Jacobson radical algebras with quadratic growth

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Abstract

In this paper, it is shown that over every countable algebraically closed field $K$ there exists a finitely generated $K$-algebra that is Jacobson radical, infinite dimensional, generated by two elements, graded, and has quadratic growth. We also propose a way of constructing examples of algebras with quadratic growth that satisfy special types of relations.

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Introduction

Algebras with linear growth were described by Small, Stafford and Warfield in [6]. In [3] (pp. 18) Bergman proved that algebras with growth function smaller than $f(n) = \frac{n(n+1)}{2}$ have linear growth. What properties would algebras with a growth function close to $f(n) = \frac{n(n+1)}{2}$ satisfy? Examples of primitive algebras with very small growth functions were constructed by

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Usi Vishne using Moore trajectories [9]. In [1] Bartholdi constructed self-similar algebras with very small growth functions over the field $F_2$ which are graded nil. In fact, all algebras constructed in [1] are primitive and hence not Jacobson radical (as mentioned in [8]).

We will construct an example with growth function bounded above by $n^2 + 4n + 3$ which are both infinite dimensional and Jacobson radical. It is unclear whether this algebra is nil. We will also present a way to construct other examples which are bounded above by the same growth function.

Recall that non-nil Jacobson radical algebras with Gelfand-Kirillov dimension two were constructed in [8], and nil algebras with Gelfand-Kirillov dimension not exceeding three were constructed in [5]. It is not known if there are nil algebras with quadratic growth, or more generally with Gelfand-Kirillov dimension two.

Our first main result is the following:

**Theorem 0.1.** Let $\mathbb{K}$ be an algebraically closed field. Let $A = \mathbb{K}\langle x, y \rangle$ to be the free noncommutative algebra generated (in degree one) by the elements $x, y$. Let $H(n) \subset A$ be the homogeneous subspace of degree $n \geq 0$. Finally, for any $F \subseteq H(n)$, let:

$$\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn+j)FA.$$

For any sequence $\{N_i\}_{i \in \mathbb{N}}$ of strictly increasing natural numbers and any sequence $\{F_i\}_{i \in \mathbb{N}}$ of homogeneous subspaces such that $F_i \subseteq H(2^{N_i})$ and $\dim F_i < \frac{1}{2}(N_i - N_{i-1} + 1)$, the quotient algebra $A/\langle \mathcal{E}(F_i) \rangle_{i \in \mathbb{N}}$ can be homomorphically mapped onto an infinite dimensional graded algebra with quadratic or linear growth. Moreover, the dimension of this algebra’s homogeneous subspace of dimension $n$ would be bounded above by $2n + 2$.

In other words, there’s a graded ideal $E \lhd A$ such that $\bigcup_{i \in \mathbb{N}} \mathcal{E}(F_i) \subseteq E$ and $A/E$ is infinite dimensional and has quadratic growth. Specifically, $1 \leq H(n)/(E \cap H(n)) \leq 2n + 2$ for each $n \geq 1$. As a corollary we get the following result.
Corollary 0.2. Over every countable, algebraically closed field $\mathbb{K}$ there exists a finitely generated $\mathbb{K}$-algebra that’s Jacobson radical, infinite dimensional, generated by two elements, graded and has quadratic growth.

We also propose a new way of constructing examples of algebras with quadratic growth satisfying special types of relations.

The general path of the proof is as follows:

- Subspaces $U(2^n), V(2^n) \subseteq H(2^n)$ are constructed, depending on $U(2^i), V(2^i)$ for $i < n$. This part bears resemblance to results from [4]. Properties that the $V(2^n)$ spaces exhibit include $V(2^{n-1})^2 \subseteq V(2^n)$ and $\dim V(2^n) = 2$, the latter being instrumental in establishing quadratic growth. We assure that sets $\{F_i\}_{i \in \mathbb{N}}$ are contained in our sets $U(2^n)$.

- In section 3 we introduce ideal $E$, whose construction uses the sets $U(2^n)$, in order to arrive at our desired quotient $A/E$. Note that the ideal $E$ is defined differently than in [4]. We then find an upper bound of the growth of $A/E$.

- In sections 4 and 5 we show that for some appropriate choice of sets $\{F_i\}$, the constructed algebra $A/E$ is Jacobson radical.

We wrap up the proof of Theorem and its corollary in section 5.

1 Notation

In what follows, $\mathbb{K}$ is a countable field and $A = \mathbb{K}(x, y)$ is the free $\mathbb{K}$-algebra in two non-commuting indeterminates $x$ and $y$. The monomials in this algebra will be the products of the form $x_1 \cdots x_n$, with each $x_i \in \{x, y\}$ (whereas the monomials with coefficient will be of the form $kx_1 \cdots x_n$ with $k \in \mathbb{K}$). The degree of a monomial is the length of this product. For any $n \geq 0$, $H(n)$ will denote the homogeneous subspace of degree $n$: the $\mathbb{K}$-space generated by the degree-$n$ monomials. Finally, $\bar{A} = \sum_{n=1}^{\infty} H(n)$ will be the $\mathbb{K}$-space of polynomials with no constant term.
2 Constructing sets $U(2^n)$ and $V(2^n)$

Suppose we have a strictly increasing sequence of naturals $\{N_i\}_{i=0}^{\infty}$ with $N_0 = 1$ and a sequence of homogeneous subspaces $\{F_i\}_{i=0}^{\infty}$ with each $F_i \subseteq 2^{N_i}$ and $F_0 = (0)$.

In this section, we address the question: does there exist, for every $i \geq 0$, a subspace $U_i \subset H(2^i)$ and two monomials (with non-zero coefficient) $v_{i,1}, v_{i,2} \in H(2^i)$ such that, for each $i \geq 0$:

1. $U_i \oplus \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2} = H(2^i)$.

2. There exists a $v \in \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2}$ such that $U_{i+1} = H(2^i)U_i + U_i H(2^i) + vH(2^i)$.

3. $F_i \subseteq U_{N_i}$.

We will eventually set $V_i = \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2}$, so that $U_i \oplus V_i = H(2^i)$.

We shall attack the problem with induction. For the base case, set $U_0$ as an arbitrary subspace of $H(1)$ with $\dim U_0 = \dim H(1) - 2$, and set $v_{0,1}$, $v_{0,2}$ as two linearly independent monomials such that $U_0 + \mathbb{K}v_{0,1} + \mathbb{K}v_{0,2} = H(1)$.

For the inductive step, assume the existence of $U_{N_i}, v_{N_i,1}, v_{N_i,2}$ for some $i \geq 0$, and find possible $U_k, v_{k,1}, v_{k,2}$ for all $N_i < k \leq N_{i+1}$.

Let $W \cong \mathbb{K}^{2(N_i+1-N_i)}$ be a $\mathbb{K}$-space with indices $\{x_{k,1}, x_{k,2}\}_{k=N_i}^{N_{i+1}-1}$, let $W_k$ be the subspace of all elements where $(x_{k,1}, x_{k,2}) = (0,0)$, and let $\overline{W} = W \setminus \bigcup_{k=N_i}^{N_{i+1}-1} W_k$.

Given some vector $\bar{w} \in \overline{W}$, define $U_k(\bar{w}), v_{k,1}(\bar{w}), v_{k,2}(\bar{w})$ recursively for each $N_i \leq k \leq N_{i+1}$, as follows: first, set $U_{N_i}(\bar{w}) = U_{N_i}$, $v_{N_i,1}(\bar{w}) = v_{N_i,1}$, $v_{N_i,2}(\bar{w}) = v_{N_i,2}$.

Then, assuming $U_k(\bar{w}), v_{k,1}(\bar{w}), v_{k,2}(\bar{w})$ are defined for some $N_i \leq k < N_{i+1}$:

$$U_{k+1}(\bar{w}) = H(2^k)U_k(\bar{w}) + U_k(\bar{w})H(2^k) + (x_{k,2}(\bar{w})v_{k,1}(\bar{w}) - x_{k,1}(\bar{w})v_{k,2}(\bar{w}))H(2^k).$$

If $x_{k,1}(\bar{w}) \neq 0$, set:

$$v_{k+1,1}(\bar{w}) = x_{k,1}(\bar{w})^{-1}v_{k,1}^2(\bar{w}),$$
Theorem 2.2. For any sequence \( \bar{w} \)

\[ v_{k+1,2}(\bar{w}) = x_{k,1}(\bar{w})^{-1}v_{k,1}(\bar{w})v_{k,2}(\bar{w}), \]
and if \( x_{k,1}(\bar{w}) = 0 \), then \( x_{k,2}(\bar{w}) \neq 0 \), so set:

\[ v_{k+1,1}(\bar{w}) = x_{k,2}(\bar{w})^{-1}v_{k,2}(\bar{w})v_{k,1}(\bar{w}), \]

\[ v_{k+1}(\bar{w}) = x_{k,2}(\bar{w})^{-1}v_{k,2}^2(\bar{w}). \]

For any \( \bar{w} \in \mathcal{W} \), this clearly satisfies conditions (1-2).

Lemma 2.1. For any \( N_i \leq k < N_{i+1} \), \( a, b \in \{1, 2\} \), \( \bar{w} \in \mathcal{W} \),

\[ v_{k,a}(\bar{w})v_{k,b}(\bar{w}) \in x_{k,a}(\bar{w})v_{k+1,b}(\bar{w}) + U_{k+1}(\bar{w}) \]

Proof. If \( x_{k,1}(\bar{w}) \neq 0 \), and \( a = 1 \), \( v_{k,a}(\bar{w})v_{k,b}(\bar{w}) = x_{k,a}(\bar{w})v_{k+1,b}(\bar{w}) \).

If \( x_{k,1}(\bar{w}) \neq 0 \), and \( a = 2 \),

\[ v_{k,a}(\bar{w})v_{k,b}(\bar{w}) = x_{k,a}(\bar{w})v_{k+1,b}(\bar{w}) + x_{k,1}(\bar{w})^{-1}(x_{k,2}(\bar{w})v_{k,1}(\bar{w}) - x_{k,1}(\bar{w})v_{k,2}(\bar{w}))v_{k,b}(\bar{w}). \]

If \( x_{k,1}(\bar{w}) = 0 \) and \( a = 1 \),

\[ v_{k,a}(\bar{w})v_{k,b}(\bar{w}) = x_{k,2}(\bar{w})^{-1}(x_{k,2}(\bar{w})v_{k,1}(\bar{w}) - x_{k,1}(\bar{w})v_{k,2}(\bar{w}))v_{k,b}(\bar{w}). \]

And if \( x_{k,1}(\bar{w}) = 0 \) and \( a = 2 \), \( v_{k,a}(\bar{w})v_{k,b}(\bar{w}) = x_{k,2}(\bar{w})v_{k+1,b}(\bar{w}). \) \( \square \)

Let \( P = \mathbb{K}[x_{k,1}, x_{k,2}]_{k=N_i}^{N_{i+1} - 1} \), i.e. the (commutative) algebra of polynomial functions \( W \to \mathbb{K} \). Let \( Q = \prod_{k=N_i}^{N_{i+1} - 1} (\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{2^k - 1} \) be a homogeneous subspace of \( P \).

Theorem 2.2. For any sequence \( \{s_k\}_{k=1}^{2^{N_{i+1} - N_i}} \) of \( \{1, 2\} \), there exists some \( p_s \in Q \) such that for any \( \bar{w} \in \mathcal{W} \),

\[ \prod_{k=1}^{2^{N_{i+1} - N_i}} v_{N_i,s_k} \bar{w} = p_s(\bar{w})v_{N_{i+1}, s_2^{N_{i+1} - N_i}}(\bar{w}) + U_{N_{i+1}}(\bar{w}). \]

Proof. We will use induction to show that, for any \( 0 \leq h \leq N_{i+1} - N_i \) and any sequence \( \{s_k\}_{k=1}^{h} \) of \( \{1, 2\} \),
Therefore, the inductive statement is true for \( h = N_{i+1} - N_i \).

The base case is simply \( v_{N_i,s_1} \in v_{N_i,s_1}(\vec{w}) + U_{N_i}(\vec{w}) \).

For the inductive step, let \( \{s_k\}_{k=1}^{2h+1} \) be a sequence of \( \{1, 2\} \), and assume the inductive statement is true for \( \{s_k\}_{k=1}^{2h} \) and \( \{s_k\}_{k=2h+1}^{2h+1} \). Lemma 2.1 shows that:

\[
v_{N_i+h, s_{2h}}(\vec{w})v_{N_i+h, s_{2h+1}}(\vec{w}) \in x_{N_i+h, s_{2h}}(\vec{w})v_{N_i+h+1, s_{2h+1}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).
\]

Therefore,

\[
\prod_{k=1}^{2h+1} v_{N_i,s_k} \in \left( \prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j}-1} x_{N_i+j, s_{2j}(2k-1)}(\vec{w}) \right) v_{N_i+h, s_{2h}}(\vec{w}) + U_{N_i+h}(\vec{w}) \cdot \\
\left( \prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j}-1} x_{N_i+j, s_{2j}(2k-1)+2h}(\vec{w}) \right) v_{N_i+h, s_{2h+1}}(\vec{w}) + U_{N_i+h}(\vec{w}) \subseteq \\
\left( \prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j}} x_{N_i+j, s_{2j}(2k-1)}(\vec{w}) \right) x_{N_i+h, s_{2h}}(\vec{w})v_{N_i+h+1, s_{2h+1}}(\vec{w}) + U_{N_i+h+1}(\vec{w}) = \\
\left( \prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j}} x_{N_i+j, s_{2j}(2k-1)}(\vec{w}) \right) v_{N_i+h+1, s_{2h+1}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).
\]

\[\square\]

**Corollary 2.3.** For any \( f \in H(2^{N_{i+1}}) \), there exists \( p, q \in Q \) such that \( \forall \vec{w} \in \overline{W}, f \in p(\vec{w})v_{N_i+1,1}(\vec{w}) + q(\vec{w})v_{N_i+1,2}(\vec{w}) + U_{N_i+1}(\vec{w}) \).

**Proof.** First, note that:

\[
H(2^{N_{i+1}}) = (U_{N_i} + \mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_i+1} - N_i} =
\]
(\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2N_i+1-N_i} + \sum_{k=1}^{2N_i+1-N_i} H((k-1)2^{N_i})U_{N_i}H(2^{N_i+1} - k2^{N_i})

And for each \( f \in H(2^{N_i+1}) \), there exists a \( f' \in (\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2N_i+1-N_i} \) such that, for any \( \bar{w} \in \overline{W} \), \( f \in f' + U_{N_i+1}(\bar{w}) \).

Since \( f' \) can be written as a linear combination of the elements of the form \( \prod_{k=1}^{2N_i+1} v_{N_i,s_k} \), it’s sufficient to prove the corollary over these elements, which is done in theorem 2.2.

Let \( d = \dim F_{i+1} \), let \( \{f_k\}_{k=1}^d \) be elements that generate \( F_{i+1} \), and let \( \{p_k, q_k\} \subseteq Q \) be such that \( \forall \bar{w} \in \overline{W}, f_k \in p_k(\bar{w})v_{N_i+1,1}(\bar{w}) + q_k(\bar{w})v_{N_i+1,2}(\bar{w}) + U_{N_i+1}(\bar{w}) \), as detailed in corollary 2.3. If there exists a \( \bar{w} \in \overline{W} \) such that each \( p_k(\bar{w}) = q_k(\bar{w}) = 0 \), then we can set \( (U_k, v_{k,1}, v_{k,2}) = (U_k(\bar{w}), v_{k,1}(\bar{w}), v_{k,2}(\bar{w})) \), and condition (4) can be satisfied.

Let \( G = \sum_{k=1}^d \mathbb{K}p_k + \mathbb{K}q_k \subseteq Q \) be the vector space generated by \( \{p_k, q_k\} \). Our remaining goal is to show \( \exists \bar{w} \in \overline{W} : G(\bar{w}) = (0) \).

Let \( R \) be the algebra generated by \( Q \), i.e. \( R = \sum_{k=1}^\infty Q^k \).

**Lemma 2.4.** If \( G, P \) are defined as above, then:

\[
R \cap GP \subseteq G + GR.
\]

**Proof.** Let \( M \) be the set of all monomials of \( P \) (without coefficient). Let \( M_Q \) be the monomials that generate \( Q \), let \( M_R = \bigcup_{j=1}^\infty M_Q^j \) be the monomials that generate \( R \), and let \( M'_R = M_R \setminus (M_R \cup \{1\}) \). \( P \) can be decomposed:

\[
P = \mathbb{K} \oplus R \oplus \mathbb{K}M'_R.
\]

Note that for any \( m \in M_Q \) and any \( m' \in M'_R \), \( mm' \in M_R \). As \( R \) is generated by monomials, \( R \cap QM'_R = (0) \).

Let \( g \in G \), and let \( p \in P \) have the decomposition \( p = k + r + s \), with \( k \in \mathbb{K} \), \( r \in R \) and \( s \in \mathbb{K}M'_R \). Suppose that \( gp \in R \). Since \( gk + gr \in R \), \( gs \in R \cap QM'_R = (0) \). Therefore, \( gp \in \mathbb{K}g + gR \), and \( R \cap GP \subseteq G + GR \).

**Theorem 2.5.** If \( \{\bar{w} \in W : G(\bar{w}) = (0)\} \subseteq W \setminus \overline{W} = \bigcup_{k=N_i}^{N_i+1-1} W_k \), then \( d \geq \frac{1}{2}(N_{i+1} - N_i + 1) \).
Proof. Let $Z$ be the affine variety function of $P$: if $I \lhd P$ is an ideal, then $Z(I) = \{ \bar{w} \in W : I(\bar{w}) = (0) \}$. It’s our goal to show that if $Z(GP) \subseteq \bigcup_{k=N_i}^{N_i+1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.

Since $Q$ annihilates each $W_k$, it must annihilate $Z(GP)$ as well. Hilbert’s nullstellensatz states that since $\mathbb{K}$ is algebraically closed, for each $q \in Q$, there must be an exponent $q^i \in GP$. Using lemma 2.4, $q^i \in R \cap GP \subseteq G + GR$, and so the quotient algebra $R/(G+GR)$ is nil. Since $G^2 \subseteq GR$, $R/GR$ is nil as well. All finitely generated commutative nil algebras are finite dimensional, so applying Lemma 3.2 in [2] several times gives $2d \geq \text{GKdim} R$. Recall that Lemma 3.2 [2] says that if $R$ is a commutative finitely generated graded algebra of Gelfand-Kirillov dimension $t$, and $I$ is a principal ideal generated by a homogeneous element then $R/I$ has Gelfand-Kirillov dimension at least $t - 1$.

Remember that for any $j \geq 0$, $Q^i = \prod_{k=N_i}^{N_i+1} (\mathbb{K}v_{k,1} + \mathbb{K}v_{k,2})^2 N_{i+1-k-1}$, and:

$$\dim Q^i = \prod_{k=N_i}^{N_i+1} (j 2^{N_{i+1-k-1}} + 1) \geq 2^{\frac{1}{2}(N_{i+1}-N_i-1)(N_{i+1}-N_i)} j N_{i+1}-N_i,$$

therefore $\text{GKdim} R \geq N_{i+1} - N_i + 1$. 

We can thus conclude that as long as $F_{i+1} < \frac{1}{2}(N_{i+1} - N_i + 1)$, there is a $\bar{w} \in \overline{W}$ such that $G(\bar{w}) = 0$, and we have appropriate spaces $\{U_k\}$ and monomials $\{v_{k,1}, v_{k,2}\}$ for all $k \leq N_{i+1}$. If this holds for all $i \geq 0$, the induction can proceed.

3 Constructing the ideal $E$

For any $i \geq 0$, let $V_i = \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$, let $v_i \in V_i$ be such that $U_{i+1} = H(2^i)U_i + U_iH(2^i) + v_i H(2^i)$, and let $Q_i = U_i + \mathbb{K}v_i$. If $v_{i,1} \notin \mathbb{K}v_i$, let $W_i = \mathbb{K}v_{i,1}$, otherwise, $W_i = \mathbb{K}v_{i,2}$. This way $Q_i \oplus W_i = H(2^i)$, $U_{i+1} = H(2^i)U_i + Q_i H(2^i)$, and $V_{i+1} = W_i V_i$. 


Proposition 3.1. For any \( j > i \) and any \( k \leq 2^{j-i} - 1 \),

\[ H(2^k)U_i H(2^j - (k + 1)2^i) \subseteq U_j \]

Proof. Apply induction on the value of \( j \) by using \( H(2^i)U_i + U_i H(2^i) \subseteq U_{i+1} \).

For any \( n > 0 \), let \( m \geq 0 \) be maximal such that \( 2^m \leq n \), and define:

\[ R(n) = \{ x \in H(n) : xH(2^{m+1} - n) \subseteq U_{m+1} \} \]

\[ L(n) = \{ x \in H(n) : H(2^{m+1} - n)x \subseteq U_{m+1} \} \]

Also, set \( R(0) = L(0) = (0) \).

Proposition 3.2. For any \( n > 0 \) and any \( M \) such that \( 2^M > n \),

\[ R(n)H(2^M - n) \subseteq U_M \]

\[ H(2^M - n)L(n) \subseteq U_M \]

Proof. Apply simple induction on \( M \), using the fact that \( H(2^M)U_M + U_M H(2^M) \subseteq U_{M+1} \).

Proposition 3.3. For any \( n > 0 \), \( R(n)H(1) \subseteq R(n+1) \) and \( H(1)L(n) \subseteq L(n+1) \).

Proof. Let \( m \geq 0 \) be maximal such that \( 2^m \leq n \). If \( 2^{m+1} - 1 < n \), then:

\[ R(n)H(1) \cdot H(2^{m+1} - n - 1) = R(n)H(2^{m+1} - n) \subseteq U_{m+1} \]

and \( R(n)H(1) \subseteq R(n+1) \).

If \( 2^{m+1} - 1 = n \), then:

\[ R(n)H(1) \cdot H(2^{m+2} - n - 1) \subseteq U_{m+1} H(2^{m+1}) \subseteq U_{m+2} \]

and \( R(n)H(1) \subseteq R(n+1) \).

By symmetry, \( H(1)L(n) \subseteq L(n+1) \).
Define the space $R'(n) \subseteq H(n)$ recursively; if $n = 0$, set $R(0) = \mathbb{K}$, and otherwise, $m$ be maximal such that $2^m \leq n$ and set:

$$R'(n) = W_m R'(n - 2^m)$$

Note that $\dim R'(n) = 1$.

**Proposition 3.4.** For any $n \geq 0$, $R(n) \oplus R'(n) = H(n)$.

**Proof.** Use induction on $n$. The base case $n = 0$ is trivial.

For the inductive step, $n \geq 0$, let $m$ be maximal such that $2^m \leq n$, and assume that $R(n - 2^m) \oplus R'(n - 2^m) = H(n - 2^m)$. Proposition 3.2 can be used to confirm that:

$$Q_m H(n - 2^m) \cdot H(2^{m+1} - n) = Q_m H(2^m) \subseteq U_{m+1},$$

$$H(2^m) R(n - 2^m) \cdot H(2^{m+1} - n) \subseteq H(2^m) U_m \subseteq U_{m+1},$$

$$R(n) + R'(n) \supseteq Q_m H(n - 2^m) + H(2^m) R(n - 2^m) + W_m R'(n - 2^m) = H(n).$$

Since $\dim R'(n) = 1$, either $R(n) \oplus R'(n) = H(n)$ or $R'(n) \subseteq R(n)$. However, the latter option implies $R(n) = H(n)$ and that $H(n) \cdot H(2^{m+1} - n) \subseteq U_{m+1}$, a clear contradiction. Therefore, $R(n) \oplus R'(n) = H(n)$. \hfill $\square$

**Proposition 3.5.** For any $n \geq 0$,

$$0 < \dim H(n)/L(n) \leq 2$$

**Proof.** Let $m$ be maximal such that $2^m \leq n$.

If $H(n)/L(n)$ were zero, then $L(n) = H(n)$ and $H(2^{m+1} - n) H(n) \subseteq U_{m+1}$, a contradiction.

Using proposition 3.2, $R(2^{m+1} - n) H(n) \subseteq U_{m+1}$. By proposition 3.4,

$$L(n) = \{ x \in H(n) : R(2^{m+1} - n) x \in U_{m+1} \}$$

Let $p \in H(2^{m+1} - n)$ be an element that generates $R'(2^{m+1} - n)$, and let $\phi : H(n) \rightarrow H(2^{m+1})/U_{m+1}$ be the $\mathbb{K}$-linear transformation:

$$\phi : x \mapsto px/U_{m+1}$$

So that $L(n) = \ker \phi$. Since the image of $\phi$ is at most dimension 2, $\dim H(n)/L(n) \leq 2$. \hfill $\square$
Let \( L'(n) \subseteq H(n) \) be a space such that \( L(n) \oplus L'(n) = H(n) \). Proposition 3.5 shows that \( \dim L'(n) \) is either 1 or 2.

Define the space \( E(n) \subseteq H(n) \) as:

\[
E(n) = \bigcap_{i=0}^{n} L(i)H(n-i) + H(i)R(n-i)
\]

**Lemma 3.1.** For any \( n > 0 \), \( E(n)H(1) + H(1)E(n) \subseteq E(n+1) \).

**Proof.** Using proposition 3.3,

\[
E(n)H(1) = \bigcap_{i=0}^{n} L(i)H(n-i) \cdot H(1) + H(i)R(n-i)H(1) \subseteq \bigcap_{i=0}^{n} L(i)H(n+1-i) + H(i)R(n+1-i).
\]

It remains to show that \( E(n)H(1) \subseteq L(n+1)H(0) + H(n+1)R(0) = L(n+1) \).

Let \( m \geq 0 \) be maximal such that \( 2^m \leq n + 1 \).

\[
H(2^{m+1} - n - 1)E(n)H(1) \subseteq H(2^{m+1} - n - 1)L(n - 2^m + 1)H(2^m) + H(2^m)R(2^m - 1)H(1) \subseteq U_mH(2^m) + H(2^m)U_m \subseteq U_{m+1}
\]

Therefore, by definition, \( E(n)H(1) \subseteq L(n+1) \).

\( H(1)E(n) \subseteq E(n+1) \) can be proven by symmetry. \( \Box \)

Let \( E = \sum_{n=1}^{\infty} E(n) \).

**Theorem 3.2.** \( E \) is an ideal of \( A \).

**Proof.** Apply lemma 3.1 to the definition of \( E \). \( \Box \)

**Proposition 3.6.** \( A/E \) is infinite dimensional.

**Proof.**

\[
\dim A/E = \sum_{n=1}^{\infty} \dim H(n)/E(n) > \sum_{n=1}^{\infty} \dim H(n)/R(n) = \sum_{n=1}^{\infty} \dim R'(n) = \infty
\]

\( \Box \)
Proposition 3.7. $A/E$ has quadratic or linear growth.

Proof. Using the fact that $(L(i)H(n-i) + H(i)R(n-i)) \oplus L'(i)R'(n-i) = H(n)$, and recalling proposition 3.5,

$$
\dim H(n)/E(n) \leq \sum_{i=0}^{n} \dim L'(i)R'(n-i) \leq \sum_{i=0}^{n} 2 = 2(n+1),
$$

$$
\sum_{i=0}^{n} \dim H(i)/E(i) \leq n^2 + 3n + 1.
$$

Proposition 3.6 shows algebra isn’t finite dimensional. Bergman’s Gap Theorem [3] proves that the only growths strictly less than quadratic are linear and finite, so $A/E$ must have quadratic or linear growth. \qed

4 \quad $E \supseteq \mathcal{E}(F_i)$

Theorem 4.1. For any $n > 0$, let $m$ be maximal such that $2^m \leq n$.

$$
\bigcap_{i=0}^{2^m+1-n} \{ x \in H(n) : H(i)xH(2^m+1-n-i) \subseteq U_mH(2^m) + H(2^m)U_m \} \subseteq E(n).
$$

Proof. It’s sufficient to show that for any $0 \leq i \leq 2^m+1-n$ and any $x \in H(n)$ such that $x \notin L(2^m-i)H(n-2^m+i) + H(2^m-i)R(n-2^m+i)$,

$$
H(i)xH(2^m+1-n-i) \notin U_mH(2^m) + H(2^m)U_m.
$$

$x$ can be uniquely decomposed into $x_1 + x_Lx_R$, with:

$$
x_1 \subseteq L(2^m-i)H(n-2^m+i) + H(2^m-i)R(n-2^m+i),
$$

$$
x_L \subseteq L'(2^m-i), \; x_R \in R'(n-2^m+i)
$$

Under our assumption, $x_Lx_R \neq 0$. However,

$$
H(i)x_1H(2^m+1-n-i) \in H(i)L(2^m-i)H(2^m) + H(2^m)R(n-2^m+i)H(2^m+1-n-i) \subseteq
$$
Therefore it’s sufficient to show there exists \( y \in H(i) \) and \( z \in H(2^{m+1} - n - i) \) such that \( yx_L x_R z \notin U_m H(2^m) + H(2^m) U_m \).

As \( x_L \notin L(2^m - i) \), there must exist a \( y \in H(i) \) such that \( yx_L \notin U_m \). Let \( yx_L = x_{LU} + x_{LV} \), with \( x_{LU} \in U_m \) and \( 0 \neq x_{LV} \in V_m \). Symmetrically, there’s a \( z \in H(2^{m+1} - n - i) \) with \( x_R = x_{RU} + x_{RV} \), \( x_{RU} \in U_m \), and \( 0 \neq x_{RV} \in V_m \).

\[
yx_L x_R z = x_{LU} x_R z + x_{LV} x_{RU} + x_{LV} x_{RV} \notin U_m H(2^m) + H(2^m) U_m
\]

\[\square\]

For any non-zero homogeneous space \( F \subseteq H(n) \), let \( \mathcal{E}(F) \) denote the space:

\[
\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA.
\]

**Proposition 4.1.** For any non-zero homogeneous space \( F \subseteq H(n) \), \( \mathcal{E}(F) \) is an ideal.

**Proof.** By the definition, it’s clear that \( \mathcal{E}(F) \) is right ideal. To prove it’s a left ideal, it’s sufficient to show that \( H(1) \mathcal{E}(F) \subseteq \mathcal{E}(F) \).

\[
H(1) \mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j + 1) FA = \bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn + j + 1) FA \cap \sum_{k=0}^{\infty} H(kn + n) FA = \bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn + j + 1) FA \cap \sum_{k=1}^{\infty} H(kn + n) FA \subseteq \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA = \mathcal{E}(F).
\]

\[\square\]

**Corollary 4.2.** For any \( i \geq 0 \), \( \mathcal{E}(F_i) \subseteq E \).
Proof. Since it’s graded, $\mathcal{E}(F_i)$ can decomposed into homogeneous subspaces.
If $n < 2^{N_i}$, $\mathcal{E}(F_i) \cap H(n) = \emptyset$, and if $n \geq 2^{N_i}$,
$$\mathcal{E}(F_i) \cap H(n) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\lfloor (n-j)2^{-N_i} - 1 \rfloor} H(k2^{N_i} + j)F_iH(n - (k + 1)2^{N_i} - j)$$

Let $n \geq 2^{N_i}$ and let $m$ be maximal such that $2^m \leq n$. For any $0 \leq j \leq 2^{m+1} - n$,
$$H(j)(\mathcal{E}(F_i) \cap H(n))H(2^{m+1} - n - j) \subseteq \sum_{k=1}^{\lfloor (n+j)2^{-N_i} - 1 \rfloor} H(k2^{N_i})F_iH(2^{m+1} - (k + 1)2^{N_i}) \subseteq H(k2^{N_i})U_{N_i}H(2^{m+1} - (k + 1)2^{N_i}).$$

Using proposition 3.1, this is contained in $U_{m+1}$, and so by theorem 4.1, $\mathcal{E}(F_i) \cap H(n) \subseteq E(n)$.  

5 Enumerating elements

To build a Jacobson radical homomorphic image through this method, we use a method very similar to used in Theorem 9 in [7], but readapted for our constraints. First, we require that the field $\mathbb{K}$ be countable, so that we can enumerate the polynomials of $\bar{A}$. For each such $f \in \bar{A}$, we will find a $g \in \bar{A}$ and a sufficiently ”small” $F$ such that $f + g - fg \in \mathcal{E}(F)$.

Let $f \subseteq \bar{A}$ be any polynomial with no constant term, and let $d$ be minimal such that $f \in \sum_{n=1}^{d} H(n)$. $f$ can be decomposed as $f = f_{(i_1)} + \cdots + f_{(i_d)}$ with each $f_{(i)} \in F(i)$. Recursively define the spaces $s(n) \subseteq H(n)$ for each $n \geq 0$ with:

- $s(0) = 1$,
- $s(n) = \sum_{i=1}^{\min\{n,d\}} f_{(i)}s(n - i)$ for $n > 0$.

This way,
$$s(n) = \sum_{k=0}^{n} \sum_{1 \leq i_1, \ldots, i_k \leq d, i_1 + \cdots + i_k = n} f_{(i_1)} \cdots f_{(i_k)}.$$
Lemma 8 from [8] can be used to prove a simple property:

**Lemma 5.1.** For any $m_1, m_2 \geq 0$ and any $n \geq m_1 + m_2 + 2d$,

$$s(n) \subseteq \sum_{a,b=1}^{d} H(m_1 + a)s(n - m_1 - m_2 - a - b + 1)H(m_2 + b - 1)$$

Using $s$, we can build our subspace $F$. Recall that $|X|$ is the number of generators of $A$.

**Theorem 5.2.** For any $N \geq 2d$, there exists a homogeneous subspace $F \subseteq H(N)$ with $\dim F \leq \left(\frac{|X|^d - 1}{|X| - 1}\right)^2$ and a polynomial $g \in \bar{A}$ such that $f + g - fg \in \mathcal{E}(F)$.

**Proof.** Let $g = -\sum_{n=1}^{2N+d} s(n)$, and let $P$ be the two-sided ideal generated by $\{s(2N + i)\}_{i=1}^{d}$. By the recursive construction of $s$,

$$g = -\sum_{n=1}^{2N+d} s(n) = -\sum_{n=1}^{2N+d \min(n,d)} \sum_{i=1}^{n} f(i)s(n - i) = -f - \sum_{i=1}^{d} \sum_{n=i+1}^{2N+d} f(i)s(n - i) = -f - \sum_{i=1}^{d} \sum_{n=1}^{N} f(i)s(n) - \sum_{i=1}^{d} \sum_{n=2N+1}^{2N+d} f(i)s(n) \in -f + fg + P$$

Now, set $F = \sum_{a,b=0}^{d-1} H(a)s(N - a - b)H(b)$. It is our goal to show that $P \subseteq \mathcal{E}(F)$. Thanks to proposition 4.1, it sufficient to show that for any $1 \leq i \leq d$, $s(2N + i) \in \mathcal{E}(F)$. Consequently, it’s sufficient to show that for any $0 \leq j < N$,

$$s(2N + i) \in H(j)FH(N + i - j) = \sum_{a,b=0}^{d-1} H(j + a)s(N - a - b)H(N + i + b - j),$$

which can be extracted easily from lemma 5.1.

Finally, recall that $\dim H(n) = |X|^n$, where $|X|$ is the number of generators of $A$.

$$\dim F \leq \sum_{a,b=0}^{d-1} \dim H(a)s(N - a - b)H(b) = \sum_{a,b=0}^{d-1} |X|^{a+b} = \left(\frac{|X|^d - 1}{|X| - 1}\right)^2.$$
In order to make our quotient algebra \( \bar{A}/E \) Jacobson radical, for every \( f \in \bar{A} \) there needs to be a \( g \in \bar{A} \) such that \( f + g - fg \in E \). As \( \bar{A} \) is countable, we can make an enumeration \( f_1, f_2, \ldots \). For each \( f_m \), let \( d_m \) be minimal such that \( f_m \in \sum_{n=1}^{d_m} H(n) \). For any \( N_m \geq 1 + \log_2 d_m \), theorem 5.2 can give us a \( g_m \in \bar{A} \) and an \( F_m \subseteq H(2^{N_m}) \) such that \( f_m + g_m - f_m g_m \in \mathcal{E}(F_m) \) and \( \dim F_m \leq \left( \frac{|X|^{d_m - 1}}{|X| - 1} \right)^2 \).

If each \( \dim F_m < \frac{1}{2}(N_m - N_{m-1} + 1) \), then we can construct the ideal \( E \) as detailed in section 3. \( A/E \) is infinite dimensional (proposition 3.6), has quadratic growth (because affine algebras with linear growth are PI by Small-Stafford-Warfield Theorem [6]) with each \( \dim H(n)/E(n) \leq 2(n + 1) \) (proposition 3.7), and contains each \( \mathcal{E}(F_m) \) (corollary 4.2). Fortunately, each \( N_m \) can be set arbitrarily high in relation to \( N_{m-1} \). The needed upper bound of dimension of \( F_m \) depends on \( d_m, |X|, N_m \) and \( N_{m-1} \), so if each \( N_m \) is set to \( \sup \{1 + \log_2 d_m, 2 \left( \frac{|X|^{d_m - 1}}{|X| - 1} \right)^2 + N_{m-1} \} \), each \( F_m \) will be ”small enough” for the construction of \( E \).

References


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