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Asymptotics of Some Convolutional Recurrences

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Abstract

We study the asymptotic behavior of the terms in sequences satisfying recurrences of the form \( a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n, k) a_k a_{n-k} \) where, very roughly speaking, \( f(n, k) \) behaves like a product of reciprocals of binomial coefficients. Some examples of such sequences from map enumerations, Airy constants, and Painlevé I equations are discussed in detail.

1 Main results

There are many examples in the literature of sequences defined recursively using a convolution. It often seems difficult to determine the asymptotic behavior of such sequences. In this note we study the asymptotics of a general class of such sequences. We prove

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subexponential growth by using an iterative method that may be useful for other recurrences. By subexponential growth we mean that, for every constant $D > 1$, $a_n = o(D^n)$ as $n \to \infty$. Thus our motivation for this note is both the method and the applications we give.

Let $d > 0$ be a fixed integer and let $f(n, k) \geq 0$ be a function that behaves like a product of some powers of reciprocals of binomial coefficients, in a general sense to be specified in Theorem 1. We deal with the sequence $a_n$ for $n \geq d$ where $a_d, a_{d+1}, \ldots, a_{2d-1} \geq 0$ are arbitrary and, when $n \geq 2d$,

$$a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n, k)a_ka_{n-k}. \quad (1)$$

Without loss of generality,

we assume that $f(n, k) = f(n, n - k)$

since we can replace $f(n, k)$ and $f(n, n - k)$ in (1) with $\frac{1}{2}(f(n, k) + f(n, n - k))$.

Theorem 1 proves subexponential growth. Theorem 2 provides more accurate estimates under additional assumptions. In Section 2, we apply the corollary to some examples.

**Theorem 1 (Subexponential growth)** Let $a_n$ be defined by recursion (1) with $a_d > 0$.

Suppose there is a function $R(x)$ defined on $(0, 1/2]$, an $\alpha > 0$ and an $r$ such that

(a) $0 < R(x) < r < 1$,

(b) $\lim_{x \to 0^+} R(x) = 0$,

(c) $0 \leq f(n, k) = O \left(n^{-\alpha}R^{k-d}(k/n)\right)$ uniformly for $d \leq k \leq n/2$.

Then $a_n$ grows subexponentially; in fact,

$$a_n = (1 + O(n^{-\alpha}))a_{n-1}. \quad (2)$$

**Proof:** We first note that the $a_n$ are non-decreasing when $n \geq 2d - 1$.

Our proof is in three steps. We first prove that $a_n = O(C^n)$ for some constant $C > 2$. We then prove that $C$ can be chosen very close to 1. Finally we deduce (2) and subexponential growth.

**First Step:** Since the bound in (c) is bounded by some constant times the geometric series $n^{-\alpha}R^{k-d}$ with ratio less than 1, $\sum_{k=d}^{n-d} f(n, k) = O(n^{-\alpha})$. Hence we can choose $M$ so large that $\sum_{k=d}^{n-d} f(n, k) < 1/4$ when $n > M$. Next choose $C \geq 2$ so large ($C = \max\{a_d, a_{d+1}, \ldots, a_{2d-1}, a_M, 2\}$ will do) that $a_n < 2C^n$ for $n \leq M$. By induction, using the recursion (1), we have for $n > M$

$$a_n < 2C^{n-1} + (1/4)4C^n \leq C^n + C^n = 2C^n.$$

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Second Step: By (b) there is a $\lambda$ in $(0, 1/2)$ such that $R(x) < \frac{1}{2x}$ for $0 < x < \lambda$. Fix any $D \leq C$ such that $a_n = O(D^n)$, which is true for $D = C$ by the First Step.

Split the sum in (1) into $\lambda n \leq k \leq (1 - \lambda)n$ and the rest, calling the first range of $k$ the “center” and the rest the “tail”. Noting $r < 1$, the center sum is bounded by

$$2 \sum_{k=\lambda n+1}^{n/2} f(n, k) a_{n-k} = O\left(D^n \sum_{k=\lambda n+1}^{n/2} r^{k-d}\right) = O\left((r^\lambda D)^n\right).$$

(3)

Since $a_j$ are increasing, the tail sum is bounded by

$$2 \sum_{k=d}^{\lambda n} f(n, k) a_{n-k} = O(n^{-\alpha} a_{n-1} \sum_{k=d}^{\lambda n} R(x)^{k-d} D^k) \quad (4)$$

$$= O(n^{-\alpha} a_{n-1} \sum_{k=d}^{\lambda n} (DR(x))^{k-d} = O(n^{-\alpha} a_{n-1}),$$

where the last equality follows from the fact that $DR(x) < 1/2$. Combining (3) and (4),

$$a_n = \left(1 + O(n^{-\alpha})\right) a_{n-1} + O((r^\lambda D)^n).$$

(5)

When $r^\lambda D > 1$, induction on $n$ easily leads to $a_n = O((r^\lambda D')^n)$ for any $D' > D$, an exponential growth rate no larger than $r^\lambda D'$.

Since $r^\lambda$ has a fixed value less than one, we can iterate this process, replacing $D$ by $r^\lambda D'$ at the start of the Second Step. We finally obtain a growth rate $D > 1$ with $r^\lambda D < 1$. This completes the second step.

Third Step: With the value of $D$ just obtained, the last term in (5) is exponentially small and hence is $O(n^{-\alpha} a_{n-1})$. Thus we obtain (2) which immediately implies subexponential growth of $a_n$, since $1 + O(n^{-\alpha}) < D$ for any $D > 1$ and sufficiently large $n$. \[\blacksquare\]

To say more than (2), we need additional information about the behavior of the $f(n, k)$.

When $f(n, k)/f(n, d)$ is small for each $k$ in the range $d + 1 \leq k \leq n - d - 1$, the first and last terms dominate the sum. The following theorem is based on this observation.

Theorem 2 (Asymptotic behavior) Assume (a)–(c) of Theorem 1 hold. Suppose further that there is a $\beta > 0$ such that

$$\frac{f(n, k)}{f(n, d)} = O(n^{-\beta} r^{k-d-1}) \quad \text{uniformly for} \quad d + 1 \leq k \leq n/2.$$ \quad (6)

Then

$$\log a_n = 2a_d \sum_{k=2d+1}^{n} f(k, d) + O\left(\sum_{k=2d}^{n} f(k, d) \left(k^{-\alpha} + k^{-\beta}\right)\right).$$ \quad (7)
Proof: We assume $n > 2d$. Remove the $k = d$ and $k = n - d$ terms from the sum in (1). We first deal with the remaining sum. Theorem 1 gives $a_k = O(D^k)$ for all $D > 1$, so we can assume $D < 1/r$. Using (6)
\[
\sum_{k=d+1}^{n-d-1} f(n,k)a_k a_{n-k} = O\left(f(n,d)n^{-\beta}a_{n-1}\right) \sum_{k=d+1}^{n/2} r^{k-d-1} D^k
\]

Combining this with (1), we obtain
\[
a_n = a_{n-1} + 2a_d f(n,d) a_{n-d} + f(n,d) O(n^{-\beta}) a_{n-1}
\]

Taking logarithms and noting for expansion purposes that $f(n,d) = O(n^{-\alpha})$, we obtain
\[
\log a_n - \log a_{n-1} = 2a_d f(n,d) + O\left((n^{-\alpha} + n^{-\beta}) f(n,d)\right).
\]

Sum over $n$ starting with $n = 2d + 1$. The theorem follows immediately when we note that the constant terms can be incorporated into the $O()$ in (7) since the sum therein is bounded below by a nonzero constant.

Corollary 1 Assume the conditions of Theorem 2 hold and $f(n,d) = \Theta(n^{-\alpha})$.

- If $\alpha < 1$, then $a_n = \exp(\Theta(n^{1-\alpha}))$.
- If $\alpha > 1$, then $a_n = K + O(n^{1-\alpha})$ for some constant $K$.
- If $f(n,d) - A/n$ are the terms of a convergent series, then $a_n \sim C n^{2Ad}$ for some positive constant $C$.

Proof: Since $\alpha > 0$ and $\beta > 0$, (7) gives $\log a_n = \Theta(\sum_{k=2d+1}^{n} k^{-\alpha})$. The case $\alpha < 1$ follows immediately; for $\alpha > 1$, we see that $a_n$ is bounded and nondecreasing and therefore has a limit $K$. For $m > n$, (2) gives $\log(a_m/a_n) = O(n^{1-\alpha})$ uniformly in $m$. Letting $m \to \infty$, we obtain the claim regarding $\alpha > 1$.

For $\alpha = 1$, the first sum in (7) is $A \log n + B + o(1)$ for some constant $B$, and the last sum in (7) converges.

2 Examples

We apply Theorem 2 and Corollary 1 to some recursions which arise from combinatorial applications. In our examples, $f(n,k)$ behaves like a product of the reciprocal of binomial coefficients, which satisfies the conditions of Theorems 1 and 2. A more general case of interest is when $f(n,k)$ takes the form of the product of functions like
\[
g(n,k) = \frac{[a]_k [a]_{n-k}}{[a]_n}
\]
for some constant $a > 0$, where $[x]_k = x(x+1) \cdots (x+k-1) = \frac{\Gamma(x+k)}{\Gamma(k)}$, the rising factorial.

We note that when $a = 1$, $g(n, k) = \binom{n}{k}^{-1}$.

We begin with some useful bounds. When $a > 0$ and $1 \leq k \leq n/2$,

$$g(n, k) = \prod_{j=0}^{k-1} \frac{a+j}{a+n-k+j} < \left( \frac{a+k}{a+n} \right)^k \leq \left( \frac{k}{n} \right)^k \left( \frac{1+a/k}{1+a/n} \right)^k = O \left( \left( \frac{k}{n} \right)^k \right) = O \left( n^{-1}(3k/2n)^{k-1} \right)$$

since $k(2/3)^{k-1}$ is bounded. So $g$ satisfies the condition on $f$ in Theorem 1(c), with $\alpha = 1$.

Similarly, when $a > 0$ and $d \leq k \leq n/2$,

$$\frac{g(n, k)}{g(n, d)} = \prod_{j=0}^{k-d-1} \frac{a+d+j}{a+n-k+d+j} = O \left( n^{-1}(3k/2n)^{k-d-1} \right).$$

This is in accordance with (6) with $\beta = 1$.

**Example 1 (Map enumeration constants)** There are numbers $t_n$ appearing in the asymptotic enumeration of maps in an orientable surface of genus $n$, whose value does not concern us here. Define $u_n$ by

$$t_n = 8 \frac{[1/5]_n [4/5]_{n-1}}{\Gamma \left( \frac{5n-1}{2} \right)} \left( \frac{25}{96} \right)^n u_n.$$

Then $u_1 = 1/10$ and $u_n$ satisfies the following recursion [3]

$$u_n = u_{n-1} + \sum_{k=1}^{n-1} f(n, k) u_k u_{n-k} \quad \text{for} \quad n \geq 2,$$

where

$$f(n, k) = \frac{[1/5]_k [1/5]_{n-k} [4/5]_{k-1} [4/5]_{n-k-1}}{[1/5]_n [4/5]_{n-1}}.$$

From the observations above, the conditions of Theorem 2 are satisfied with $d = 1$, $R(\lambda) = (3\lambda/2)^2$ and $\alpha = \beta = 2$. Hence, $u_n \sim K$ for some constant $K$. Unlike the proof in [3], this does not depend on the value of $u_1$. [1]

**Example 2 (Airy constants)** The Airy constants $\Omega_n$ are determined by $\Omega_1 = 1/2$ and the recurrence [7]

$$\Omega_n = (3n-4)\Omega_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} \Omega_k \Omega_{n-k} \quad \text{for} \quad n \geq 2.$$
Let $\Omega_n = n! [2/3]_{n-1} 3^n a_n$. Then $a_n$ satisfies (1) with $d = 1$ and

$$f(n, k) = \frac{[2/3]_{k-1} [2/3]_{n-k-1}}{[2/3]_{n-1}}.$$ 

Theorem 2 applies with $d = 1$, $R(\lambda) = 3\lambda/2$ and $\alpha = \beta = 1$. Since

$$f(n, 1) = \frac{1}{n - 4/3} = \frac{1}{n} + \frac{4/3}{n(n - 4/3)}$$

and $a_1 = 1/6$, we have $a_n \sim C n^{1/3}$ for some constant $C$.

We note that it is possible to apply the result of Olde Daalhuis [13] to obtain a full asymptotic expansion for $\Omega_n$. Let

$$A_n = \frac{\Omega_n}{3^n n!}.$$ 

Then the recursion for $\Omega_n$ becomes

$$A_n = (n - 4/3) A_{n-1} + \sum_{k=1}^{n-1} A_k A_{n-k}, \quad n \geq 2.$$ 

It follows that the formal series

$$F(z) = \sum_{n \geq 1} \frac{A_n}{z^n}$$

satisfies the Riccati equation

$$F'(z) + \left(1 + \frac{1}{3z}\right) F(z) - F^2(z) - \frac{1}{6z} = 0.$$ 

It then follows from the result of Olde Daalhuis [13] that

$$A_n \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} b_k \Gamma(n - k), \quad \text{as } n \to \infty,$$

where $b_0 = 1$ and $b_k$ can be computed using the recursion

$$b_k = \frac{-2}{k} \sum_{j=2}^{k+1} b_{k+1-j} A_j, \quad k \geq 1.$$ 

In particular, we have

$$\Omega_n \sim \frac{1}{2\pi} \Gamma(n) 3^n n! = \frac{1}{2\pi n} (n!)^{2/3} 3^n, \quad \text{as } n \to \infty.$$ 

It is well known that solutions to the Riccati equation have infinitely many singularities, hence $F(z)$ (via its Borel transform [2]) cannot satisfy a linear ODE with polynomial coefficients. This implies that the sequence $A_n$ (and hence $\Omega_n$) is not holonomic.
Example 3  The following recursion, with \( \ell > 0 \) and \( \ell \neq 1/2 \), appeared in [6]. The Airy constants are the special case \( \ell = 1 \). The case \( \ell = 2 \) corresponds to the recursion studied in [9, 10], which arises in the study of the Wiener index of Catalan trees. We have \( C_1 = \frac{\Gamma(\ell-1/2)}{\sqrt{\pi}} \) and, for \( n \geq 2 \),

\[
C_n = n \frac{\Gamma(n\ell + (n/2) - 1)}{\Gamma((n-1)\ell + (n/2) - 1)} C_{n-1} + \frac{1}{4} \sum_{k=1}^{n-1} \binom{n}{k} C_k C_{n-k}. \tag{11}
\]

Define \( a_n \) by \( C_n = n! \cdot g(n) a_n \) where \( g(1) = 1 \) and

\[
g(m) = \prod_{k=2}^{m} \frac{\Gamma(k\ell + (k/2) - 1)}{\Gamma((k-1)\ell + (k/2) - 1)}.
\]

Then (11) becomes

\[
a_n = a_{n-1} + \sum_{k=1}^{n-1} \frac{g(k)g(n-k)}{4g(n)} a_k a_{n-k},
\]

so \( f(n,k) = g(k)g(n-k)/4g(n) \).

With \( a \) fixed and \( x \to \infty \) and using 6.1.47 on p.257 of [1] (or using Stirling’s formula), we have

\[
\frac{\Gamma(x+a)}{\Gamma(x)} = x^a \left( 1 + \frac{a(a-1)}{2x} + O(1/x^2) \right)
= x^a \left( 1 + \frac{a-1}{2x} \right)^a \left( 1 + O(1/x^2) \right)
= \left( x + \frac{a-1}{2} \right)^a \left( 1 + O(1/x^2) \right). \tag{12}
\]

When \( m > 1 \), (13) gives us

\[
g(m) = \prod_{k=2}^{m} \left( \frac{(2\ell + 1)k - \ell - 3}{2} \right)^\ell \left( 1 + O(1/k^2) \right)
= \Theta(1) \left( (\ell + 1/2)^m \prod_{k=2}^{m} \left( k - \frac{\ell + 3}{2\ell + 1} \right) \right)^\ell
= \Theta(1) \left( (\ell + 1/2)^m [a]_{m-1} \right)^\ell, \text{ where } a = \frac{3\ell - 1}{2\ell + 1}.
\]

Hence

\[
f(n, k) = \Theta(1) \left| \frac{[a]_{k-1}[a]_{n-k-1}}{[a]_{n-1}} \right|^\ell.
\]

where the absolute values have been introduced to allow for \( a < 0 \). A slight adjustment of the argument leading to (8) and (9) leads to

\[
f(n,k) = O(n^{-\ell}(3k/2n)\ell(k-1)) \quad \text{and} \quad \frac{f(n,k)}{f(n,1)} = O(n^{-\ell}(3k/2n)\ell(k-1)).
\]
for $1 \leq k \leq n/2$. Hence Theorem 2 applies with $\alpha = \beta = \ell$, and $a_n$ converges to a constant when $\ell > 1$ by Corollary 1.  

It is interesting to note that there is a simple relation between the sequence $u_n$ in Example 1 and the sequence $a_n$ in Example 3 with $\ell = 2$. It is not difficult to check that the $f(n,k)$ defined in Example 3 is exactly five times the $f(n,k)$ in Example 1: since $a_1 = 5u_1$, we have $a_n = 5u_n$ for all $n \geq 1$. This simple relation suggests a relationship between the number of maps on an orientable surface of genus $g$ and the $g$th moment of a particular toll function on a certain type of trees. Using a bijective approach, Chapuy [4] recently found an expression for $t_g$ as the $g$th moment of the labels in a random well-labelled tree.

3 A convolutional recursion arising from Painlevé I

The following is recursion (44) in [11].

$$\alpha_n = (n - 1)^2 \alpha_{n-1} + \sum_{k=2}^{n-2} \alpha_k \alpha_{n-k}, \quad n \geq 1, \quad n \geq 3. \quad (14)$$

It follows from Proposition 14 of [11] that, for $0 < \alpha_1 < 1$ and $\alpha_2 = \alpha_1 - \alpha_1^2$,

$$\alpha_n = c(\alpha_1)((n - 1)!)^2 \left(1 - \frac{2\alpha_2(n-3)}{3(n-1)^2(n-2)^2} + \delta_n\right), \quad (15)$$

where $c(\alpha_1)$ depends only on $\alpha_1$, and

$$\delta_n = O(1/n^4).$$

We note that $\alpha_n$ for $n \geq 3$ depends only on $\alpha_2$. The proof of (15) relies on the fact that $0 < \alpha_2 < 1/4$ for $0 < \alpha_1 < 1$. It is conjectured in [11] that the asymptotic expression (15) actually holds for a wider range of values of $\alpha_1$.

For $n \geq 1$, let

$$p_n = \frac{\alpha_n}{((n - 1)!)^2}.$$ 

Then, as shown in [11], $p_n$ satisfies recursion (1) with $d = 2$ and

$$f(n,k) = \left(\frac{(n-k-1)!k!}{(n-1)!}\right)^2.$$ 

We note here $f(n,2) = O(n^{-4})$. It follows from Theorem 2 that

$$p_n = p(1 + \epsilon_n) \quad \text{for any } \alpha_2 > 0,$$

where $p = p(\alpha_2)$ is a positive constant and $\epsilon_n = O(1/n^3)$. 

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It is also interesting to note that, with $\alpha_1 = 1/50$, $\alpha_2 = 49/2500$, the sequence $\alpha_n$ is related to the sequence $u_n$ in Example 1 by

$$\alpha_n = [1/5]_n [4/5]_{n-1} u_n.$$ 

As mentioned in [11], the formal series $v(t) = \sum_{n \geq 1} \alpha_n t^{-n}$ satisfies

$$t^2 v'' + tv' - (t + 2\alpha_1)v + tv^2 + \alpha_1 = 0,$$  \quad (16)

and hence, with

$$t = \frac{8\sqrt{6}}{25} x^{5/2},$$

$$y(x) = (x/6)^{1/2}(1 - 2v(t))$$

satisfies the following Painlevé I:

$$y'' = 6y^2 - x.$$ 

This connection with Painlevé I is used in [8] to show that the sequence $\alpha_n$ is not holonomic (It follows that $u_n$ and $t_n$ in Example 1 are also not holonomic). The proof uses the fact that solutions to Painlevé I have infinitely many singularities and hence cannot satisfy a linear ODE with polynomial coefficients.

In the following we apply the techniques of [14] to prove that (15) holds for any complex constant $\alpha_1$. It is convenient to introduce the formal series

$$u_0(z) = v(z^2) = \sum_{n=2}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} \alpha_n z^{-2n}.$$ 

It follows from (16) that $u = u_0(z)$ is a formal solution to the differential equation

$$\frac{1}{4} u'' + \frac{1}{4z} u' - \left( 1 + \frac{2\alpha_1}{z^2} \right) u + u^2 + \frac{\alpha_1}{z^2} = 0.$$ 

The Stokes lines for this differential equation are the positive and the negative real axes. When the negative real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_1(z) = Ke^{2z}z^{-1/2} \sum_{n=0}^{\infty} c_n z^{-n},$$

in which the Stokes multiplier $K$ is a constant (depending on the constant $\alpha_1$). To determine the coefficients $c_n$ we observe that $u_1$ is a solution of the linear differential equation

$$\frac{1}{4} u_1'' + \frac{1}{4z} u_1' - \left( 1 + \frac{2\alpha_1}{z^2} - 2u_0 \right) u_1 = 0.$$ 

Hence, for the coefficients $c_n$ we can take $c_0 = 1$ and for the others we have

$$nc_n = \frac{1}{4} \left( n - \frac{1}{2} \right)^2 c_{n-1} + 2 \sum_{k=4}^{n+1} b_k c_{n+1-k}, \quad n \geq 1.$$
The first five coefficients are
\[c_0 = 1, \quad c_1 = \frac{1}{16}, \quad c_2 = \frac{9}{512}, \quad c_3 = \frac{75}{8192} + \frac{2}{3} \alpha_2, \quad c_4 = \frac{3675}{524288} + \frac{13}{24} \alpha_2.\]

In a similar manner it can be shown that when the positive real axis is crossed the Stokes phenomenon switches on a divergent series
\[u_2(z) = iKe^{-2z}z^{-1/2} \sum_{n=0}^{\infty} (-1)^n c_n z^{-n}.\]

This is all the information that is needed to conclude that
\[\alpha_n = b_{2n} \sim \frac{K}{\pi} \sum_{k=0}^{\infty} (-1)^k c_k \frac{\Gamma(2n - k - \frac{1}{2})}{2^{2n-k-(1/2)}}, \quad \text{as} \quad n \to \infty.\]

By taking the first 4 terms in this expansion we can verify that (15) holds for any complex constant \(\alpha_1\).

For more details see [12], [13] and [14]. (It’s best to get the version of the first reference on the website http://www.maths.ed.ac.uk/ adri/public.htm.)

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**References**


