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**Citation for published version:**

Huangfu, Q & Hall, J 2015, 'Novel update techniques for the revised simplex method', *Computational optimization and applications*, vol. 60, no. 3, pp. 587-608. <https://doi.org/10.1007/s10589-014-9689-1>

**Digital Object Identifier (DOI):**

[10.1007/s10589-014-9689-1](https://doi.org/10.1007/s10589-014-9689-1)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

Computational optimization and applications

**Publisher Rights Statement:**

The final publication is available at Springer via <http://dx.doi.org/10.1007/s10589-014-9689-1>

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# Novel update techniques for the revised simplex method

Qi Huangfu · J. A. Julian Hall

7 July 2013

Technical Report ERGO-13-001

**Abstract** This paper introduces three novel techniques for updating the invertible representation of the basis matrix when solving practical sparse linear programming (LP) problems using a high performance implementation of the dual revised simplex method, being of particular value when suboptimization is used. Two are variants of the product form update and the other permits multiple Forrest-Tomlin updates to be performed. Computational results show that one of the product form variants is significantly more efficient than the traditional approach, with its performance approaching that of the Forrest-Tomlin update for some problems. The other is less efficient, but valuable in the context of the dual revised simplex method with suboptimization. Results show that the multiple Forrest-Tomlin updates are performed with no loss of serial efficiency.

**Keywords** Linear programming · dual revised simplex method · sparse matrices · basis inverse update techniques

**Mathematics Subject Classification (2000)** 90C05 · 65F05

## 1 Introduction

For particular classes of linear programming (LP) problems and families of related LP problems, the revised simplex method is generally the preferred solution technique. The opportunities and limitations of modern computer architectures, together with the desire to solve ever larger problem instances, continue to drive developments in high performance computing techniques for the simplex method [14, 19].

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The principal computational challenge when implementing the revised simplex method is the efficient solution of linear systems of equations whose matrix of coefficients is the simplex basis matrix or its transpose. There are standard efficient techniques [22, 24] to obtain an invertible representation of a particular basis matrix and many more have been developed to update this representation by exploiting the relation between successive basis matrices, whereby one column is replaced as a result of each simplex iteration. The original and simplest technique for updating the invertible representation is the product form update of Dantzig and Orchard-Hays [6]. The class of updates that is widely accepted as being the most efficient derive from the work of Bartels and Golub [1], most notable being the techniques of Forrest and Tomlin [9], Reid [20] and Suhl and Suhl [23]. Several other schemes have been proposed, and of particular relevance to this paper is the work by Saunders *et al.* [8, 12] and Hall [13] on generalising the Schur complement update of Bisschop and Meeraus [3]. A recent review of simplex update procedures is given by Elble and Sahinidis [7].

Motivated by the requirements of the high performance implementation of the revised simplex method (`hso1`) developed by Huangfu and Hall [16], this paper introduces three novel procedures for updating the invertible representation of the simplex basis matrix. Section 2 introduces the background necessary to discuss the new update procedures in Section 3. Results in Section 4 show that the serial performance of two of them is competitive with traditional techniques and conclusions are offered in Section 5.

## 2 Background

An LP problem in standard form is

$$\text{minimize } c^T x \quad \text{subject to } Ax = b, \quad x \geq 0,$$

where  $x \in \mathbb{R}^n$  and  $b \in \mathbb{R}^m$ . It may be assumed that the matrix  $A$  is of full rank. In the simplex method, the indices of variables are partitioned into sets  $\mathcal{B}$  corresponding to  $m$  basic variables and  $\mathcal{N}$  corresponding to  $n - m$  nonbasic variables such that the basis matrix  $B$  formed from the columns of  $A$  corresponding to  $\mathcal{B}$  is nonsingular. Each iteration of the simplex method chooses an index  $q \in \mathcal{N}$  to exchange with the  $p^{\text{th}}$  entry of  $\mathcal{B}$ . The corresponding change in the basis matrix  $B$  may be expressed as

$$\bar{B} = B + (a_q - B e_p) e_p^T. \quad (1)$$

In the (primal or dual) revised simplex method the indices  $p$  and  $q$  are identified by algorithmic techniques dependent on data obtained by solving linear systems of equations of the form

$$B \hat{x} = x \quad (2)$$

$$\hat{x}^T B = x^T \quad (\text{or } B^T \hat{x} = x), \quad (3)$$

each at least once per iteration. The linear systems (2) and (3) are referred to as the *forward* and *transposed* system respectively. In particular, each revised simplex iteration requires the solution of the forward system

$$B \hat{a}_q = a_q \quad (4)$$

and the transposed system

$$\hat{e}_p^T B = e_p^T. \quad (5)$$

Such linear systems of equations are solved using an invertible representation of  $B$  whose efficient identification and use represent the principal computational challenge when implementing the revised simplex method.

There are standard efficient techniques [22,24] to obtain an invertible representation of a particular basis matrix directly, all of which use Gaussian elimination to identify an LU decomposition of  $B$ . The operation of finding the invertible representation directly in this way is referred to as INVERT and the matrix to which it is applied is denoted by  $B_0$ . Although INVERT involves row and column permutations, for convenience and with no loss of generality, they may be ignored so  $B_0 = L_0 U_0$ . This decomposition is represented as a product of elementary *eta* matrices of the form

$$E = \begin{bmatrix} 1 & \eta_1 & & & \\ & \ddots & \vdots & & \\ & & \eta_p & & \\ & & \vdots & \ddots & \\ & & \eta_m & & 1 \end{bmatrix}, \quad (6)$$

where  $\eta_p$  is referred to as the *pivotal entry* or simply the *pivot*, and the remaining entries in the  $p^{\text{th}}$  *pivotal* column form the *eta vector*  $\eta$ , for which the  $p^{\text{th}}$  entry is zero.

The solution procedure for the forward system (2) is referred to as FTRAN and, within it, the operation with  $E^{-1}$  is given by

$$E^{-1}x: \quad x_p := x_p/\eta_p \quad \text{and then} \quad x := x - x_p \eta. \quad (7)$$

For the backward system (3), the solution procedure is referred to as BTRAN and the corresponding operation with  $E^{-1}$  is given by

$$x^T E^{-1}: \quad y = x^T \eta \quad \text{and then} \quad x_p := (x_p - y)/\eta_p. \quad (8)$$

When the vector  $x$  is sparse the FTRAN operation (7) is easily skipped when  $x_p$  is zero. However, for BTRAN (8), the corresponding null operation occurs when  $y$  is zero. This is just as likely to occur but cannot be identified so easily. However, it is possible to represent the LU factors as a product of row eta matrices, each of which is of the form

$$R = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & r_1 & \dots & r_m \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \quad (9)$$

Thus the backward system (3) may be solved via a sequence of FTRAN operations (7) using the row-wise representation of the LU factors. This is observed, for example, by Hall and McKinnon [15].

## 2.1 Updating the invertible representation of $B$

Obtaining an invertible representation of the basis matrix following the column replacement (1) by updating the existing invertible representation is essential for efficiency reasons and is an operation referred to as UPDATE. The first and simplest such operation is the product form update of Dantzig and Orchard-Hays [6]. Many others have been proposed, but the update of Forrest and Tomlin [9] and the related Suhl update [23] are widely accepted as being the most efficient.

### *The product form update*

The product form (PF) update rearranges (1) so that

$$\bar{B} = B + (a_q - Be_p)e_p^T = B(I + (\hat{a}_q - e_p)e_p^T) = BE$$

where  $E = I + (\hat{a}_q - e_p)e_p^T$  is an eta matrix whose vector  $\hat{a}_q$  is naturally available as the result of the forward system (4). Applying this  $k$  times yields the following representation of the basis matrix and its inverse.

$$B_k = B_0 E_1 E_2 \dots E_k \quad \Rightarrow \quad B_k^{-1} = E_k^{-1} \dots E_2^{-1} E_1^{-1} B_0^{-1} \quad (10)$$

Thus the original decomposition  $B_0 = L_0 U_0$  remains unaltered when the basis matrix changes.

### *The Forrest-Tomlin update*

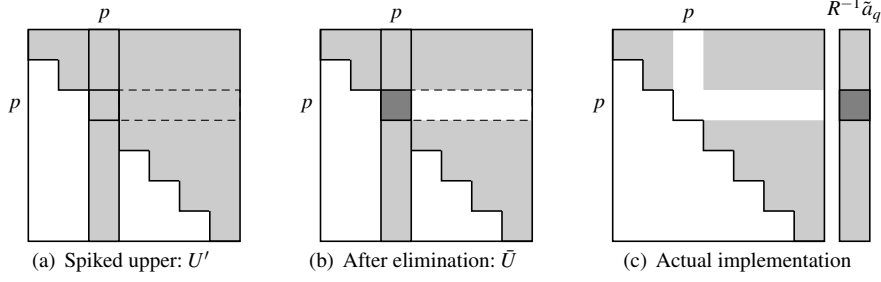
By allowing the decomposition  $B = LU$  to be modified, the Forrest-Tomlin (FT) update [9] generally achieves greater efficiency with respect to sparsity than the PF update. This is done by working on the following rearrangement of the basis matrix update equation.

$$\begin{aligned} \bar{B} &= B + (a_q - Be_p)e_p^T \\ \Rightarrow \quad L^{-1}\bar{B} &= U + (L^{-1}a_q - Ue_p)e_p^T \\ &= U + (\tilde{a}_q - u_p)e_p^T = U' \end{aligned} \quad (11)$$

Whilst the basis update equation (1) replaces column  $p$  of basis matrix  $B$  by  $a_q$  in (11), column  $p$  of the factor  $U$  is replaced by the partial FTRAN result  $\tilde{a}_q = L^{-1}a_q$ . As illustrated in Figure 1(a), the replacement yields a spiked upper factor  $U'$ .

From  $U'$ , the Forrest-Tomlin update restores triangularity by elimination. Specifically, it uses other rows to eliminate the off-diagonal entries of row  $p$ , yielding a permuted triangular matrix  $\bar{U}$  as shown in Figure 1(b).

The elimination process can be represented by a single row transformation  $R^{-1}$ , so that  $\bar{U} = R^{-1}U'$ . Note that this row eta matrix  $R$  is a special case of (9) since it has pivotal entry  $r_p = 1$  so  $R$  and its inverse can be expressed as  $R = I + e_p r^T$  and  $R^{-1} = I - e_p r^T$  respectively. The computation of the eta vector  $r$  was identified by Forrest and Tomlin [9] as an additional partial BTRAN operation  $r^T = \tilde{u}_p^T U^{-1}$ , where



**Fig. 1** Forrest-Tomlin update: column spike and elimination

$\bar{u}_p^T$  is row  $p$  of  $U$  without the diagonal entry  $u_{pp}$ . It can easily be verified that applying  $R^{-1}$  to  $U$  as  $R^{-1}U = U - e_p \bar{u}_p^T$  modifies only the entries in row  $p$ , eliminating all its off-diagonal entries. Since  $U$  and  $U'$  differ only in their  $p^{\text{th}}$  column, applying  $R^{-1}$  to  $U'$  also eliminates its off-diagonal entries in row  $p$ . Meanwhile, applying  $R^{-1}$  to column  $p$  of  $U'$  modifies only the  $p^{\text{th}}$  entry of the newly inserted vector  $\tilde{a}_q$  which becomes  $\tilde{a}_{pq} := \tilde{a}_{pq} - r^T \tilde{a}_q$ . Therefore, applying the row transformation  $R^{-1}$  to  $U'$  yields the permuted triangular matrix  $\bar{U}$  as shown in Figure 1(b).

As identified later by Tomlin [25], the additional partial BTRAN operation can be avoided by forming  $r$  from the intermediate result  $\tilde{e}_p^T = e_p^T U^{-1}$  of the regularly solved transposed system (5). This can be identified by observing that  $\bar{u}_p^T = e_p^T U - u_{pp} e_p^T$  so

$$r^T = \bar{u}_p^T U^{-1} = (e_p^T U - u_{pp} e_p^T) U^{-1} = e_p^T - u_{pp} \tilde{e}_p^T. \quad (12)$$

Since its  $p^{\text{th}}$  entry is zero, it follows that  $r$  is given by scaling  $\tilde{e}_p$  by  $-u_{pp}$  and setting the  $p^{\text{th}}$  entry to zero. Thus, if the partial BTRAN result  $\tilde{e}_p$  is stored, it can be assumed that the eta vector  $r$  is available at negligible cost.

Clearly  $\bar{U} = R^{-1}U'$  is not a triangular matrix, but it can be put into this form via a symmetric cyclic permutation of rows and columns  $p$  to  $m$ . However, in practice, no permutations are performed since all that is required is an invertible representation of  $\bar{U}$  rather than an explicit triangular matrix. Within an implementation, triangular matrices in basis matrix decompositions are represented in product form as sequences of eta vectors, pivotal entries and pivotal indices. Thus the representation of  $\bar{U}$  is obtained by deleting the eta vector corresponding to column  $p$  of  $U$ , setting all entries corresponding to row  $p$  of  $U$  to zero and appending a new eta vector  $R^{-1} \tilde{a}_q$ , pivotal entry  $\tilde{a}_{pq}$  and index  $p$  to the sequence, as illustrated in Figure 1(c).

Combining  $\bar{B} = LU'$  (11) and  $\bar{U} = R^{-1}U'$  yields the following updated representation of the basis matrix and its inverse.

$$\bar{B} = LR\bar{U} \quad \text{and} \quad \bar{B}^{-1} = \bar{U}R^{-1}L^{-1}$$

Repeating these operations, after  $k$  updates, the basis matrix  $B_k$  and its inverse can be expressed as

$$B_k = LR_1 R_2 \dots R_k U_k \quad \Rightarrow \quad B_k^{-1} = U_k^{-1} R_k^{-1} \dots R_2^{-1} R_1^{-1} L^{-1}. \quad (13)$$

The essential difference between the FT and PF updates is that the former stores two partially transformed results  $\tilde{a}_q$  and  $\tilde{e}_p$  and deletes one row and one column from the eta file, whereas the PF update simply stores the final FTRAN result  $\hat{a}_q$ . These deletions and the sparsity of the partially transformed results relative to  $\hat{a}_q$  is such that the FT update frequently has a significantly lower storage requirement than the PF update. This, in turn, leads to generally superior performance when the invertible representation based on the FT update is used to solve linear systems. However, the operations of the FT update require dynamic data structures to accommodate deletion and insertion, making it significantly more difficult to implement than the PF update.

For several reasons, the FT update is used as a benchmark for the experiments in Section 4 and as the basis of the CFT update introduced in Section 3.2. It is preferred to other FT-like updates such as that of Suhl and Suhl [23] since the latter's efficiency is compromised by the requirement for the BTRAN operation which, as observed above, can be avoided with the FT update. The FT update is chosen over the Reid update [20] since the former is preferable when solving hyper-sparse LP problems. The results in Section 4 comparing Huangfu and Hall's FT-based solver `hso1` to `clp` and `cp1ex` give an empirical justification for this choice of update technique.

### 3 New update techniques

This section introduces two novel variants of the product form update and an extension of the Forrest-Tomlin update. Although applicable in general, all are motivated by the requirements of the high performance implementation by Hall and Huangfu [14, 16] of the dual revised simplex method with suboptimization introduced by Rosander [21].

#### 3.1 Variants of the product form update

Although the variants of the product form update set out below are relatively simple, the authors are unaware of them having been described before. This is possibly because, until now, the scope for them to be useful has not arisen.

##### *The elementary matrix for a general rank-one update*

The updates introduced below are based on elementary matrices for rank-one updates which are more general than the eta matrices of LU factors or the PF and FT updates. This general elementary matrix can be expressed as

$$T = I + uv^T, \quad (14)$$

where  $u$  and  $v$  are arbitrary compatible vectors. When  $v = e_p$ ,  $T$  is an eta matrix  $E$  (6) and when  $u = e_p$ ,  $T$  is a row-wise eta matrix  $R$  (9). When  $T$  is nonsingular,

$$T^{-1} = I - \frac{1}{\mu} uv^T, \quad (15)$$

where  $\mu = 1 + v^T u$ . Clearly  $T$  and its inverse can be represented by  $u$  and  $v$ . Although strictly unnecessary, when operating with  $T^{-1}$  it is also convenient to record the value of  $\mu$ . Solving linear systems with  $T$  is straightforward, the operations with  $T^{-1}$  for FTRAN being

$$T^{-1}x = x - \frac{v^T x}{\mu} u, \quad (16)$$

and those for BTRAN

$$x^T T^{-1} = x^T - \frac{x^T u}{\mu} v^T. \quad (17)$$

### 3.1.1 Alternate product form update

Working from the basis update expression (1), if  $B$  is taken out as a factor on the *right* then

$$\bar{B} = (I + (a_q - B e_p) \hat{e}_p^T) B = (I + (a_q - a_{p'}) \hat{e}_p^T) B,$$

where  $p'$  is the index within  $A$  of column  $p$  of  $B$ . The matrix  $T = I + (a_q - a_{p'}) \hat{e}_p^T$  is in the general form (14) so, using (15), the inverse of the updated basis matrix is given by

$$\bar{B}^{-1} = B^{-1} \left( I - \frac{1}{\mu} (a_q - a_{p'}) \hat{e}_p^T \right),$$

where  $\mu = 1 + \hat{e}_p^T (a_q - a_{p'}) = \hat{a}_{pq}$  is the  $p^{\text{th}}$  entry of  $\hat{a}_q$ . In contrast to the PF update, which requires the FTRAN result  $\hat{a}_q$ , the new update formula uses the BTRAN result  $\hat{e}_p$  and thus is called the *alternate* product form (APF) update. It follows from (16) that the operation with  $T^{-1}$  for FTRAN is

$$T^{-1}x: \quad y = \hat{e}_p^T x / \mu \quad \text{and then} \quad x := x - y(a_q - a_{p'})$$

and, from (17), that the corresponding operation for BTRAN is

$$x^T T^{-1}: \quad y = x^T (a_q - a_{p'}) / \mu \quad \text{and then} \quad x^T := x^T - y \hat{e}_p^T.$$

After  $k$  APF updates, the basis matrix and its inverse can be expressed as

$$B_k = T_k T_{k-1} \dots T_1 B_0 \quad \Rightarrow \quad B_k^{-1} = B_0^{-1} T_1^{-1} \dots T_{k-1}^{-1} T_k^{-1}.$$

Note that, in contrast to the PF update, the elementary matrices which constitute the update are applied before rather than after the LU decomposition of  $B_0$  when solving forward systems with  $B_k^{-1}$ .



### Discussion

The relation between the APF and PF updates may be viewed as being analogous to that between the *alternative block LU* update (introduced and implemented by Hall [13]) and the *Block LU* (BLU) update of Eldersveld and Saunders [8]. However, the APF is distinctive since it avoids the requirement for an invertible representation of a Schur complement. This yields a significant overhead when large numbers of updates are performed.

Relative to the PF update, trading operations with  $x_p$  for operations with  $a_q - a_{p'}$  makes the APF update appear unattractive. However, there is minimal additional storage overhead since  $a_q$  and  $a_{p'}$  are columns from the coefficient matrix  $A$  so may be represented by the indices  $q$  and  $p'$ . Hall and McKinnon [15] also identify (classes of) LP problems where it is typical for  $\hat{a}_q$  to be dense but  $\hat{e}_p$  to be sparse, in which case the APF update will require significantly less storage. Performance-wise, unless  $\hat{e}_p$  is significantly more sparse than  $\hat{a}_q$ , the overhead of the operations with  $a_q - a_{p'}$  is such that using the APF rather than the PF update can be expected to be less efficient.

Since the cost of using the APF update corresponds to the density of the final BTRAN result  $\hat{e}_p$ , rather than partial results  $\tilde{a}_q$  and  $\tilde{e}_p$  that underpin the FT update, it is expected that, like the PF update, the storage requirement and performance of the APF update will be inferior to those of the FT update. However, it is shown by Huangfu and Hall in [16] that the APF is particularly valuable in the context of a high performance parallel scheme for the dual simplex method since it permits particular multiple FTRAN operations to be performed as a single FTRAN. The number of elementary APF operations that must be performed is of the same order as the number of FTRAN operations. Thus the saving of all but one of these FTRANs that is achieved by using the APF update is not expected to be compromised by the overhead of maintaining or applying APF updates rather than an alternative scheme.

#### 3.1.2 Middle product form update

The *middle product form* (MPF) update inserts the updates in product form into the middle of factors  $L$  and  $U$ . In detail, the MPF update is derived as follows from the basis update expression (1), assuming that  $B = LU$ .

$$\begin{aligned}\bar{B} &= LU + (a_q - Be_p)e_p^T \\ &= LU + LL^{-1}(a_q - Be_p)e_p^T U^{-1}U \\ &= L(I + (\tilde{a}_q - Ue_p)\tilde{e}_p^T)U \\ &= L(I + (\tilde{a}_q - u_p)\tilde{e}_p^T)U\end{aligned}$$

where  $\tilde{a}_q = L^{-1}a_q$  and  $\tilde{e}_p^T = e_p^T U^{-1}$  are partial FTRAN and BTRAN results respectively, and  $u_p = Ue_p$  is the  $p^{\text{th}}$  column of  $U$ . The matrix  $T = I + (\tilde{a}_q - u_p)\tilde{e}_p^T$  is in the general form (14) so, using (15), the inverse of the updated basis matrix is given by

$$\bar{B}^{-1} = U^{-1}\left(I - \frac{1}{\mu}(\tilde{a}_q - u_p)\tilde{e}_p^T\right)L^{-1},$$

where  $\mu = 1 + \tilde{e}_p^T(\tilde{a}_q - u_p) = \hat{a}_{pq}$  is the  $p^{\text{th}}$  entry of  $\hat{a}_q$ . It follows from (16) that the operation with  $T^{-1}$  for FTRAN is

$$T^{-1}x: \quad y = \tilde{e}_p^T x / \mu \quad \text{and then} \quad x := x - y(\tilde{a}_q - u_p)$$

and, from (17), that the corresponding operation for BTRAN is

$$x^T T^{-1}: \quad y = x^T (\tilde{a}_q - u_p) / \mu \quad \text{and then} \quad x^T := x^T - y \tilde{e}_p^T.$$

Note that  $T$  is inserted after  $L$  so the invertible representation of  $\bar{B}$  is given by  $\bar{L} = LT$  and  $\bar{U} = U$ . Since  $U$  remains unchanged, its  $p^{\text{th}}$  column is always available directly. After  $k$  updates, the basis matrix and its inverse can be expressed as

$$B_k = L_0 T_1 T_2 \dots T_k U_0 \quad \Rightarrow \quad B_k^{-1} = U_0^{-1} T_k^{-1} \dots T_2^{-1} T_1^{-1} L_0^{-1}.$$

### Discussion

The relation between the MPF and PF updates is analogous to that between the *partitioned LU* update (introduced by Gill *et al.* [12] but not implemented) and the BLU update of Eldersveld and Saunders [8]. However, like the APF, the MPF is distinctive since it avoids a Schur complement.

The MPF update is comparable with the FT update since both are based on the partial FTRAN result  $\tilde{a}_q$  and partial BTRAN result  $\tilde{e}_p$ . However the MPF has no deletion corresponding to that of the FT update so will have greater storage overhead. In terms of update efficiency, the MPF update is preferable since it only involves storing operations, whereas the FT update needs additional deletion and insertion work to update the factor  $U$ . In terms of solving efficiency, it is expected to be preferable to operate with the FT update since it replaces  $u_p$  by  $\tilde{a}_q$  and stores only  $\tilde{e}_p$  as an eta matrix  $R$ . Thus solving with the FT update requires only additional simple eta matrix operations with  $R$ , while solving with the MPF requires slightly more complicated and time-consuming operations with  $T$ .

Concerns over the numerical stability of product form representations of  $B^{-1}$  motivated the introduction of LU decompositions into simplex implementations by Bartels and Golub [2] and partial pivoting for stability when updating the LU decomposition [1]. Although some FT-like update procedures derived from the work of Bartels and Golub have incorporated threshold pivoting, modern descriptions of such update procedures and codes based on them [5, 18, 20, 23] include no numerical pivoting. Thus the superior numerical performance of FT-like updates must be assumed to derive from the following two observations. The first is the fact that FT-like updates may delete etas corresponding to ill-conditioned bases, whereas PF-like updates retain all etas. The second follows from the observation (see results in Section 4) that FT-like updates have lower storage requirements, implying fewer floating-point operations and, hence, a lower probability of numerical instability actually occurring. In practice, numerical difficulties with PF-like updates are not encountered when solving most classes of LP problem, particularly those which are hyper-sparse since little numerical elimination is necessary. Any poor numerical behaviour is readily identified and rectified by reinversion so it is very rare for a good PF-based solver to fail to

solve an LP problem. Finally, in the context of Huangfu and Hall's FT-based simplex solver `hso1` [16], PF-like updates are only used for very limited numbers of basis changes and no numerical instability has been observed.

### 3.2 Collective Forrest-Tomlin update

The collective Forrest-Tomlin (CFT) update is designed to perform a set of  $t$  Forrest-Tomlin operations simultaneously to obtain the FT invertible representation of  $B_{k+t}$  directly from that of  $B_k$ . This requirement arises naturally within the dual simplex method with suboptimization.

#### *The dual simplex method with suboptimization*

The details of this simplex variant are given by Huangfu and Hall [16]. However, it is necessary to set out some of its data requirements to motivate the collective Forrest-Tomlin update described below. Suboptimization in the dual simplex method requires the vectors  $\hat{e}_p^T = e_p^T B_k^{-1}$  for  $p$  in a small subset  $\mathcal{P}$  of the rows. Iterations of suboptimization then identify  $t$  basis changes given by  $\{(p_i, q_i)\}_{i=0}^{t-1}$ , where  $\{p_i\}_{i=0}^{t-1} \subseteq \mathcal{P}$ . When, during the course of suboptimization, the vector  $e_{p_i}^T B_{k+i}^{-1}$  is required, it is obtained from  $\hat{e}_{p_i}$  by a few APF operations. Following the suboptimization iterations, updating data for the whole LP problem requires the pivotal columns  $\hat{a}_{q_i} = B_{k+i}^{-1} a_{q_i}$  for  $i = 0, \dots, t-1$ , which are also obtained in two steps as  $\hat{a}_q = B_k^{-1} a_q$  and then regular PF operations.

#### *The Forrest-Tomlin update after suboptimization*

To perform Forrest-Tomlin updates after suboptimization requires

$$\tilde{a}_{q_i} = R_{k+i}^{-1} \dots R_k^{-1} \dots R_1^{-1} L^{-1} a_{q_i} \quad \text{and} \quad \tilde{e}_{p_i}^T = e_{p_i}^T U_{k+i}^{-1}, \quad (18)$$

for  $i = 0, \dots, t-1$ , so that the next elimination matrix  $R_{k+i+1}$  can be constructed using  $\tilde{e}_{p_i}^T$  and the new column transformation can be formed as  $R_{k+i+1}^{-1} \tilde{a}_{q_i}$ . In the sequential simplex method, they are naturally available as partial results of the routinely solved forward (4) and transposed (5) systems. In suboptimization, initialisation of  $\hat{e}_p^T = e_p^T B_k^{-1}$  and the subsequent computation of  $\hat{a}_q = B_k^{-1} a_q$  yield only

$$\bar{a}_{q_i} = R_k^{-1} \dots R_1^{-1} L^{-1} a_{q_i} \quad \text{and} \quad \bar{e}_{p_i}^T = e_{p_i}^T U_k^{-1}. \quad (19)$$

The challenge, therefore, is to obtain the partial results for the appropriate basis (18) from the partial results (19) obtained naturally, for  $i = 1, \dots, t-1$ . The results for  $i = 0$  are known. The rest of this section will discuss how to achieve this efficiently using simple linear algebra operations.

### 3.2.1 Updating the partial FTRAN results

Updating the partial FTRAN results  $\bar{a}_{q_i}$  to  $\tilde{a}_{q_i}$  is a relatively straightforward task. By comparing the available (19) and the required (18) results, the updating operation is readily identified as applying new row eta transformations after  $R_k$ ,

$$\tilde{a}_{q_i} = R_{k+i}^{-1} \cdots R_{k+2}^{-1} R_{k+1}^{-1} \bar{a}_{q_i},$$

where each new row transformation  $R_{k+j}$ , ( $0 < j \leq i$ ) is available once  $\tilde{e}_{p_{j-1}}$  is computed. Thus the computation of  $\tilde{a}_{q_i}$  is scheduled after the computation of  $\tilde{e}_{p_{i-1}}$ , which corresponds to the latest required row transformation  $R_{k+i}$ .

### 3.2.2 Updating the partial BTRAN results

Compared to updating the partial FTRAN results, updating the partial BTRAN results from

$$\bar{e}_{p_i}^T = e_{p_i}^T U_k^{-1} \quad \text{to} \quad \tilde{e}_{p_i}^T = e_{p_i}^T U_{k+i}^{-1}$$

is more complicated because the difference between  $U_k$  and  $U_{k+i}$  when  $i \geq 1$  involves (multiple) replacements and eliminations. However, by carefully rearranging the replacement and elimination involved, computation of  $\tilde{e}_{p_i}$  can still be achieved by simple linear algebra operations. Their derivation is explored in detail by updating  $\bar{e}_{p_1}$  to  $\tilde{e}_{p_1}$ , with reference to the first two upper factors  $U_k$  and  $U_{k+1}$ .

Recall that the Forrest-Tomlin update derives  $U_{k+1}$  via the following elimination operations on the spiked matrix  $U'_k$

$$U_{k+1} = R_{k+1}^{-1} U'_k, \quad (20)$$

where  $U'_k = U_k + (\tilde{a}_{q_0} - U_k e_{p_0}) e_{p_0}^T$ . Rearranging  $U'_k$  by taking out  $U_k$  as a factor on the left, it follows that

$$\begin{aligned} U'_k &= U_k + U_k (U_k^{-1} \tilde{a}_{q_0} - e_{p_0}) e_{p_0}^T \\ &= U_k (I + (\hat{a}_{q_0} - e_{p_0}) e_{p_0}^T) \\ &= U_k E_{k+1}, \end{aligned}$$

where  $E_{k+1} = I + (\hat{a}_{q_0} - e_{p_0}) e_{p_0}^T$  is the eta matrix used in the PF update (10). By substituting  $U'_k = U_k E_{k+1}$  into (20),  $U_{k+1}$  and its inverse can be represented by

$$U_{k+1} = R_{k+1}^{-1} U_k E_{k+1} \quad \text{and} \quad U_{k+1}^{-1} = E_{k+1}^{-1} U_k^{-1} R_{k+1}. \quad (21)$$

By using this new representation for  $U_{k+1}^{-1}$ , the calculation of  $\tilde{e}_{p_1}$  can be considered in three steps applying  $E_{k+1}^{-1}$ ,  $U_k^{-1}$  and  $R_{k+1}$  respectively. Using  $y$  to represent intermediate results, starting from  $y = e_{p_1}$ , the following three-step process is given by (21).

(1) Applying  $E_{k+1}^{-1}$  updates  $y$  by

$$y^T := y^T E_{k+1}^{-1} = e_{p_1}^T E_{k+1}^{-1} = \begin{cases} e_{p_1}^T + \mu e_{p_0}^T & p_1 \neq p_0 \\ \alpha e_{p_1}^T & p_1 = p_0 \end{cases}$$

where  $\alpha = 1/\hat{a}_{p_0q_0}$  and  $\mu = -\hat{a}_{p_1q_0}/\hat{a}_{p_0q_0}$  are the pivotal entry and the  $p_1^{\text{th}}$  entry of the eta vector of  $E_{k+1}^{-1}$ . The first situation  $p_1 \neq p_0$  is the common case, which is implied in the suboptimization framework, as each row in  $\mathcal{S}$  is used only once. In more general applications, it is possible to have  $p_1 = p_0$ . However, since this case corresponds to  $\mu = 0$  and a scaling of  $y$ , it is reasonable and convenient to omit it from the following analysis.

(2) Applying  $U_k^{-1}$  to  $y$  gives

$$y^T := y^T U_k^{-1} = (e_{p_1}^T + \mu e_{p_0}^T) U_k^{-1} = \bar{e}_{p_1}^T + \mu \bar{e}_{p_0}^T.$$

This expresses  $y$  in terms of the available partial BTRAN results  $\bar{e}_{p_0}$  and  $\bar{e}_{p_1}$ , where both are computed with  $U_k^{-1}$ .

(3) Applying  $R_{k+1}$  to  $y$  completes the process, giving the required partial BTRAN result  $\tilde{e}_{p_1}$  thus.

$$\tilde{e}_{p_1}^T = y^T R_{k+1} = \bar{e}_{p_1}^T R_{k+1} + \mu \bar{e}_{p_0}^T R_{k+1} \quad (22)$$

When transposed as  $R_{k+1}^T \bar{e}_{p_1}$ , calculation of the first term in (22) is seen to be a standard FTRAN operation (7) which adds a multiple  $[\bar{e}_{p_1}]_{p_0}$  of  $r_{k+1}$  to  $\bar{e}_{p_1}$  so

$$\bar{e}_{p_1}^T R_{k+1} = \bar{e}_{p_1}^T + [\bar{e}_{p_1}]_{p_0} r_{k+1}^T. \quad (23)$$

As for the second term in (22), by substituting for  $R_{k+1}$  using (9) and  $r_{k+1}$  using (12),

$$\begin{aligned} \mu \bar{e}_{p_0}^T R_{k+1} &= \mu \bar{e}_{p_0}^T (I + e_{p_0} r_{k+1}^T) \\ &= \mu \bar{e}_{p_0}^T + \mu (e_{p_0}^T - u_{p_0 p_0} \bar{e}_{p_0}^T) / u_{p_0 p_0} \\ &= \frac{\mu}{u_{p_0 p_0}} e_{p_0}^T \end{aligned} \quad (24)$$

so the second term in (22) is seen as adding the scalar value  $\mu/u_{p_0 p_0}$  to  $y_{p_0}$ . Substituting (23) and (24) for the two terms in (22) yields the following update formula.

$$\tilde{e}_{p_1}^T = \bar{e}_{p_1}^T + [\bar{e}_{p_1}]_{p_0} r_{k+1}^T + \frac{\mu}{u_{p_0 p_0}} e_{p_0}^T \quad (25)$$

In practice, to obtain  $\mu = -\hat{a}_{p_1q_0}/\hat{a}_{p_0q_0}$  requires only one inner product,  $\hat{a}_{p_1q_0} = \bar{e}_{p_1}^T \tilde{a}_{q_0}$ , since  $\hat{a}_{p_0q_0}$  is available as the simplex pivotal entry. Thus updating the partial BTRAN result from  $\bar{e}_{p_1}$  to  $\tilde{e}_{p_1}$  involves only one vector addition and the evaluation of one inner product.

The partial BTRAN result  $\tilde{e}_{p_2}^T = e_{p_2}^T U_k^{-1}$  corresponding to the third pivot choice, can be updated in two steps, firstly to  $e_{p_2}^T U_{k+1}^{-1}$  by using (25), and then to  $e_{p_2}^T U_{k+2}^{-1}$  by repeating the same operations with partial results  $\tilde{e}_{p_1}$  and  $\tilde{a}_{q_1}$ . Using this stepwise updating procedure for  $i = 1, \dots, t-1$ , the partial BTRAN result  $\tilde{e}_{p_i}$  corresponding to the  $(i+1)^{\text{st}}$  pivot choice can be updated to the required form  $\tilde{e}_{p_i}$  in  $i$  steps of the form (25). Note that to apply (25) requires the previous partial FTRAN results  $\tilde{a}_{p_j}$  for  $j = 0, \dots, i-1$ . Thus the updating of  $\tilde{e}_{p_i}$  is scheduled after the computation of last required partial FTRAN result  $\tilde{a}_{q_{i-1}}$ . Therefore, together with the data requirement of operations with  $\tilde{a}_i$  discussed previously, the calculation for updating the two types of partial transformed results is interlaced as the sequence  $\tilde{a}_{q_1}, \tilde{e}_{p_1}, \tilde{a}_{q_2}, \tilde{e}_{p_2}$ , etc.

### 3.3 New techniques and hyper-sparsity

For hyper-sparse LP problems, it is generally possible to skip most of the operations with elementary matrices in the invertible representation of  $B_k$ , so it is important to identify the operations which must be performed at a cost proportional to the number of resulting floating-point operations. This is readily achieved by established techniques [11, 15] if the elementary matrices are column (6) or row (9) eta matrices. Thus hyper-sparsity may be exploited fully with the PF or FT updates. However, in the case of the APF and MPF updates, the form of the general elementary matrix (14) prohibits the extension of these techniques within the phase of FTRAN and BTRAN associated with UPDATE.

## 4 Results

The serial efficiency of the novel updates introduced in Section 3 is assessed in this section using the set of 30 LP test problems in Table 1.

### *The LP problems and testing environment*

The LP test problems are drawn from the Netlib [10] and Kennington [4] test sets, as well as other publicly-available collections. Although not particularly large, the range of algebraic properties exhibited by these problems is representative of practical and large scale problems. The experiments are performed using modules of Huangfu's dual revised simplex solver `hso1` which uses the Forrest-Tomlin update. As a measure of the computational efficiency of these modules, results in Table 2 give the solution time, iteration count and iteration time of `hso1` and the dual revised simplex solvers of `c1p` 1.15 [5] and `cp1ex` 12.4 [17]. These results are illustrated using the performance profiles in Figures 2 and 3 which show, for each solver, the proportion of problems for which the particular measure relative to the best solver is within the factor on the horizontal axis. It is clear from these results that `hso1` is competitive with `cp1ex` and `c1p`, both overall and per iteration, so the observations on the results presented below for the update techniques developed in this paper are applicable to the most efficient of simplex solvers.

Of particular relevance to the experiments discussed below is the average relative size of the matrix  $B_0$  and its invertible representation, given in the column of Table 1 headed "Fill-in", and the proportion of the results of FTRAN and BTRAN with density less than 10%, given in the corresponding columns of Table 1. Hall and McKinnon [15] introduced this measure of an LP problem's hyper-sparsity and discuss reasons for the extreme variance between these values for some problems. The claim that the LP test set is representative is based on the range of values for these three measures, as well as problem size. Unsurprisingly there is clear correlation between the fill-in during INVERT and the measures of hyper-sparsity.

For a given LP problem and simplex update technique there is an optimal interval between successive INVERT operations which balances the cost of INVERT and the overheads associated with having an updated invertible representation. The particular

| Name         | Rows   | Nz/Col | Fill-in | FTRAN | BTRAN | Interval | PF          | APF          | MPF            | FT      | CFT     |
|--------------|--------|--------|---------|-------|-------|----------|-------------|--------------|----------------|---------|---------|
| CRE-B        | 9649   | 4.53   | 1.01    | 100   | 83    | 175      | 0.44        | 0.86         | <b>0.37</b>    | 0.59    | 0.40    |
| DANO3MIP-LP  | 3203   | 5.74   | 1.42    | 1     | 7     | 101      | 24.99       | 26.77        | <b>10.89</b>   | 7.93    | 7.26    |
| DCP2         | 32388  | 26.53  | 1.02    | 100   | 97    | 437      | 13.63       | 7.49         | <b>6.23</b>    | 2.96    | 3.00    |
| DFL001       | 6072   | 3.42   | 1.29    | 38    | 62    | 153      | 11.59       | 15.91        | <b>5.07</b>    | 4.01    | 3.61    |
| FOME12       | 24285  | 3.42   | 1.29    | 50    | 63    | 312      | 82.62       | 126.04       | <b>40.23</b>   | 28.26   | 25.55   |
| GEN4         | 1538   | 25.64  | 2.22    | 1     | 15    | 123      | 1.13        | <b>0.77</b>  | 1.20           | 0.78    | 0.78    |
| KEN-18       | 105128 | 3.31   | 1.00    | 100   | 100   | 942      | 9.61        | 9.95         | <b>4.03</b>    | 2.36    | 2.34    |
| L30          | 2702   | 4.21   | 2.06    | 10    | 8     | 106      | 6.95        | 6.60         | <b>2.69</b>    | 2.10    | 2.00    |
| LINE_520C    | 93326  | 8.21   | 2.04    | 10    | 11    | 548      | 17367.06    | 12358.58     | <b>3530.39</b> | 1333.97 | 1278.56 |
| LP22         | 2959   | 5.88   | 1.82    | 16    | 25    | 132      | 12.47       | 12.51        | <b>6.89</b>    | 4.73    | 4.40    |
| MAROS-r7     | 3137   | 16.06  | 2.00    | 5     | 13    | 136      | 7.37        | <b>5.35</b>  | 5.69           | 3.58    | 3.52    |
| MOD2         | 35665  | 6.94   | 1.03    | 46    | 68    | 274      | 70.84       | 58.26        | <b>19.03</b>   | 15.99   | 13.75   |
| NS1688926    | 32768  | 103.22 | 1.00    | 72    | 100   | 346      | 44.74       | <b>8.62</b>  | 9.28           | 8.94    | 8.90    |
| OSA-60       | 10281  | 7.00   | 1.00    | 100   | 99    | 247      | 0.05        | 0.09         | <b>0.04</b>    | 0.03    | 0.03    |
| PDS-20       | 33875  | 2.88   | 1.01    | 100   | 98    | 452      | 2.93        | 5.56         | <b>1.80</b>    | 1.35    | 1.20    |
| PDS-40       | 66845  | 2.85   | 1.01    | 100   | 98    | 641      | 16.88       | 33.82        | <b>8.42</b>    | 5.90    | 5.46    |
| PDS-100      | 156244 | 2.75   | 1.00    | 100   | 99    | 988      | 47.50       | 156.00       | <b>26.39</b>   | 15.36   | 14.67   |
| PLOT87       | 2031   | 15.11  | 2.06    | 10    | 19    | 144      | 4.12        | 4.04         | <b>2.95</b>    | 1.79    | 1.74    |
| QAP12        | 3193   | 5.00   | 4.13    | 2     | 16    | 214      | 151.22      | 167.04       | <b>74.47</b>   | 41.82   | 39.94   |
| SELF         | 960    | 156.01 | 1.68    | 0     | 2     | 131      | <b>5.02</b> | 6.00         | 5.09           | 4.14    | 4.15    |
| SGPF5Y6      | 246077 | 2.68   | 1.00    | 100   | 100   | 1444     | 24.87       | 20.39        | <b>11.49</b>   | 5.17    | 6.00    |
| STAT96V1     | 5995   | 2.98   | 1.58    | 24    | 4     | 119      | 22.12       | 23.41        | <b>5.36</b>    | 4.60    | 4.19    |
| STAT96V4     | 3173   | 7.88   | 1.25    | 74    | 31    | 189      | 15.39       | 28.97        | <b>9.03</b>    | 6.90    | 6.41    |
| STORMG2-125  | 66186  | 3.36   | 1.00    | 100   | 100   | 730      | 4.98        | 2.06         | <b>1.76</b>    | 0.93    | 1.10    |
| STORMG2-1000 | 528186 | 3.36   | 1.00    | 100   | 100   | 2020     | 386.03      | <b>55.56</b> | 60.43          | 29.78   | 30.89   |
| STR3D        | 159488 | 3.23   | 1.04    | 97    | 71    | 487      | 461.42      | 906.36       | <b>128.10</b>  | 124.00  | 103.14  |
| TRUSS        | 1001   | 4.16   | 1.37    | 41    | 2     | 68       | 1.20        | 2.12         | <b>0.92</b>    | 0.73    | 0.63    |
| WATSON_1     | 201156 | 2.74   | 1.00    | 100   | 100   | 1531     | 31.08       | 16.00        | <b>11.17</b>   | 4.08    | 4.75    |
| WATSON_2     | 352013 | 2.74   | 1.00    | 100   | 100   | 2132     | 45.93       | 16.70        | <b>15.89</b>   | 5.20    | 6.49    |
| WORLD        | 35511  | 6.74   | 1.04    | 41    | 61    | 256      | 81.94       | 73.98        | <b>22.50</b>   | 19.30   | 16.33   |

**Table 1** LP test problems, fill-in factor for the representation of  $B_0^{-1}$ , sparsity of FTRAN and BTRAN results, reinversion interval and CPU time performing FTRAN and BTRAN and UPDATE operations for different simplex update techniques

| Problem      | hso1          |                 |                | clp           |                 |                | cplex         |                 |                |
|--------------|---------------|-----------------|----------------|---------------|-----------------|----------------|---------------|-----------------|----------------|
|              | Solution Time | Iteration Count | Iteration Time | Solution Time | Iteration Count | Iteration Time | Solution Time | Iteration Count | Iteration Time |
| CRE-B        | 6.4           | 11599           | 551            | 4.2           | 11489           | 368            | 2.1           | 16030           | 130            |
| DANO3MIP_LP  | 53.2          | 60161           | 885            | 31.8          | 42506           | 747            | 8.8           | 23608           | 372            |
| DCP2         | 12.9          | 25360           | 508            | 15.3          | 24456           | 626            | 5.6           | 23432           | 237            |
| DFL001       | 16.0          | 26322           | 607            | 17.9          | 26866           | 667            | 9.8           | 20255           | 483            |
| FOME12       | 101.1         | 103005          | 982            | 66.9          | 84818           | 789            | 78.6          | 90732           | 867            |
| GEN4         | 5.3           | 1107            | 4757           | 16.7          | 4157            | 4025           | 16.5          | 5276            | 3125           |
| KEN-18       | 13.6          | 107471          | 127            | 19.3          | 106158          | 182            | 8.5           | 96533           | 88             |
| L30          | 11.2          | 10290           | 1091           | 11.5          | 10233           | 1124           | 8.1           | 10857           | 742            |
| LINF_520C    | 3646.8        | 132244          | 27576          | 3349.1        | 140119          | 23902          | 8242.8        | 117755          | 70000          |
| LP22         | 22.2          | 25080           | 883            | 21.0          | 22963           | 913            | 14.9          | 24308           | 612            |
| MAROS-R7     | 11.5          | 6025            | 1914           | 12.0          | 5531            | 2172           | 3.9           | 6585            | 596            |
| MOD2         | 53.2          | 43386           | 1226           | 31.3          | 35308           | 886            | 26.2          | 47995           | 547            |
| NS1688926    | 25.0          | 13849           | 1804           | 4424.7        | 196353          | 22535          | 52.8          | 23819           | 2217           |
| OSA-60       | 6.5           | 4694            | 1394           | 2.1           | 4792            | 434            | 0.8           | 5014            | 158            |
| PDS-20       | 6.9           | 38872           | 177            | 8.6           | 40183           | 213            | 4.5           | 22825           | 199            |
| PDS-40       | 28.8          | 94914           | 304            | 36.5          | 104217          | 350            | 15.6          | 56126           | 278            |
| PDS-100      | 87.4          | 234184          | 373            | 194.7         | 319231          | 610            | 56.6          | 163076          | 347            |
| PILOT87      | 7.0           | 7240            | 970            | 7.2           | 7179            | 1009           | 4.7           | 7285            | 645            |
| QAP12        | 158.3         | 128131          | 1235           | 146.7         | 89194           | 1645           | 139.9         | 140184          | 998            |
| SELF         | 40.2          | 4738            | 8483           | 30.2          | 4713            | 6406           | 54.4          | 15281           | 3559           |
| SGPF5Y6      | 180.2         | 348115          | 518            | 242.1         | 360809          | 671            | 7.9           | 97918           | 81             |
| STAT96V1     | 97.6          | 16904           | 5777           | 82.5          | 17984           | 4588           | 51.9          | 14226           | 3646           |
| STAT96V4     | 157.5         | 72531           | 2172           | 52.3          | 34657           | 1510           | 79.0          | 43124           | 1832           |
| STORMG2-125  | 9.5           | 81869           | 116            | 23.5          | 89988           | 262            | 4.4           | 92755           | 47             |
| STORMG2-1000 | 530.7         | 658534          | 806            | 1876.8        | 730471          | 2569           | 228.2         | 873329          | 261            |
| STP3D        | 602.5         | 130689          | 4610           | 350.0         | 115329          | 3035           | 328.7         | 114533          | 2870           |
| TRUSS        | 8.2           | 18929           | 433            | 5.5           | 18817           | 290            | 4.2           | 19693           | 213            |
| WATSON_1     | 50.4          | 238973          | 211            | 65.8          | 223522          | 294            | 9.6           | 87148           | 110            |
| WATSON_2     | 55.1          | 334733          | 165            | 12976.9       | 498860          | 26013          | 35.3          | 358621          | 99             |
| WORLD        | 66.1          | 47104           | 1402           | 41.0          | 41279           | 993            | 22.2          | 50538           | 438            |

**Table 2** Solution time (s), iteration count and iteration time ( $\mu$ s) for the hso1, cplex and clp dual revised simplex solvers

optimal interval can be expected to vary as the simplex method solves a given LP. In theory this interval should be chosen so that the time associated with INVERT equals the total time forming and operating with the etas associated with UPDATE. To ensure deterministic performance, in hso1 the interval is determined using a flop-based pseudo-clock and practical experience shows that performance is not sensitive to the parameters in the model. The sequence of optimal reinversion intervals was estimated for the FT update, and the average value is given in the column headed "Interval".

Experiments with the PF, APF, MPF, FT and CFT updates were performed using the same sequence of basis changes from a "logical" basis  $B = I$  to an optimal solution of the LP. This eliminates variance in the results due to the fact that a change in update technique typically leads to the simplex method taking a different path to an optimal solution. The same sequence of optimal (FT) reinversion intervals was also used for all experiments with a given problem. This facilitates some comparisons between methods and ensures that the FT update is not disfavoured. Results for each update



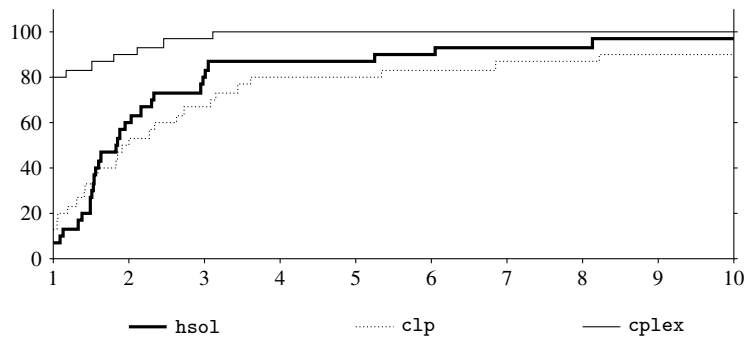


Fig. 2 Performance profiles for the solution time of hsol, clp and cplex on the test problems

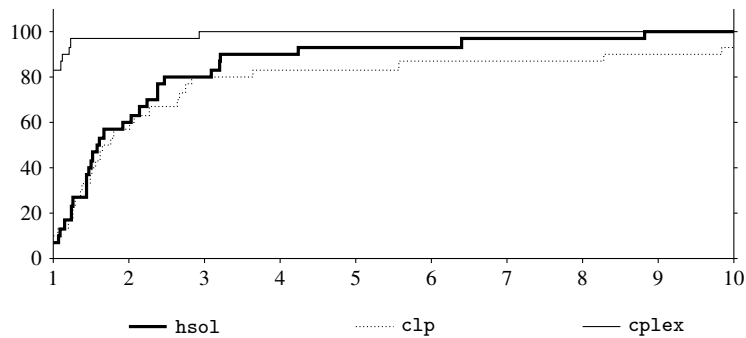


Fig. 3 Performance profiles for the iteration time of hsol, clp and cplex on the test problems

technique's particular sequence of optimal reinversion intervals were obtained but did not lead to deeper insight or any alternative conclusions so are omitted.

### Results and analysis

The principal measure of the efficiency of a particular update procedure is the total CPU time to perform the update and then perform FTRAN and BTRAN for each basis in the sequence. This is given in the columns headed PF, APF, MPF, FT and CFT in Table 1 where, for each LP problem, the best performance of the three product form updates is highlighted in bold.

The superiority of the MPF update over the PF and APF updates is clear: it is the best for 26 of the 30 problems. Of the other four problems, the APF update is the best for three and in all of these cases the proportion of sparse BTRAN results is at least that of FTRAN. However, on 16 of the 30 problems the APF update is the worst, with the PF update being the worst for all but one of the others. A further measure of the superiority of the MPF update is obtained by considering the CPU time relative to the better of the CPU time for the PF and APF updates. The geometric mean of this ratio shows the use of the MPF update to be 40% more efficient.

The FT update is better than the best of the product form updates for 27 of the problems and, on average, is 61% more efficient than the better of the PF and APF

updates. However, the FT update is only 34% more efficient than the MPF update. Thus the efficiency of the MPF update is rather closer to that of the FT update than it is to the other product form updates.

The performance when using the collective FT update (CFT) is very similar to that when using the standard FT update, the former being 4% faster. This demonstrates that the CFT update incurs no significant overhead.

A secondary measure of efficiency is the storage requirement so, for each LP problem and each update technique, the relative size of  $B_k$  and its invertible representation was calculated. The relative performances of the update procedures was similar to the CPU results, so details are not given. Using the MPF update was 31% more efficient than the better of the PF and APF updates. Using the FT update was 16% more efficient than the MPF update and 42% more efficient than the better of the PF and APF updates. There was no significant difference between the average fill-in for the FT and CFT updates.

Although the deletion and insertion operations of the FT update result in a CPU overhead that is not shared with the product form updates, the overall computational efficiency when using the FT update rather than the MPF or the best of the product form updates is greater than the storage efficiency. This is due to the fact that hyper-sparsity may be exploited fully when using the FT update, but only when operating with  $B_0^{-1}$  in the case of the APF and MPF updates.

Further insight into why the MPF update is significantly more efficient than the PF and APF updates is given in the results presented in Table 3. Although it is unsurprising that the average density of the vectors  $\tilde{a}_q$  and  $\tilde{e}_p$  is lower than the average density of  $\hat{a}_q$  and  $\hat{e}_p$  for the PF and APF updates respectively, the relative densities are generally so much lower that it is much more efficient to store and apply  $\tilde{a}_q$  and  $\tilde{e}_p$  rather than just one of  $\hat{a}_q$  and  $\hat{e}_p$ . An intuitive explanation follows from the observation that  $\tilde{a}_q$  and  $\tilde{e}_p$  are the solution of triangular rather than square systems of equations. Specifically, the initial fill-in which occurs when forming  $\tilde{a}_q = L^{-1}a_q$  and  $\tilde{e}_p^T = e_p^T U^{-1}$  is likely to imply the need to apply etas in the representation of  $U^{-1}$  and  $L^{-1}$  when forming  $\hat{a}_q = U^{-1}\tilde{a}_q$  and  $\hat{e}_p^T = \tilde{e}_p^T L^{-1}$ , resulting in significant further fill-in. However, the initial fill-in is very much less likely to require the application of further etas in the representation of  $L^{-1}$  and  $U^{-1}$  when forming  $\tilde{a}_q = L^{-1}a_q$  and  $\tilde{e}_p^T = e_p^T U^{-1}$  since much of the fill-in occurs in components for which any corresponding eta occurs earlier in the representation so need not be applied. Table 3 also illustrates the greater efficiency of the FT update with respect to sparsity.

## 5 Conclusions

Three novel update procedures have been introduced in this paper. Two of them are variants of the product form and one is an extension of the Forrest-Tomlin (FT) update.

Of the two product form variants, the middle product form (MPF) update is generally much more efficient than the alternate product form (APF) update. The APF is very inefficient for some problems and is not recommended as a valuable general technique. However, its limited use in a variant of the dual revised simplex method

| Model        | Average density |             |             |             |             |             |
|--------------|-----------------|-------------|-------------|-------------|-------------|-------------|
|              | PF              | APF         | MPF         |             | FT          |             |
|              | $\hat{a}_q$     | $\hat{e}_p$ | $\hat{a}_q$ | $\hat{e}_p$ | $\hat{a}_q$ | $\hat{e}_p$ |
| CRE-B        | 0.313           | 4.237       | 0.137       | 0.022       | 0.229       | 0.038       |
| DANO3MIP_LP  | 47.206          | 56.684      | 10.373      | 0.721       | 2.524       | 1.332       |
| DCP2         | 3.194           | 0.877       | 1.223       | 0.026       | 0.141       | 0.062       |
| DFL001       | 26.926          | 31.374      | 4.888       | 0.213       | 0.667       | 0.441       |
| FOME12       | 6.893           | 8.503       | 1.776       | 0.056       | 0.197       | 0.151       |
| GEN4         | 97.135          | 31.433      | 86.840      | 16.533      | 26.823      | 20.709      |
| KEN-18       | 0.030           | 0.104       | 0.010       | 0.002       | 0.009       | 0.002       |
| L30          | 88.009          | 75.486      | 17.018      | 3.224       | 3.876       | 4.506       |
| LINF_520C    | 67.949          | 26.318      | 7.101       | 0.575       | 0.645       | 0.789       |
| LP22         | 62.458          | 42.579      | 19.856      | 2.679       | 2.952       | 3.582       |
| MAROS-R7     | 79.525          | 36.413      | 45.516      | 10.664      | 9.487       | 11.569      |
| MOD2         | 9.101           | 7.441       | 0.802       | 0.030       | 0.125       | 0.088       |
| NS1688926    | 15.797          | 0.329       | 1.127       | 0.059       | 0.249       | 0.140       |
| OSA-60       | 0.123           | 0.489       | 0.112       | 0.019       | 0.086       | 0.036       |
| PDS-20       | 0.239           | 0.947       | 0.103       | 0.008       | 0.051       | 0.013       |
| PDS-40       | 0.232           | 0.825       | 0.085       | 0.005       | 0.034       | 0.008       |
| PDS-100      | 0.071           | 0.387       | 0.036       | 0.002       | 0.016       | 0.003       |
| PILOT87      | 66.639          | 47.045      | 32.635      | 5.465       | 8.173       | 5.949       |
| QAP12        | 88.641          | 73.825      | 29.858      | 3.930       | 4.271       | 4.650       |
| SELF         | 98.875          | 85.018      | 96.353      | 26.404      | 49.351      | 28.758      |
| SGPF5Y6      | 0.004           | 0.012       | 0.003       | 0.001       | 0.002       | 0.001       |
| STAT96V1     | 67.349          | 74.287      | 5.261       | 0.754       | 1.218       | 1.219       |
| STAT96V4     | 20.358          | 40.531      | 4.483       | 0.912       | 0.980       | 0.870       |
| STORMG2-125  | 0.034           | 0.021       | 0.012       | 0.004       | 0.007       | 0.005       |
| STORMG2-1000 | 0.028           | 0.003       | 0.003       | 0.001       | 0.001       | 0.001       |
| STP3D        | 2.235           | 4.943       | 0.134       | 0.006       | 0.021       | 0.015       |
| TRUSS        | 24.768          | 77.844      | 10.812      | 1.132       | 3.661       | 1.966       |
| WATSON_1     | 0.021           | 0.016       | 0.006       | 0.001       | 0.004       | 0.001       |
| WATSON_2     | 0.007           | 0.002       | 0.002       | 0.000       | 0.001       | 0.000       |
| WORLD        | 9.945           | 8.938       | 0.849       | 0.032       | 0.140       | 0.097       |

**Table 3** Average density of eta vectors for the PF, MPF and APF updates

that is amenable to parallelisation yields a valuable computational saving. The performance of the MPF is generally very much better than that of the APF and original product form update. Indeed, its performance frequently approaches that of the FT update. Since the MPF update does not require any elimination operations or dynamic data structures it is very much easier to implement than the FT update so is an attractive update procedure when developing a simple, relatively efficient implementation of the revised simplex method for either research studies or in an industrial context where an independent code-base is necessary.

The collective Forrest-Tomlin (CFT) update organises the calculations required to perform multiple FT updates with the same efficiency as the corresponding sequence of standard FT updates. This ensures that there is no loss of serial efficiency when the CFT is used in the context of the dual revised simplex method with suboptimization.

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