ and $r \geq 2$. Fix an integer $\epsilon$ such that

$$|\Sigma| < (2r - 1)(r - \epsilon),$$

and $\epsilon \geq 0$. We prove the following result, which implies Proposition 2.4.

**Proposition 2.4.** The points of the set $\Sigma$ impose independent linear conditions on homogeneous forms of degree $3r - 4 - \epsilon$.

Fix an arbitrary point $P$ of the set $\Sigma$. To prove Proposition 2.4 it is enough to construct a surface\(^2\) in $\mathbb{P}^3$ of degree $3r - 4 - \epsilon$ that contains $\Sigma \setminus P$ and does not contain $P$.

We may assume that $r \geq 3$ and $\epsilon \leq r - 3$, because the claim of Proposition 2.4 follows from Theorem 2 in \([9]\) and Theorem 2.2 in the case when $r \leq 3$ or $\epsilon \geq r - 3$.

**Lemma 2.5.** Suppose that $\Sigma \subset \Pi$, where $\Pi$ is a hyperplane in $\mathbb{P}^3$. Then there is a surface of degree $3r - 4 - \epsilon$ in $\mathbb{P}^3$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$.

**Proof.** Suppose that $|\Sigma \setminus P| > [(3r - 1 - \epsilon)/2]^2$. Then

$$\begin{align*}
(2r - 1)(r - \epsilon) - 2 &\geq |\Sigma \setminus P| \geq \left[\frac{3r - 1 - \epsilon}{2}\right]^2 + 1 \geq \frac{(3r - 2 - \epsilon)^4}{4} + 1,
\end{align*}$$

\(^2\)For simplicity we consider homogeneous forms on $\mathbb{P}^3$ as surfaces in $\mathbb{P}^3$. 

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which implies that $(r - 4)^2 + 2r + \epsilon^2 \leq 0$. We have $r = 4$ and $\epsilon = 0$, which implies that
\[
|\Sigma \setminus P| \leq \frac{3r - 1 - \epsilon}{2} \left( 3r - 1 - \epsilon - \frac{3r - 1 - \epsilon}{2} \right).
\]

Thus, we proved that in every possible case the inequality
\[
|\Sigma \setminus P| \leq \max \left( \frac{3r - 1 - \epsilon}{2} \left( 3r - 1 - \epsilon - \frac{3r - 1 - \epsilon}{2} \right) , \frac{3r - 1 - \epsilon}{2} \right)^2
\]
holds, but at most $3r - 4 - \epsilon$ points of $\Sigma \setminus P$ can lie on a line, because $3r - 4 - \epsilon \geq 2r - 1$.

Let us prove that at most $k(3r - 1 - \epsilon - k) - 2$ points of the set $\Sigma \setminus P$ can lie on a curve of degree $k \leq (3r - 1 - \epsilon)/2$. It is enough to show that
\[
k(3r - 1 - \epsilon - k) - 2 \geq k(2r - 1)
\]
for all $k \leq (3r - 1 - \epsilon)/2$. We must prove the latter inequality only for $k > 1$ such that
\[
k(3r - 1 - \epsilon - k) - 2 < |\Sigma \setminus P| \leq (2r - 1)(r - \epsilon) - 2,
\]
because otherwise the condition that at most $k(3r - 1 - k) - 2$ points of $\Sigma \setminus P$ can lie on a curve of degree $k$ is vacuous. In particular, we may assume that $k < r - \epsilon$, but
\[
k(3r - 1 - \epsilon - k) - 2 \geq k(2r - 1) \iff r > k - \epsilon,
\]
which implies that at most $k(3r - 1 - \epsilon - k) - 2$ points of $\Sigma \setminus P$ lie on a curve of degree $k$.

It follows from Theorem 2.2 that there is a curve $C \subset \Pi$ of degree $3r - 4 - \epsilon$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$. Let $Y$ be a sufficiently general cone in $\mathbb{P}^3$ over the curve $C$. Then $Y$ is the required surface. \hfill \Box

Fix a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.

**Lemma 2.6.** Suppose that at most $(2r - 1)k$ points of the set $\Sigma'$ lie on a possibly reducible curve in $\Pi$ of degree $k$. Then there is a surface in $\mathbb{P}^3$ of degree $3r - 4 - \epsilon$ that contains all points of the set $\Sigma \setminus P$ but does not contain the point $P$.

**Proof.** Arguing as in the proof of Lemma 2.5 we obtain a curve $C \subset \Pi$ of degree $3r - 4 - \epsilon$ that contains $\Sigma' \setminus P'$ and does not pass through $P'$. Let $Y$ be the cone in $\mathbb{P}^3$ over the curve $C$ with the vertex $O$. Then $Y$ is the required surface. \hfill \Box

To conclude the proof of Proposition 2.1 we may assume that at least $(2r - 1)k + 1$ points of $\Sigma'$ lie on a curve of degree $k$, where $k$ is the smallest number of such property.

**Lemma 2.7.** The inequality $k \geq 3$ holds.

**Proof.** Let $\Phi \subset \Sigma$ be a subset such that $|\Phi| > 2(2r - 1)$, but the set $\psi(\Phi)$ is contained in a conic $C \subset \Pi$. Then the conic $C$ is irreducible. Let $D$ be a linear system of quadric surfaces in $\mathbb{P}^3$ containing $\Phi$. Then the base locus of $D$ is zero-dimensional by Lemma 2.3.

The inequality $k \geq 2$ holds by Lemma 2.3, which implies $r \geq 3$.

Let $W$ be a cone in $\mathbb{P}^3$ over $C$ with the vertex $\Omega$. Then
\[
8 = D_1 \cdot D_2 \cdot W \geq \sum_{\omega \in \Phi} \text{mult}_\omega(D_1) \text{mult}_\omega(D_2) \geq |\Phi| > 2(2r - 1) \geq 8,
\]
where $D_1$ and $D_2$ are general divisors in $D$, which is a contradiction. \hfill \Box
There is a subset $\Lambda^1_k \subseteq \Sigma$ such that $|\Lambda^1_k| > (2r - 1)k$, but $\psi(\Lambda^1_k)$ is contained in an irreducible curve of degree $k$. Similarly, we get a disjoint union $\cup_{j=k}^{l} \cup_{i=1}^{c_j} \Lambda^j_i$, where $\Lambda^j_i$ is a subset of $\Sigma$ such that $|\Lambda^j_i| > (2r - 1)j$, the points of the subset $\psi(\Lambda^j_i)$ lie on an irreducible reduced curve in $\Pi$ of degree $j$, and at most $(2r - 1)\zeta$ points of the subset

$$
\psi \left( \Sigma \setminus \left( \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda^j_i \right) \right) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
$$

lie on a curve in $\Pi$ of degree $\zeta$. Put $\Lambda = \cup_{j=k}^{l} \cup_{i=1}^{c_j} \Lambda^j_i$. Let $\Xi^i_j$ be the base locus of the linear system of surfaces of degree $j$ that contains $\Lambda^j_i$. Then $\Xi^i_j$ is a finite set by Lemma 2.13 and

$$(2.8) \quad 0 \leq |\Sigma \setminus \Lambda| < (2r - 1)(r - \epsilon) - 1 - \sum_{i=k}^{l} c_i(2r - 1)i < (2r - 1)(r - \epsilon - \sum_{i=k}^{l} ic_i).$$

**Corollary 2.9.** The inequality $\sum_{i=k}^{l} ic_i \leq r - \epsilon - 1$ holds.

We have $\Lambda_j^j \subseteq \Xi^i_j$ by construction, but the points of the set $\Xi^i_j$ impose independent linear conditions on homogeneous forms of degree $3(j - 1)$ by the following result.

**Lemma 2.10.** Let $\mathcal{M}$ be a linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(\lambda)|$ such that the base locus of the linear system $\mathcal{M}$ is zero-dimensional. Then the points of the base locus of $\mathcal{M}$ impose independent linear conditions on homogeneous forms of degree $n(\lambda - 1)$.

**Proof.** See Lemma 22 in [2] or Theorem 3 in [6].

Put $\Xi = \cup_{j=k}^{l} \cup_{i=1}^{c_j} \Xi^i_j$. Then $\Lambda \subseteq \Xi$.

**Lemma 2.11.** Suppose that $\Sigma \subseteq \Xi$. Then there is a surface in $\mathbb{P}^3$ of degree $3r - 4 - \epsilon$ that contains all points of the set $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$.

**Proof.** It follows from Lemma 2.10 that for every set $\Xi^i_j$ containing the point $P$ there is a surface in $\mathbb{P}^3$ of degree $3(j - 1)$ that contains the set $\Xi^i_j \setminus P$ and does not contain the point $P$. For every set $\Xi^i_j$ not containing the point $P$ there is a surface of degree $j$ that contains the set $\Xi^i_j$ and does not contain $P$ by the definition of the set $\Xi^i_j$.

The inequality $j < 3(j - 1)$ holds, because $k > 2$. Therefore, for every $\Xi^i_j \neq \emptyset$ there is a surface $F^i_j \subset \mathbb{P}^3$ of degree $3(j - 1)$ that contains the set $\Xi^i_j \setminus (\Xi^i_j \cap P)$ and does not contain the point $P$. The union $\cup_{j=k}^{l} \cup_{i=1}^{c_j} F^i_j$ is a surface of degree

$$
\sum_{i=k}^{l} 3(i-1)c_i \leq \sum_{i=k}^{l} 3ic_i - 3c_k \leq 3r - 6 - 3\epsilon \leq 3r - 4 - \epsilon
$$

that contains all points of the set $\Sigma \setminus P$ and does not contain the point $P$. □

The proof of Lemma 2.11 implies that there is surface of degree $\sum_{i=k}^{l} 3(i-1)c_i$ that contains $(\Xi \cap \Sigma) \setminus (\Xi \cap P)$ and does not contain $P$, and there is a surface of degree $\sum_{i=k}^{l} ic_i$ that contains $\Xi \cap \Sigma$ and does not contain any point of the set $\Sigma \setminus (\Xi \cap \Sigma)$.

**Lemma 2.12.** Let $\Lambda$ and $\Delta$ be disjoint finite subsets in $\mathbb{P}^n$ such that there is a hypersurface of degree $\zeta \leq \xi$ that contains $\Lambda$ and does not contain any point in $\Delta$, the points of $\Lambda$ impose independent linear conditions on hypersurfaces of degree $\xi$, the points of $\Delta$ impose independent linear conditions on hypersurfaces of degree $\xi - \zeta$. Then the points of $\Lambda \cup \Delta$ impose independent linear conditions on hypersurfaces of degree $\xi$. 
Lemma 2.14. \( \text{Let } Q \text{ be a point in } \Lambda \cup \Delta. \text{ To conclude the proof we must find a hypersurface of degree } \xi \text{ that contains } (\Lambda \cup \Delta) \setminus Q \text{ and does not contain } Q. \text{ We may assume that } Q \in \Lambda. \) 

Let \( F \) be the homogenous form of degree \( \xi \) that vanishes at \( \Lambda \setminus Q \) and does not vanish at \( Q \). Put \( \Delta = \{Q_1, \ldots, Q_4\} \), where \( Q_i \) is a point. Then there is a homogenous form \( G_i \) of degree \( \xi \) that vanishes at \( (\Lambda \cup \Delta) \setminus Q_i \) and does not vanish at \( Q_i \). We have

\[
F(Q_i) + \mu_i G_i(Q_i) = 0
\]

for some \( \mu_i \in \mathbb{C} \), because \( g_i(Q_i) \neq 0 \). Then the homogenous form \( F + \sum_{i=1}^{\delta} \mu_i G_i \) vanishes at every point of the set \( (\Lambda \cup \Delta) \setminus Q \) and does not vanish at the point \( Q \). \( \square \)

We have \( \sum l \) points of the set \( \Sigma \) lie on a line in \( \Pi \). Then there is \( 4(2r-1)(r-\Delta) - 12 \geq (3r - 2 - \Delta)^2 \), which implies that \( r^2 - 8r + 16 + 2r\Delta + \Delta^2 \leq 0 \), which is a contradiction. \( \square \)

**Lemma 2.13.** The inequality \( |\Sigma| \leq \lfloor (d + 3)/2 \rfloor^2 \) holds.

**Proof.** Suppose that the inequality \( |\Sigma| \geq \lfloor (d + 3)/2 \rfloor^2 + 1 \) holds. Then

\[
(2r - 1)(r - \epsilon - \sum_{i=k} l c_i) - 2 \geq |\Sigma| \geq \left\lfloor \frac{d + 2}{4} \right\rfloor^2 + 1 \geq \frac{(3r - 2 - \epsilon - \sum_{i=k} l c_i)^2}{4} + 1
\]

by Corollary 2.9. Put \( \Delta = \epsilon + \sum_{i=k} l c_i \). Then \( \Delta \geq k \geq 3 \) and

\[
4(2r - 1)(r - \Delta) - 12 \geq (3r - 2 - \Delta)^2,
\]

which implies that \( r^2 - 8r + 16 + 2r\Delta + \Delta^2 \leq 0 \), which is a contradiction. \( \square \)

The inequality \( d \geq 3 \) holds by Corollary 2.9 because \( r \geq 3 \).

**Lemma 2.14.** Suppose that at least \( d + 1 \) points of the set \( \Sigma \) lie on a line. Then there is a surface in \( \mathbb{P}^3 \) of degree \( 3r - 4 - \epsilon \) containing \( \Sigma \setminus P \) and not passing through the point \( P \).

**Proof.** We have \( |\Sigma| \geq d + 1 \). Hence, it follows from the inequalities 2.8 that

\[
3r - 3 - \epsilon - \sum_{i=k} l c_i < (2r - 1)(r - \epsilon) - 1 - \sum_{i=k} l c_i (2r - 1)i,
\]

which gives \( \sum_{i=k} l c_i \neq r - \epsilon - 1 \). Now it follows from Corollary 2.9 that \( \sum_{i=k} l c_i \leq r - \epsilon - 2 \), but \( 2r - 1 \geq 3r - 3 - \epsilon - \sum_{i=k} l c_i \), which implies that \( \sum_{i=k} l c_i = r - \epsilon - 2 \) and \( d = 2r - 2 \).

We have a surface of degree \( \sum_{i=k} l (i-1)c_i \leq 3r - 4 - \epsilon \) that contains \( (\Xi \cap \Sigma) \setminus (\Xi \cap P) \) and does not contain the point \( P \), and we have a surface of degree \( r - \epsilon - 2 \) that contains all points of the set \( \Xi \cap \Sigma \) and does not contain any point of the set \( \Sigma \setminus (\Xi \cap \Sigma) \).

The set \( \Sigma \setminus (\Xi \cap \Sigma) \) contains at most \( 4r - 4 \) points, but at most \( 2r - 1 \) points of \( \Sigma \) lie on a line. The points of \( \Sigma \setminus (\Xi \cap \Sigma) \) impose independent linear conditions on homogeneous forms of degree \( 2r - 2 \) by Theorem 2 in \( [9] \), which concludes the proof by Lemma 2.12. \( \square \)

Therefore, we may assume that at most \( d \) points of the set \( \Sigma \) lie on a line in \( \Pi \).

**Lemma 2.15.** At most \( t(d + 3 - t) - 2 \) points of \( \Sigma \) lie on a curve in \( \Pi \) of degree \( t \leq (d + 3)/2 \).

**Proof.** At most \( (2r - 1)t \) of the points of \( \Sigma \) lie on a curve in \( \Pi \) of degree \( t \), which implies that to conclude the proof it is enough to show that the inequality

\[
t(d + 3 - t) - 2 \geq (2r - 1)t
\]
Lemma 3.2. Suppose that there are plane

\[
\text{that contains } \sum_{i=k}^{l} ic_i \leq t \leq \frac{d+3}{2}
\]

hold. Let \( g(x) = x(d+3-x) - 2 \). Then \( g(x) \) is increasing for every \( x < (d+3)/2 \), which implies that \( g(t) \geq g(r - \epsilon - \sum_{i=k}^{l} ic_i) \). Now the inequalities \( 2.8 \) imply that

\[
(2r - 1) \left( r - \epsilon - \sum_{i=k}^{l} ic_i \right) - 2 \geq |\Sigma| > g(t) \geq \left( r - \epsilon - \sum_{i=k}^{l} ic_i \right) (2r - 1) - 2,
\]

which is a contradiction. \( \square \)

We can apply Theorem \( 2.2 \) to the blow up of \( \Pi \) at the points of \( \Sigma \) and the integer \( d \), which implies the existence of a surface in \( \mathbb{P}^3 \) of degree \( 3r - 4 - \epsilon \) that contains every point of the set \( \Sigma \setminus P \) and does not contain the point \( P \) by Lemma \( 2.12 \).

3. Auxiliary result.

In this section we prove Theorem \( 1.6 \). Let \( \pi : X \to \mathbb{P}^3 \) be a double cover branched over a surface \( S \) of degree \( 2r \geq 4 \) with isolated ordinary double points.

Lemma 3.1. Let \( F \) be a hypersurface in \( \mathbb{P}^n \) of degree \( d \) such that \( F \) has isolated singularities, and \( C \) be a curve in \( \mathbb{P}^n \) of degree \( k \). Then \( C \) contains at most \( k(d-1) \) singular points of the hypersurface \( F \), and the equality \( |\text{Supp}(C) \cap \text{Sing}(F)| = k(d-1) \) implies that every singular point of the hypersurface \( F \) contained in \( C \) is non-singular on the curve \( C \).

Proof. Let \( f(x_0, \ldots, x_n) \) be the homogeneous form of degree \( d \) such that \( f(x_0, \ldots, x_n) = 0 \) defines the hypersurface \( F \), where \( (x_0 : \ldots : x_n) \) are homogeneous coordinates on \( \mathbb{P}^n \). Put

\[
\mathcal{D} = \left\{ \sum_{i=0}^{n} \lambda_i \frac{\partial f}{\partial x_i} = 0 \right\} \subset \mathcal{O}_{\mathbb{P}^n}(d-1),
\]

where \( \lambda_i \in \mathbb{C} \). Then the base locus of the linear system \( \mathcal{D} \) consists of singular points of the hypersurface \( F \). Therefore, the curve \( C \) intersects a generic member of the linear system \( \mathcal{D} \) at most \( (d-1)k \) times, which implies the claim. \( \square \)

Lemma 3.2. Suppose that there are plane \( \Pi \subset \mathbb{P}^3 \) and a reduced curve \( C \subset \Pi \) of degree \( r \) that contains \( (2r-1)r \) singular points of \( S \). Then \( S \) can be defined by the equation \( f(x) = 0 \).

Proof. Let \( S|_{\Pi} = \sum_{i=1}^{\alpha} m_i C_i \), where \( C_i \) is an irreducible reduced curve, and \( m_i \) is a natural number. We may assume that \( C_i \neq C_j \) for \( i \neq j \), and \( C = \sum_{i=1}^{\beta} C_i \), where \( \beta \leq \alpha \). Then

\[
(3.3) \quad \sum_{i=1}^{\beta} \deg(C_i) = r = \frac{\sum_{i=1}^{\alpha} m_i \deg(C_i)}{2},
\]

which implies that the curve \( C_i \) contains exactly \((2r-1)\deg(C_i)\) singular points of the surface \( S \) for every \( i \leq \beta \) due to Lemma \( 3.1 \). Moreover, the curve \( C \) is smooth at every singular point of the surface \( S \) that is contained in the curve \( C \) by Lemma \( 3.1 \).
Suppose that \( m_\gamma = 1 \) for some \( \gamma \leq \beta \). Then \( C_\gamma \) contains \( (2r - 1)\deg(C_\gamma) \) singular points of the surface \( S \), but the curve \( S|_\Pi \) must be singular at every singular point of the surface \( S \) that is contained in \( C_\gamma \). Thus, we have

\[
\text{Sing}(S) \cap \text{Supp}(C_\gamma) \subseteq \bigcup_{i \neq \gamma} C_i \cap C_\gamma,
\]

but \( |C_i \cap C_\gamma| \leq (C_i \cdot C_\gamma)_\Pi = \deg(C_i)\deg(C_\gamma) \) for \( i \neq \gamma \). Hence, we have

\[
\sum_{i \neq \gamma} \deg(C_i)\deg(C_\gamma) \geq (2r - 1)\deg(C_\gamma),
\]

but on the plane \( \Pi \) we have the equalities

\[
(2r - \deg(C_\gamma))\deg(C_\gamma) = (S|_\Pi - C_\gamma) \cdot C_\gamma = \sum_{i \neq \gamma} m_i\deg(C_i)\deg(C_\gamma),
\]

which implies that \( \deg(C_\gamma) = 1 \) and \( m_i = 1 \) for every \( i \). Now the equalities \( \ref{eq:3.3} \) imply that the equality \( \beta < \alpha \) holds, but every singular point of the surface \( S \) that is contained in the curve \( C \) must be an intersection point of \( C \) and the curve \( \sum_{i = \beta + 1}^\alpha C_i \), which consists of at most \( r^2 \) points, which is a contradiction.

Hence, we have \( m_i \geq 2 \) for every \( i \leq \beta \). Therefore, it follows from the equalities \( \ref{eq:3.3} \) that \( \alpha = \beta \) and \( m_i = 2 \) for every \( i \).

Let \( f(x, y, z, w) \) be the homogeneous form of degree \( 2r \) such that \( f = 0 \) defines the surface \( S \), where \( (x : y : z : w) \) are homogeneous coordinates on \( \mathbb{P}^3 \). We may assume that the plane \( \Pi \) is given by the equation \( x = 0 \). Then \( f(0, y, z, w) = g_2^r(y, z, w) \), where \( g_r \) is a homogeneous polynomial of degree \( r \) such that \( C \) is given by \( x = g_r = 0 \), which implies that the surface \( S \) can be defined by the equation \( \ref{eq:3.6} \).

It follows from Lemma \( \ref{lem:3.1} \) that at most \( (2r - 1)k \) singular points of the surface \( S \) can lie on a curve of degree \( k \). However, the claim of Lemma \( \ref{lem:3.1} \) can be improved for curves that are not contained in two-dimensional linear subspaces of \( \mathbb{P}^3 \).

**Lemma 3.4.** Let \( C \) be an irreducible reduced curve in \( \mathbb{P}^3 \) of degree \( k \) that is not contained in a hyperplane. Then \( |C \cap \text{Sing}(S)| \leq (2r - 1)k - 2 \).

**Proof.** Suppose that the curve \( C \) contains at least \( (2r - 1)k - 1 \) singular points of the surface \( S \). Then \( C \subset S \), because otherwise we have

\[
2rk = \deg(C)\deg(S) \leq 2(2r - 1)k - 2 = 4rk - 2k - 2,
\]

which leads to \( 2k(r - 1) \leq 2 \), but \( r \geq 2 \) and \( k \geq 3 \).

Let \( O \) be a sufficiently general point of the curve \( C \), and \( \psi : \mathbb{P}^3 \rightarrow \Pi \) be a projection from the point \( O \), where \( \Pi \) is a sufficiently general plane in \( \mathbb{P}^3 \). Then \( \psi|_C \) is a birational morphism, because \( C \) is not a plane curve. Put \( Z = \psi(C) \). Then \( Z \) has degree \( k - 1 \).

Let \( Y \) be a cone in \( \mathbb{P}^3 \) over \( Z \) with the vertex \( O \). Then \( C \subset Y \).

It follows from the generality of the point \( O \) that the point \( O \) is not contained in a hyperplane in \( \mathbb{P}^3 \) that is tangent to the surface \( S \) at some point of the curve \( C \), because the curve \( C \) is not contained in a hyperplane. Therefore, the cone \( Y \) does not tangent the surface \( S \) along the curve \( C \).

Put \( S|_Y = C + R \), where \( R \) is a curve of degree \( 2rk - k - 2r \). Then the generality of the point \( O \) implies that the curve \( R \) does not contains rulings of the cone \( Y \).
Let $\alpha : \bar{Z} \to Z$ be the normalization of $Z$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\beta} & Y \\
\downarrow{\pi} & & \downarrow{\psi_Y} \\
\bar{Z} & \xrightarrow{\alpha} & Z,
\end{array}
$$

where $\beta$ is a birational morphism, $\bar{Y}$ is smooth, and $\pi$ is a $\mathbb{P}^1$-bundle. Let $L$ be a general fiber of $\pi$, and $E$ be a section of $\pi$ such that $\beta(E) = O$. Then $E^2 = -k + 1$ on $\bar{Y}$.

Let $C$ and $\bar{R}$ be proper transforms of the curves $C$ and $R$ on the surface $\bar{Y}$ respectively, and $Q$ be an arbitrary point of the set $\text{Sing}(S) \cap C$. Then there is a point $\bar{Q} \in \bar{Y}$ such that $\beta(\bar{Q}) = Q$ and $\bar{Q} \in \text{Supp}(C \cdot \bar{R})$, but

$$
\bar{R} \equiv (2r - 2)E + (2rk - k - 2r)L
$$

and $\bar{C} \equiv E + kL$. Therefore, we have $(2r - 1)k - 2 = \bar{C} \cdot \bar{R} \geq (2r - 1)k - 1$. \hfill $\square$

Now we prove Theorem 1.6 by reductio ad absurdum. Put $\Sigma = \text{Sing}(S)$, and suppose that the following conditions hold:

- the inequalities $|\Sigma| \leq (2r - 1)r + 1$ and $r \geq 3$ hold;
- the surface $S$ can not be defined by the equation 1.5;
- the threefold $X$ is not factorial, which implies that there is a point $P \in \Sigma$ such that every surface in $\mathbb{P}^3$ of degree $3r - 4$ containing $\Sigma \setminus P$ contains the point $P$.

We assume that $r \geq 4$, because the case $r = 3$ is done in [11].

**Lemma 3.5.** Let $\Pi$ be a two-dimensional linear subspace in $\mathbb{P}^3$. Then $|\Pi \cap \Sigma| \leq 2r$.

*Proof.* Suppose that $|\Pi \cap \Sigma| > 2r$. Let us show that this assumption leads to a contradiction. Let $\Gamma$ be the subset of the set $\Sigma$ that consists of all points that are not contained in the plane $\Pi$. Then $\Gamma$ contains at most $(2r - 1)(r - 1) - 1$ points, which impose independent linear conditions on homogeneous forms of degree $3r - 5$ by Proposition 2.4.

Suppose that $P \not\in \Pi$. Then there is a surface $F \subset \mathbb{P}^3$ of degree $3r - 5$ that contains the set $\Gamma \setminus P$ and does not contain the point $P$. Hence, the union $F \cup \Pi$ is the surface of degree $3r - 4$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$, which is impossible due to our assumptions. Therefore, we have $P \in \Pi$.

The curve $\Pi \cap S$ is singular in every point of the set $\Pi \cap \Sigma$. Thus, it follows from the proof of Lemma 3.1 that $|\Pi \cap \Sigma| \leq (2r - 1)r$, but Lemma 3.2 implies that $\Pi \cap \Sigma$ is not contained in a curve of degree $r$ if $|\Pi \cap \Sigma| = (2r - 1)r$. The proof of Lemma 2.5 implies that there is a surface of degree $3r - 4$ that contains the set $(\Pi \cap \Sigma) \setminus P$ and does not contain the point $P$, which concludes the proof by Lemma 2.12. \hfill $\square$

The inequality $|\Sigma| \geq (2r - 1)r$ holds by Proposition 2.1.

**Lemma 3.6.** Let $L_1$ and $L_2$ be distinct lines in $\mathbb{P}^3$. Then $|(L_1 \cup L_2) \cap \Sigma| < 4r - 2$.

*Proof.* Suppose that $|(L_1 \cup L_2) \cap \Sigma| \geq 4r - 2$. Then $|L_i \cap \Sigma| = 2r - 1$ by Lemma 3.1 and the lines $L_1$ and $L_2$ are not contained in one hyperplane by Lemma 3.5.

Fix two points $Q_1$ and $Q_2$ in $\Sigma \setminus ((L_1 \cup L_2) \cap \Sigma)$ different from $P$ such that $Q_1 \not= Q_2$, and let $\Pi_i$ be a plane in $\mathbb{P}^3$ that contains $L_i$ and $Q_i$. Then $|\Pi_i \cap \Sigma| = 2r$ by Lemma 3.5.

Suppose that $P \not\in \Pi_1 \cup \Pi_2$. Then there is a surface $F \subset \mathbb{P}^3$ of degree $3r - 6$ that does not contain the point $P$ and contains all points of the set

$$
\left( \Sigma \setminus \left( \Sigma \cap (\Pi_1 \cup \Pi_2) \right) \right) \setminus P
$$

10
by Proposition 2.4. Hence, the surface $F \cup \Pi_1 \cup \Pi_2$ is a surface in $\mathbb{P}^3$ of degree $3r - 4$ that contains all points of the set $\Sigma \setminus P$ and does not contain the point $P$, which contradicts to our assumption. Therefore, we have $P \in \Pi_1 \cup \Pi_2$.

The set $\Sigma \cap (\Pi_1 \cup \Pi_2)$ consists of $4r$ points by Lemma 3.5. Therefore, the points of the set $\Sigma \cap (\Pi_1 \cup \Pi_2)$ impose independent linear conditions on homogeneous forms $\mathbb{P}^3$ of degree $3r - 4$ by Theorem 2 in [9]. On the other hand, the inequality

$$\left| \Sigma \setminus (\Sigma \cap (\Pi_1 \cup \Pi_2)) \right| < (2r - 1)(r - 2)$$

holds, and the points of $\Sigma \setminus (\Sigma \cap (\Pi_1 \cup \Pi_2))$ impose independent linear conditions homogeneous forms of degree $3r - 4$ by Proposition 2.4, which is impossible by Lemma 2.12. □

**Lemma 3.7.** Let $C$ be a curve in $\mathbb{P}^3$ of degree $k \geq 2$. Then $|C \cap \Sigma| < (2r - 1)k$.

**Proof.** Suppose that $|C \cap \Sigma| \geq (2r - 1)k$. Let us show that this assumption leads to a contradiction. We have $|C \cap \Sigma| = (2r - 1)k$ by Lemma 3.1 and the curve $C$ is not contained in a hyperplane by Lemma 3.5. Therefore, the curve $C$ is reducible by Lemma 3.4.

Let us put $C = \sum_{i=1}^{\alpha} C_i$, where $\alpha \geq 2$ and $C_i$ is an irreducible curve. Then $k = \sum_{i=1}^{\alpha} d_i$, where $d_i$ is the degree of the curve $C_i$, which implies $|C_i \cap \Sigma| = (2r - 1)d_i$ by Lemma 3.1.

The curve $C_i$ is contained in a hyperplane in $\mathbb{P}^3$ by Lemma 3.5. So, the equalities $d_i = 1$ and $\alpha = k$ hold by Lemma 3.5 for all $i$, which contradicts Lemma 3.6 because $k \geq 2$. □

**Lemma 3.8.** Let $L$ be a line in $\mathbb{P}^3$. Then $|L \cap \Sigma| \leq 2r - 2$.

**Proof.** Suppose that $|L \cap \Sigma| \geq 2r - 1$ holds. Let us show that this assumption leads to a contradiction. We have $|L \cap \Sigma| = 2r - 1$ by Lemma 3.4.

Let $\Phi$ be a hyperplane in $\mathbb{P}^3$ such that $\Phi$ contains the line $L$, and $\Phi$ contains an arbitrary point of the set $\Sigma \setminus (L \cap \Sigma)$. Then $\Phi$ contains $2r$ points of the set $\Sigma$ by Lemma 3.5.

Put $\Delta = \Sigma \setminus (\Phi \cap \Sigma)$. Then $|\Delta| \leq (2r - 1)(r - 1)$.

Suppose that the points of the set $\Delta$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 5$. Then it follows from Lemma 2.12 that the points of the set $\Sigma$ impose independent linear conditions on homogeneous forms of degree $3r - 4$, because the points of the set $\Phi \cap \Sigma$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4$. Therefore, the points of the set $\Delta$ impose dependent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 5$.

There is a point $Q \in \Delta$ such that every surface of degree $3r - 5$ containing $\Delta \setminus Q$ must contain $Q$, which implies $|\Delta| = (2r - 1)(r - 1)$ and $|\Sigma| = (2r - 1)r + 1$ by Proposition 2.4.

Fix sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$ and a point $O \in \mathbb{P}^3$. Let $\psi: \mathbb{P}^3 \rightarrow \Pi$ be a projection from the point $O$. Put $\Delta' = \psi(\Delta)$ and $Q' = \psi(Q)$. Then at most $2r - 2$ points of the set $\Delta'$ lie on a line by Lemmas 2.3 and 3.6.

Suppose that at most $(2r - 1)k$ points of the set $\Delta'$ lie on any curve in $\Pi$ of degree $k$ for every natural number $k$, and there is a curve $Z \subset \Pi$ of degree $r - 1$ that contains the whole set $\Delta'$. Then the points of the set $\Delta$ impose dependent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 5$ by Lemmas 2.3, 2.10 and 3.7 in the case when the curve $Z$ is irreducible. So, we have $Z = \sum_{i=1}^{\alpha} Z_i$, where $\alpha \geq 2$, and $Z_i$ is an irreducible curve of degree $d_i$. Then $r = \sum_{i=1}^{\alpha} d_i$, which implies that $Z_i$ contains $(2r - 1)d_i$ points of the set $\Delta'$, and every point of the set $\Delta'$ is contained in one irreducible component of the curve $Z$. In particular, we have $d_i \neq 1$ for every $i$.

Let $Z_\beta$ be the component of $Z$ containing $Q'$, and $\Gamma$ be a subset of $\Delta$ such that

$$\psi(\Gamma) = \Delta' \cap Z_\beta \subset \Pi \cong \mathbb{P}^2,$$
which implies \( Q \in \Gamma \). There is a surface \( F_\beta \subset \mathbb{P}^3 \) of degree \( 3(d_3 - 1) \) that contains all point of the set \( \Gamma \setminus Q \) and does not contain \( Q \) by Lemmas 2.3, 2.10, and 3.7. Let \( Y_i \) be a cone over the curve \( Z_i \), whose vertex is the point \( O \). Then the union \( F_\beta \cup \cup_{i \neq \beta} Y_i \) is a surface of degree \( 3d_i - 3 + \sum_{i \neq \beta} d_i = 2d_i + r - 4 \) that contains \( \Delta \setminus Q \) and does not contain \( Q \), which is impossible, because \( 2d_i + r - 4 \leq 3r - 5 \). Hence, we proved that

- either at least \( (2r - 1)k + 1 \) points of \( \Delta' \) lie on a curve in \( \Pi \) of degree \( k \);
- or there is no curve in \( \Pi \) of degree \( r - 1 \) that contains the whole set \( \Delta' \).

Suppose that at most \( (2r - 1)k \) points of the set \( \Delta' \) lie on every curve in \( \Pi \) of degree \( k \) for every natural \( k \). Then the points of the set \( \Delta' \setminus Q' \) and the number \( 3r - 5 \) satisfy all hypotheses of Theorem 2.2 because there is no curve in \( \Pi \) of degree \( r - 1 \) that contains the set \( \Delta' \). Hence, we can apply Theorem 2.2 to the blow up of the plane \( \Pi \) at the points of the set \( \Delta' \setminus Q' \) to prove the existence of a curve in the plane \( \Pi \) of degree \( 3r - 5 \) that contains the set \( \Delta' \setminus Q' \) and does not contains the point \( Q' \), which is a contradiction.

Therefore, at least \( (2r - 1)k + 1 \) points of the set \( \Delta' \) lie on a curve in \( \Pi \) of degree \( k \), where \( k \geq 3 \) by Lemma 2.7. Thus, the proof of Proposition 2.4 implies the existence of a subset \( \Xi \subset \Delta \) such that the following conditions hold:

- the points of \( \Xi \) impose independent linear conditions on surfaces of degree \( 3r - 5 \);
- at most \( (2r - 1)k \) points of the set \( \psi(\Delta \setminus \Xi) \) lie on a curve in \( \Pi \) of degree \( k \);
- there is a surface of degree \( \mu \leq r - 2 \) that contains all points of the set \( \Xi \) and does not contain any point of the set \( \Delta \setminus \Xi \);
- the inequality \( |\Delta \setminus \Xi| \leq (2r - 1)(r - 1 - \mu) - 1 \) holds.

Put \( \Delta = \psi(\Delta \setminus \Xi) \) and \( d = 3r - 5 - \mu \). Then the points of \( \Delta \) impose dependent linear conditions on homogeneous forms of degree \( d \) by Lemma 2.12 which implies that there is a point \( Q \in \Delta \) such that \( \Delta \setminus Q \) and \( d \) do not satisfy the hypotheses of Theorem 2.2.

We have \( d \geq 3 \), because \( r \geq 4 \). The proof of Lemma 2.13 gives

\[
|\overline{\Delta} \setminus \overline{Q}| \leq \left\lfloor \frac{d + 3}{2} \right\rfloor,
\]

which implies that at least \( t(d + 3 - t) - 1 \) points of the finite set \( \overline{\Delta} \setminus \overline{Q} \) lie on a curve of degree \( t \) for some natural number \( t \) such that \( t \leq (d + 3)/2 \).

Suppose that \( t = 1 \). At least \( d + 1 \) points of \( \Delta \) lie on a line, but at most \( 2r - 2 \) points of \( \Delta' \) lie on a line by Lemmas 2.3 and 3.6 which implies that \( d = 2r - 3 \) and \( |\Delta| = 2r - 2 \) points, which is impossible because the points of the set \( \overline{\Delta} \) impose dependent linear conditions on homogeneous forms of degree \( d \). Therefore, we see that \( t \geq 2 \).

At least \( t(d + 3 - t) - 1 \) points of \( \overline{\Delta} \setminus \overline{Q} \) lie on a curve of degree \( t \geq 2 \). Then

\[
t(d + 3 - t) - 1 \leq |\overline{\Delta} \setminus \overline{Q}| \leq (2r - 1)(r - 1) - 2 - \mu(2r - 1),
\]

but \( t(d + 3 - t) - 1 \leq (2r - 1)t \), because at most \( (2r - 1)t \) points of \( \Delta \) lie on a curve of degree \( t \). Hence, we have \( t \geq r - 1 - \mu \), which gives

\[
(2r - 1)(r - 1 - \mu) - 2 \geq |\overline{\Delta} \setminus \overline{Q}| \geq t(d + 3 - t) - 1 \geq (r - 1 - \mu)(2r - 1) - 1,
\]

which is a contradiction. \( \square \)

**Corollary 3.9.** Let \( C \) be any curve in \( \mathbb{P}^3 \) of degree \( k \). Then \( |C \cap \Sigma| < (2r - 1)k \).

Fix a hyperplane \( \Pi \subset \mathbb{P}^3 \) and a general point \( O \in \mathbb{P}^3 \). Let

\[
\psi : \mathbb{P}^3 \rightarrow \Pi \subset \mathbb{P}^3
\]

be a projection from \( O \). Put \( \Sigma' = \psi(\Sigma) \) and \( P' = \psi(P) \). Then \( \psi|_{\Sigma} : \Sigma \rightarrow \Sigma' \) is a bijection.
Lemma 3.10. Let $C$ be an irreducible curve in $\Pi$ of degree $r$. Then $|C \cap \Sigma'| < (2r - 1)r$.

Proof. Suppose that $|C \cap \Sigma'| \geq (2r - 1)r$. Let us show that this assumption leads to a contradiction. Let $\Psi$ be a subset of the set $\Sigma$ consisting of the points that are mapped to the curve $C$ by the projection $\psi$. Then $|\Psi| \geq (2r - 1)r$, but less than $(2r - 1)r$ points of the set $\Sigma$ lie on a curve of degree $r$ by Corollary 3.6.

Let $\mathcal{H}$ be a linear system of surfaces in $\mathbb{P}^3$ of degree $r$ that contains $\Psi$, and $\Phi$ be the base locus of $\mathcal{H}$. Then $\Phi$ is finite Lemma 2.5. Put $\Upsilon = \Sigma \cap \Phi$. Then the points of $\Upsilon$ impose independent linear conditions on homogeneous forms of degree $3r - 3$ by Lemma 2.10.

Let $\Gamma$ be a subset in $\Upsilon$ such that $\Upsilon \setminus \Gamma$ consists of $4r - 6$ points. Then

$$|\Gamma| \leq 2r^2 - 5r - 5 \leq \frac{(r + 2)(r + 1)r}{6} - 1,$$

because $r \geq 4$, which implies that there is a surface $F \subset \mathbb{P}^3$ of degree $r - 1$ that contains all points of the set $\Gamma$. Let $\Theta$ be a subset of the set $\Upsilon$ such that $\Theta$ consists of all points that are contained in the surface $F$. Then the points of the set $\Theta$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4$ by Theorem 3 in [9].

Put $\Delta = \Upsilon \setminus \Theta$. Then the points of $\Delta$ impose independent linear conditions on homogeneous forms of degree $2r - 3$ by Theorem 2 in [9] and Lemmas 3.5 and 3.8. So, the points of the set $\Upsilon$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ by Lemma 2.12, which also follows from Theorem 3 in [9], because $(2r - 1)r + 1 < r^3$.

We have $|\Sigma \setminus \Upsilon| \leq 1$. Thus, the points of the set $\Sigma$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ by Lemma 2.12 which is impossible. □

Lemma 3.11. There is a curve $Z \subset \Pi$ of degree $k$ such that $|Z \cap \Sigma'| \geq (2r - 1)k + 1$.

Proof. Suppose that no $(2r - 1)k + 1$ points of the set $\Sigma'$ lie on a curve of degree $k$ for every natural number $k$. Let us show that this assumption leads to a contradiction.

The finite subset $\Sigma' \setminus P' \subset \Pi$ and the natural number $3r - 4$ does not satisfy at least one of the hypotheses of Theorem 2.2, because every surface in $\mathbb{P}^3$ of degree $3r - 4$ containing all points of the set $\Sigma \setminus P$ must contain the point $P$. However, the inequalities

$$|\Sigma' \setminus P'| \leq (2r - 1)r \leq \max \left( \left\lfloor \frac{3r - 1}{2} \right\rfloor \left(3r - 1 - \left\lfloor \frac{3r - 1}{2} \right\rfloor \right), \left\lfloor \frac{3r - 1}{2} \right\rfloor^2 \right)$$

hold, and at most $3r - 4$ points of the set $\Sigma' \setminus P'$ can lie on a line, because $3r - 4 \geq 2r - 1$ and at most $2r - 1$ points of the set $\Sigma'$ can lie on a line by Lemma 2.3.

We see that at least $k(3r - 1 - k) - 1$ points of the set $\Sigma' \setminus P'$ lie on a curve of degree $k$ such that $2 \leq k \leq (3r - 1)/2$, which implies that $k = r$, because at most $k(2r - 1)$ points of the set $\Sigma'$ lie on a curve of degree $k$, and $|\Sigma' \setminus P'| \leq (2r - 1)r$. Thus, we conclude that there is a curve $C \subset \Pi$ of degree $r$ that contains at least $(2r - 1)r - 1$ points of $\Sigma' \setminus P'$.

The curve $C$ contains $P'$, because otherwise there is a curve in $\Pi$ of degree $3r - 4$ that contains the set $\Sigma' \setminus P'$ and does not contain the point $P'$. Hence, the curve $C$ contains at least $(2r - 1)r$ points of the set $\Sigma'$. Thus, the curve $C$ is reducible by Lemma 3.10.

Let $C = \sum_{i=1}^{\alpha} C_i$, where $C_i$ is an irreducible curve of degree $d_i \geq 1$ and $\alpha \geq 2$. Then

$$(2r - 1)r \leq |C \cap \Sigma'| \leq \sum_{i=1}^{\alpha} |C_i \cap \Sigma'| \leq \sum_{i=1}^{\alpha} (2r - 1)\deg(C_i) = (2r - 1)r,$$

which implies that the curve $C_i$ contains $(2r - 1)d_i$ points of the set $\Sigma$, and every point of the set $\Sigma$ is contained in at most one curve $C_i$. 

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Let $C_{i}$ be the irreducible component of the curve $C$ that contains $P'$, and $\Upsilon$ be a subset of the set $\Sigma$ that contains all points of the set $\Sigma$ that are mapped to the curve $C_{i}$ by the projection $\psi$. Then $|\Upsilon| = (2r - 1)d_{i}$, but less than $(2r - 1)d_{i}$ points of the set $\Sigma$ lie on a curve of degree $d_{i}$. Hence, the points of the set $\Upsilon$ impose independent linear conditions on the homogeneous forms of degree $3(d_{i} - 1)$ by Lemmas 2.3 and 2.10.

There is a surface $F \subset \mathbb{P}^{3}$ of degree $3(d_{i} - 1)$ that contains the set $\Upsilon \setminus P$ and does not contain the point $P$. Let $Y_{i}$ be a cone in $\mathbb{P}^{3}$ over $C_{i}$ with the vertex $O$. Then the surface

$$F \cup \bigcup_{i \neq j} Y_{i} \in |\mathcal{O}_{\mathbb{P}^{3}}(2d_{i} - 3 + r)|$$

contains $\Sigma \setminus P$ and does not contain $P$, but $2d_{i} - 3 + r \leq 3r - 4$, which is a contradiction. □

There is a disjoint union $\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_{j}} \Lambda_{j}^{i} \subset \Sigma$, where $\Lambda_{j}^{i}$ is a subset of the set $\Sigma$ such that the inequality $|\Lambda_{j}^{i}| > (2r - 1)j$ holds, all points of the subset $\psi(\Lambda_{j}^{i})$ is contained in an irreducible curve in $\Pi$ of degree $j$, and at most $(2r - 1)t$ points of the subset

$$\psi(\Sigma \setminus \left( \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_{j}} \Lambda_{j}^{i} \right)) \subset \Sigma' \subset \Pi \cong \mathbb{P}^{2}$$

lie on a curve in $\Pi$ of degree $t$. Then $k \geq 3$ by Lemma 2.4 and $k < r$ by Lemma 3.10.

Put $\Lambda = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_{j}} \Lambda_{j}^{i}$. Let $\Xi_{j}^{i}$ be the base locus of the linear system of surfaces in $\mathbb{P}^{3}$ of degree $j$ that contains all points of the set $\Lambda_{j}^{i}$. Then $\Xi_{j}^{i}$ is a finite set by Lemma 2.3 and

$$|\Sigma \setminus \Lambda| \leq (2r - 1)r + 1 - \sum_{i=k}^{l} c_{i} \left( (2r - 1)i + 1 \right) \leq (2r - 1) \left( r - \sum_{i=k}^{l} ic_{i} \right),$$

which implies that $\sum_{i=k}^{l} ic_{i} \leq r$.

Remark 3.13. The inequality $\sum_{i=k}^{l} ic_{i} \leq r - 1$ holds, because the equality $\sum_{i=k}^{l} ic_{i} = r$ and the inequalities (3.12) imply that $k = l = r$, but $k < r$ by Lemma 3.10.

It follows from Lemma 2.10 that the points of $\Xi_{j}^{i}$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^{3}$ of degree $3(j - 1)$. Put $\Xi = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_{j}} \Xi_{j}^{i}$. Then

$$|\Sigma \setminus (\Xi \cap \Sigma)| \leq (2r - 1)r - \sum_{i=k}^{l} c_{i} (2r - 1)i.$$ 

There are surfaces $F$ and $G$ in $\mathbb{P}^{3}$ of degree $\sum_{i=k}^{l} 3(i - 1)c_{i}$ and $\sum_{i=k}^{l} ic_{i}$ respectively such that $F$ contains $(\Xi \cap \Sigma) \setminus P$ and does not contain $P$, but $G$ contains $\Xi \cap \Sigma$ and does not contain any point in $\Sigma \setminus (\Xi \cap \Sigma)$. In particular, we have $\Sigma \nsubseteq \Xi$, because

$$\sum_{i=k}^{l} 3(i - 1)c_{i} \leq \sum_{i=k}^{l} 3ic_{i} - 3c_{k} \leq 3r - 6 < 3r - 4.$$

Put $\bar{\Sigma} = \psi(\Sigma \setminus (\Xi \cap \Sigma))$ and $d = 3r - 4 - \sum_{i=k}^{l} ic_{i}$. Then it follows from Lemma 2.12 that there is a point $\bar{Q} \in \Sigma$ such that every curve in $\Pi$ of degree $d$ that contains $\bar{\Sigma} \setminus \bar{Q}$ must pass through the point $\bar{Q}$ as well. Therefore, we can not apply Theorem 2.2 to the points of the subset $\bar{\Sigma} \setminus \bar{Q} \subset \Pi$ and the natural number $d$. 
The proof of Lemma 2.13 implies that the inequality
\[
|\Sigma \setminus \bar{Q}| \leq (2r - 1)\left(r - \sum_{i=1}^l c_i\right) - 1 \leq \left\lfloor \frac{d + 3}{2} \right\rfloor^2
\]
holds, but \(d = 3r - 4 - \sum_{i=1}^l c_i \geq 2r - 3 \geq 3\), because \(\sum_{i=1}^l c_i \leq r - 1\), which implies that at least \(t(d + 3 - t) - 1\) points of \(\Sigma \setminus \bar{Q}\) lie on a curve in \(\Pi\) of degree \(t \leq (d + 3)/2\).

**Lemma 3.15.** The inequality \(t \neq 1\) holds.

**Proof.** Suppose that \(t = 1\). Then at least \(d + 1\) points of the set \(\Sigma \setminus \bar{Q}\) lie on a line, which implies the inequality \(d + 1 \leq 2r - 2\) by Lemmas 2.3 and 3.8.

The inequality \(d + 1 \leq 2r - 2\) implies that \(\sum_{i=1}^l c_i = r - 1\) and \(d = 2r - 3\).

It follows from the inequality 3.14 that \(|\Sigma \setminus (\Xi \cap \Sigma)| \leq 2r - 1\), which implies that the points of the set \(\Sigma \setminus (\Xi \cap \Sigma)\) impose independent linear conditions on the homogeneous forms of degree \(2r - 3\) by Theorem 2 in [9], which is impossible by Lemma 2.12.

There is a curve \(C \subset \Pi\) of degree \(t \geq 2\) that contains at least \(t(d + 3 - t) - 1\) points of the set \(\Sigma \setminus \bar{Q}\), which implies that \(t(d + 3 - t) - 1 \leq |\Sigma \setminus \bar{Q}|\) and \(t(d + 3 - t) - 1 \leq (2r - 1)t\), because at most \((2r - 1)t\) points of the set \(\Sigma\) lie on a curve of degree \(t\). Therefore, we see that \(t \geq r - \sum_{i=1}^l c_i\), because \(t \geq 2\). It follows from the inequalities 3.12 that
\[
(2r - 1)\left(r - \sum_{i=1}^l c_i\right) - 1 \geq |\Sigma \setminus \bar{Q}| \geq t(d + 3 - t) - 1 \geq \left(r - \sum_{i=1}^l c_i\right)(2r - 1) - 1,
\]
which implies that \(t = r - \sum_{i=1}^l c_i\), the curve \(C\) contains all points of the set \(\Sigma \setminus \bar{Q}\), and the inequalities 3.12 are actually equalities. Namely, we have \(\Sigma \cap \Xi = \Lambda\) and
\[
|\Sigma \setminus \Lambda| = (2r - 1)r + 1 - \sum_{i=1}^l c_i(2r - 1)i + 1 = (2r - 1)\left(r - \sum_{i=1}^l c_i\right),
\]
which implies that \(l = k\), \(c_k = 1\), \(d = 3r - 4 - k\) and \(\sum_{i=1}^l c_i = k\).

**Lemma 3.16.** The curve \(C\) contains all points of the set \(\bar{\Sigma}\).

**Proof.** Suppose that \(C\) does not contain the set \(\bar{\Sigma}\). Then \(C\) does not contain \(\bar{Q}\), which implies that there is a curve in \(\Pi\) of degree \(r - k\) that contains the set \(\Sigma \setminus \bar{Q}\) does not contain the point \(\bar{Q}\), which is impossible, because \(d \geq r - k\).

Thus, the curve \(C\) is a curve of degree \(r - k\) that contains the set \(\psi(\Sigma \setminus \Lambda)\), which consists of exactly \((r - k)(2r - 1)\) points of the set \(\psi(\Sigma)\). On the other hand, there is an irreducible curve \(Z \subset \Pi\) of degree \(k\) that contains all points of the set \(\psi(\Lambda)\), which consists of exactly \(k(2r - 1) + 1\) points of the set \(\psi(\Sigma)\). In particular, we have
\[
|\Sigma| = |\Sigma \setminus \Lambda| + |\Lambda| = (r - k)(2r - 1) + k(2r - 1) + 1 = (2r - 1)r + 1.
\]

**Lemma 3.17.** The curve \(C\) is reducible.

**Proof.** Suppose that \(C\) is irreducible. Then the points of the set \(\Sigma \setminus \Lambda\) impose independent linear conditions on surfaces of degree \(3(r - k - 1)\) by Lemmas 2.3, 2.10 and 3.7, but the points of the set \(\Lambda\) impose independent linear conditions on surfaces of degree \(3(k - 1)\) by Lemmas 2.3 and 2.10, which implies that the points of the set \(\Sigma\) impose independent linear conditions on homogeneous forms of degree \(3r - 4\) by Lemma 2.12.
Therefore, we have $C = \sum_{i=1}^{\alpha} C_i$, where $C_i$ is an irreducible curve of degree $d_i$, which implies that $r - k = \sum_{i=1}^{\alpha} d_i$, the curve $C_i$ contains $(2r - 1)d_i$ points of the set $\bar{\Sigma}$ for every $i$, and every point of the set $\bar{\Sigma}$ is contained in a single irreducible component of $C$.

**Lemma 3.18.** The curve $Z$ contains the point $P'$.

*Proof.* Suppose that $P' \notin Z$. Let $C_\nu$ be an irreducible component of the curve $C$ that contains the point $P'$, and $\Upsilon$ be a subset of the set $\Sigma$ that contains all points that are mapped to the curve $C_\nu$ by the projection $\psi$. Then $\Upsilon$ contains $(2r - 1)d_\nu$ points.

The points of the set $\Upsilon$ impose independent linear conditions on the homogeneous forms of degree $3(d_\nu - 1)$ by Lemmas 2.3 and 2.10. Therefore, there is a surface $F \subset \mathbb{P}^3$ of degree $3(d_\nu - 1)$ that contains $\Upsilon \setminus P$ and does not contain $P$.

Let $Y_i$ and $Y$ be the cones in $\mathbb{P}^3$ over the curves $C_i$ and $Z$, respectively, whose vertex is the point $O$. Then the union $F \cup \Upsilon \cup \bigcup_{i \neq \nu} Y_i$ is a surface of degree $2d_\nu - 3 + r \leq 3r - 4$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$, which is a contradiction. □

The proof of Lemma 3.18 implies that the points of the set $\Sigma \setminus \Lambda$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4 - k$, but we already know that the points of the set $\Lambda$ impose independent linear conditions on homogeneous forms of degree $3(k - 1)$ by Lemmas 2.3 and 2.10 which is impossible by Lemma 2.12.

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