Asymptotics of Some Convolutional Recurrences

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Abstract

We study the asymptotic behavior of the terms in sequences satisfying recurrences of the form $a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n,k)a_k a_{n-k}$ where, very roughly speaking, $f(n,k)$ behaves like a product of reciprocals of binomial coefficients. Some examples of such sequences from map enumerations, Airy constants, and Painlevé I equations are discussed in detail.

1 Main results

There are many examples in the literature of sequences defined recursively using a convolution. It often seems difficult to determine the asymptotic behavior of such sequences. In this note we study the asymptotics of a general class of such sequences. We prove
subexponential growth by using an iterative method that may be useful for other recurrences. By subexponential growth we mean that, for every constant \( D > 1 \), \( a_n = o(D^n) \) as \( n \to \infty \). Thus our motivation for this note is both the method and the applications we give.

Let \( d > 0 \) be a fixed integer and let \( f(n, k) \geq 0 \) be a function that behaves like a product of some powers of reciprocals of binomial coefficients, in a general sense to be specified in Theorem 1. We deal with the sequence \( a_n \) for \( n \geq d \) where \( a_d, a_{d+1}, \ldots, a_{2d-1} \geq 0 \) are arbitrary and, when \( n \geq 2d \),

\[
a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n, k)a_ka_{n-k}.
\]

Without loss of generality,

we assume that \( f(n, k) = f(n, n - k) \)

since we can replace \( f(n, k) \) and \( f(n, n - k) \) in (1) with \( \frac{1}{2}(f(n, k) + f(n, n - k)) \).

Theorem 1 proves subexponential growth. Theorem 2 provides more accurate estimates under additional assumptions. In Section 2, we apply the corollary to some examples.

**Theorem 1 (Subexponential growth)** Let \( a_n \) be defined by recursion (1) with \( a_d > 0 \). Suppose there is a function \( R(x) \) defined on \((0, 1/2]\), an \( \alpha > 0 \) and an \( r \) such that

(a) \( 0 < R(x) < r < 1 \),

(b) \( \lim_{x \to 0^+} R(x) = 0 \),

(c) \( 0 \leq f(n, k) = O\left(n^{-\alpha}R^{k-d}(k/n)\right) \) uniformly for \( d \leq k \leq n/2 \).

Then \( a_n \) grows subexponentially; in fact,

\[
a_n = (1 + O(n^{-\alpha}))a_{n-1}.
\]

**Proof:** We first note that the \( a_n \) are non-decreasing when \( n \geq 2d - 1 \).

Our proof is in three steps. We first prove that \( a_n = O(C^n) \) for some constant \( C > 2 \). We then prove that \( C \) can be chosen very close to 1. Finally we deduce (2) and subexponential growth.

**First Step:** Since the bound in (c) is bounded by some constant times the geometric series \( n^{-\alpha}r^{k-d} \) with ratio less than 1, \( \sum_{k=d}^{n-d} f(n, k) = O(n^{-\alpha}) \). Hence we can choose \( M \) so large that \( \sum_{k=d}^{n-d} f(n, k) < 1/4 \) when \( n > M \). Next choose \( C > 2 \) so large (\( C = \max\{a_d, a_{d+1}, \ldots, a_{2d-1}, a_M, 2\} \) will do) that \( a_n < 2C^n \) for \( n \leq M \). By induction, using the recursion (1), we have for \( n > M \)

\[
a_n < 2C^{n-1} + (1/4)4C^n \leq C^n + C^n = 2C^n.
\]
Second Step: By (b) there is a \( \lambda \in (0, 1/2) \) such that \( R(x) < \frac{1}{2x} \) for \( 0 < x < \lambda \). Fix any \( D \leq C \) such that \( a_n = O(D^n) \), which is true for \( D = C \) by the First Step.

Split the sum in (1) into \( \lambda n \leq k \leq (1 - \lambda)n \) and the rest, calling the first range of \( k \) the “center” and the rest the “tail”. Noting \( r < 1 \), the center sum is bounded by

\[
2 \sum_{k=\lambda n+1}^{\lambda n/2} f(n, k) a_k a_{n-k} = O(D^n \sum_{k=\lambda n+1}^{\lambda n/2} r^{k-d}) = O((r^\lambda D)^n).
\] (3)

Since \( a_j \) are increasing, the tail sum is bounded by

\[
2 \sum_{k=d}^{\lambda n} f(n, k) a_k a_{n-k} = O(n^{-\alpha}a_{n-1} \sum_{k=d}^{\lambda n} R(x)^{k-d}D^k)
\] (4)

\[= O(n^{-\alpha}a_{n-1} \sum_{k=d}^{\lambda n} (DR(x))^{k-d} = O(n^{-\alpha}a_{n-1}),
\]

where the last equality follows from the fact that \( DR(x) < 1/2 \). Combining (3) and (4),

\[
a_n = \left(1 + O(n^{-\alpha})\right) a_{n-1} + O((r^\lambda D)^n).
\] (5)

When \( r^\lambda D > 1 \), induction on \( n \) easily leads to \( a_n = O((r^\lambda D')^n) \) for any \( D' > D \), an exponential growth rate no larger than \( r^\lambda D' \).

Since \( r^\lambda \) has a fixed value less than one, we can iterate this process, replacing \( D \) by \( r^\lambda D' \) at the start of the Second Step. We finally obtain a growth rate \( D > 1 \) with \( r^\lambda D < 1 \). This completes the second step.

Third Step: With the value of \( D \) just obtained, the last term in (5) is exponentially small and hence is \( O(n^{-\alpha}a_{n-1}) \). Thus we obtain (2) which immediately implies subexponential growth of \( a_n \), since \( 1 + O(n^{-\alpha}) < D \) for any \( D > 1 \) and sufficiently large \( n \).

To say more than (2), we need additional information about the behavior of the \( f(n, k) \). When \( f(n, k)/f(n, d) \) is small for each \( k \) in the range \( d+1 \leq k \leq n-d-1 \), the first and last terms dominate the sum. The following theorem is based on this observation.

Theorem 2 (Asymptotic behavior) Assume (a)–(c) of Theorem 1 hold. Suppose further that there is a \( \beta > 0 \) such that

\[
\frac{f(n, k)}{f(n, d)} = O(n^{-\beta}r^{k-d-1}) \quad \text{uniformly for} \quad d+1 \leq k \leq n/2.
\] (6)

Then

\[
\log a_n = 2a_d \sum_{k=2d+1}^{n} f(k, d) + O\left( \sum_{k=2d}^{n} f(k, d) (k^{-\alpha} + k^{-\beta}) \right).
\] (7)
Proof: We assume \( n > 2d \). Remove the \( k = d \) and \( k = n - d \) terms from the sum in (1). We first deal with the remaining sum. Theorem 1 gives \( a_k = O(D^k) \) for all \( D > 1 \), so we can assume \( D < 1/r \). Using (6)

\[
\sum_{k=d+1}^{n-d-1} f(n, k)a_k a_{n-k} = O(f(n, d)n^{-\beta}a_{n-1}) \sum_{k=d+1}^{n/2} r^{k-d-1}D^k
\]

Combining this with (1), we obtain

\[
a_n = a_{n-1} + 2a_d f(n, d)a_{n-d} + f(n, d)O(n^{-\beta})a_{n-1}
\]

Taking logarithms and noting for expansion purposes that \( f(n, d) = O(n^{-\alpha}) \), we obtain

\[
\log a_n - \log a_{n-1} = 2a_d f(n, d) + O((n^{-\alpha} + n^{-\beta}) f(n, d)).
\]

Sum over \( n \) starting with \( n = 2d + 1 \). The theorem follows immediately when we note that the constant terms can be incorporated into the \( O() \) in (7) since the sum therein is bounded below by a nonzero constant.

Corollary 1 Assume the conditions of Theorem 2 hold and \( f(n, d) = \Theta(n^{-\alpha}) \).

- If \( \alpha < 1 \), then \( a_n = \exp(\Theta(n^{1-\alpha})) \).
- If \( \alpha > 1 \), then \( a_n = K + O(n^{1-\alpha}) \) for some constant \( K \).
- If \( f(n, d) - A/n \) are the terms of a convergent series, then \( a_n \sim Cn^{2Aa_d} \) for some positive constant \( C \).

Proof: Since \( \alpha > 0 \) and \( \beta > 0 \), (7) gives \( \log a_n = \Theta(\sum_{k=2d+1}^n k^{-\alpha}) \). The case \( \alpha < 1 \) follows immediately; for \( \alpha > 1 \), we see that \( a_n \) is bounded and nondecreasing and therefore has a limit \( K \). For \( m > n \), (2) gives \( \log(a_m/a_n) = O(n^{1-\alpha}) \) uniformly in \( m \). Letting \( m \to \infty \), we obtain the claim regarding \( \alpha > 1 \).

For \( \alpha = 1 \), the first sum in (7) is \( A \log n + B + o(1) \) for some constant \( B \), and the last sum in (7) converges.

2 Examples

We apply Theorem 2 and Corollary 1 to some recursions which arise from combinatorial applications. In our examples, \( f(n, k) \) behaves like a product of the reciprocal of binomial coefficients, which satisfies the conditions of Theorems 1 and 2. A more general case of interest is when \( f(n, k) \) takes the form of the product of functions like

\[
g(n, k) = \frac{[a]_k [a]_{n-k}}{[a]_n}
\]
for some constant \( a > 0 \), where \([x]_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}\), the rising factorial.

We note that when \( a = 1 \), \( g(n, k) = \binom{n}{k}^{-1} \).

We begin with some useful bounds. When \( a > 0 \) and \( 1 \leq k \leq n/2 \),

\[
g(n, k) = \prod_{j=0}^{k-1} \frac{a+j}{a+n-k+j} < \left( \frac{a+k}{a+n} \right)^k
\]

\[
\leq \left( \frac{k}{n} \right)^k \left( 1 + \frac{a}{k} \right)^k = O \left( \left( \frac{k}{n} \right)^k \right) = O \left( n^{-1}(3k/2n)^{k-1} \right)
\]

since \( k(2/3)^{k-1} \) is bounded. So \( g \) satisfies the condition on \( f \) in Theorem 1(c), with \( \alpha = 1 \).

Similarly, when \( a > 0 \) and \( d \leq k \leq n/2 \),

\[
g(n, k) = \frac{g(n, k)}{g(n, d)} = \prod_{j=0}^{k-d-1} \frac{a+d+j}{a+n-k+d+j} = O \left( n^{-1}(3k/2n)^{k-d-1} \right).
\]

This is in accordance with (6) with \( \beta = 1 \).

Example 1 (Map enumeration constants) There are numbers \( t_n \) appearing in the asymptotic enumeration of maps in an orientable surface of genus \( n \), whose value does not concern us here. Define \( u_n \) by

\[
t_n = 8 \frac{\left[ \frac{1}{5} \right]_n \left[ \frac{4}{5} \right]_{n-1}}{\Gamma \left( \frac{5n-1}{2} \right)} \left( \frac{25}{96} \right)^n u_n.
\]

Then \( u_1 = 1/10 \) and \( u_n \) satisfies the following recursion [3]

\[
u_n = u_{n-1} + \sum_{k=1}^{n-1} f(n, k) u_k u_{n-k} \quad \text{for} \quad n \geq 2,
\]

where

\[
f(n, k) = \frac{\left[ \frac{1}{5} \right]_k \left[ \frac{1}{5} \right]_{n-k} \left[ \frac{4}{5} \right]_{k-1} \left[ \frac{4}{5} \right]_{n-k-1}}{\left[ \frac{1}{5} \right]_n \left[ \frac{4}{5} \right]_{n-1}}.
\]

From the observations above, the conditions of Theorem 2 are satisfied with \( d = 1 \), \( R(\lambda) = (3\lambda/2)^2 \) and \( \alpha = \beta = 2 \). Hence, \( u_n \sim K \) for some constant \( K \). Unlike the proof in [3], this does not depend on the value of \( u_1 \).  

Example 2 (Airy constants) The Airy constants \( \Omega_n \) are determined by \( \Omega_1 = 1/2 \) and the recurrence [7]

\[
\Omega_n = (3n-4)n\Omega_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} \Omega_k \Omega_{n-k} \quad \text{for} \quad n \geq 2.
\]
Let $\Omega_n = n! \left[\frac{2}{3}\right]_{n-1} 3^n a_n$. Then $a_n$ satisfies (1) with $d = 1$ and
\[
f(n, k) = \frac{\left[\frac{2}{3}\right]_{k-1} \left[\frac{2}{3}\right]_{n-k-1}}{\left[\frac{2}{3}\right]_{n-1}}.
\]

Theorem 2 applies with $d = 1$, $R(\lambda) = 3\lambda/2$ and $\alpha = \beta = 1$. Since
\[
f(n, 1) = \frac{1}{n - 4/3} = \frac{1}{n} + \frac{4/3}{n(n - 4/3)}
\]
and $a_1 = 1/6$, we have $a_n \sim Cn^{1/3}$ for some constant $C$.

We note that it is possible to apply the result of Olde Daalhuis [13] to obtain a full asymptotic expansion for $\Omega_n$. Let
\[
A_n = \frac{\Omega_n}{3^n n!}.
\]

Then the recursion for $\Omega_n$ becomes
\[
A_n = (n - 4/3) A_{n-1} + \sum_{k=1}^{n-1} A_k A_{n-k}, \ n \geq 2.
\]

It follows that the formal series
\[
F(z) = \sum_{n \geq 1} A_n \frac{z^n}{n!}
\]
satisfies the Riccati equation
\[
F'(z) + \left(1 + \frac{1}{3z}\right) F(z) - F^2(z) - \frac{1}{6z} = 0.
\]

It then follows from the result of Olde Daalhuis [13] that
\[
A_n \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} b_k \Gamma(n - k), \ \text{as} \ n \to \infty,
\]
where $b_0 = 1$ and $b_k$ can be computed using the recursion
\[
b_k = -\frac{2}{k} \sum_{j=2}^{k+1} b_{k+1-j} A_j, \ k \geq 1.
\]

In particular, we have
\[
\Omega_n \sim \frac{1}{2\pi} \frac{1}{\Gamma(n) 3^n n!} = \frac{1}{2\pi n} \frac{(n!)^2 3^n}{n^n}, \ \text{as} \ n \to \infty.
\]

It is well known that solutions to the Riccati equation have infinitely many singularities, hence $F(z)$ (via its Borel transform [2]) cannot satisfy a linear ODE with polynomial coefficients. This implies that the sequence $A_n$ (and hence $\Omega_n$) is not holonomic.
Example 3 The following recursion, with \( \ell > 0 \) and \( \ell \neq 1/2 \), appeared in [6]. The Airy constants are the special case \( \ell = 1 \). The case \( \ell = 2 \) corresponds to the recursion studied in [9, 10], which arises in the study of the Wiener index of Catalan trees. We have 
\[ C_1 = \frac{\Gamma((\ell - 1/2))}{\sqrt{\pi}} \] 
and, for \( n \geq 2 \),
\[ C_n = n \frac{\Gamma(n\ell + (n/2) - 1)}{\Gamma((n - 1)\ell + (n/2) - 1)} C_{n-1} + \frac{1}{4} \sum_{k=1}^{n-1} \binom{n}{k} C_k C_{n-k}. \] (11)
Define \( a_n \) by 
\[ C_n = n! g(n) a_n \] 
where \( g(1) = 1 \) and
\[ g(m) = \prod_{k=2}^{m} \frac{\Gamma(k\ell + (k/2) - 1)}{\Gamma((k-1)\ell + (k/2) - 1)}. \]
Then (11) becomes
\[ a_n = a_{n-1} + \sum_{k=1}^{n-1} \frac{g(k) g(n-k)}{4g(n)} a_k a_{n-k}, \]
so \( f(n,k) = g(k) g(n-k)/4g(n) \).

With \( a \) fixed and \( x \to \infty \) and using 6.1.47 on p.257 of [1] (or using Stirling’s formula), we have
\[ \frac{\Gamma(x+a)}{\Gamma(x)} = x^a \left( 1 + \frac{a(a-1)}{2x} + O(1/x^2) \right) \]
\[ = x^a \left( 1 + \frac{a-1}{2x} \right)^a \left( 1 + O(1/x^2) \right) \] (12)
\[ = \left( x + \frac{a-1}{2} \right)^a \left( 1 + O(1/x^2) \right). \] (13)
When \( m > 1 \), (13) gives us
\[ g(m) = \prod_{k=2}^{m} \left( \frac{(2\ell + 1)k - \ell - 3}{2} \right)^\ell \left( 1 + O(1/k^2) \right) \]
\[ = \Theta(1) \left( (\ell + 1/2)^m \prod_{k=2}^{m} \left( k - \frac{\ell + 3}{2\ell + 1} \right)^\ell \right) \]
\[ = \Theta(1) \left( (\ell + 1/2)^m [a]_{m-1} \right)^\ell, \] where \( a = \frac{3\ell - 1}{2\ell + 1}. \)

Hence
\[ f(n,k) = \Theta(1) \left| [a]_{k-1} [a]_{n-k-1} \right|^\ell \left| [a]_{n-1} \right|. \]
where the absolute values have been introduced to allow for \( a < 0 \). A slight adjustment of the argument leading to (8) and (9) leads to
\[ f(n,k) = O(n^{-\ell}(3k/2n)^{\ell(k-1)}) \] and \[ \frac{f(n,k)}{f(n,1)} = O(n^{-\ell}(3k/2n)^{\ell(k-d-1)}) \]
for \(1 \leq k \leq n/2\). Hence Theorem 2 applies with \(\alpha = \beta = \ell\), and \(a_n\) converges to a constant when \(\ell > 1\) by Corollary 1.

It is interesting to note that there is a simple relation between the sequence \(u_n\) in Example 1 and the sequence \(a_n\) in Example 3 with \(\ell = 2\). It is not difficult to check that the \(f(n,k)\) defined in Example 3 is exactly five times the \(f(n,k)\) in Example 1: since \(a_1 = 5u_1\), we have \(a_n = 5u_n\) for all \(n \geq 1\). This simple relation suggests a relationship between the number of maps on an orientable surface of genus \(g\) and the \(g\)th moment of a particular toll function on a certain type of trees. Using a bijective approach, Chapuy [4] recently found an expression for \(t_g\) as the \(g\)th moment of the labels in a random well-labelled tree.

### 3 A convolutional recursion arising from Painlevé I

The following is recursion (44) in [11].

\[
\alpha_n = (n-1)^2\alpha_{n-1} + \sum_{k=2}^{n-2} \alpha_k \alpha_{n-k}, \quad n \geq 1, \quad n \geq 3. 
\]

(14)

It follows from Proposition 14 of [11] that, for \(0 < \alpha_1 < 1\) and \(\alpha_2 = \alpha_1 - \alpha_1^2\),

\[
\alpha_n = c(\alpha_1)((n-1)!)^2 \left(1 - \frac{2\alpha_2(n-3)}{3(n-1)^2(n-2)^2} + \delta_n\right),
\]

(15)

where \(c(\alpha_1)\) depends only on \(\alpha_1\), and

\[\delta_n = O(1/n^4).\]

We note that \(\alpha_n\) for \(n \geq 3\) depends only on \(\alpha_2\). The proof of (15) relies on the fact that \(0 < \alpha_2 < 1/4\) for \(0 < \alpha_1 < 1\). It is conjectured in [11] that the asymptotic expression (15) actually holds for a wider range of values of \(\alpha_1\).

For \(n \geq 1\), let

\[p_n = \frac{\alpha_n}{((n-1)!)^2}.
\]

Then, as shown in [11], \(p_n\) satisfies recursion (1) with \(d = 2\) and

\[f(n,k) = \left(\frac{(n-k-1)!(k-1)!}{(n-1)!}\right)^2.
\]

We note here \(f(n,2) = O(n^{-4})\). It follows from Theorem 2 that

\[p_n = p(1 + \epsilon_n) \quad \text{for any } \alpha_2 > 0,
\]

where \(p = p(\alpha_2)\) is a positive constant and \(\epsilon_n = O(1/n^3)\).
It is also interesting to note that, with $\alpha_1 = 1/50$, $\alpha_2 = 49/2500$, the sequence $\alpha_n$ is related to the sequence $u_n$ in Example 1 by

$$\alpha_n = \left[\frac{1}{5}\right]_n \left[\frac{4}{5}\right]_{n-1} u_n.$$  

As mentioned in [11], the formal series $v(t) = \sum_{n \geq 1} \alpha_n t^{-n}$ satisfies

$$t^2 v'' + tv' - (t + 2\alpha_1)v + tv^2 + \alpha_1 = 0,$$

and hence, with

$$t = \frac{8\sqrt{6}}{25} x^{5/2},$$

$y(x) = (x/6)^{1/2}(1 - 2v(t))$ satisfies the following Painlevé I:

$$y'' = 6y^2 - x.$$  

This connection with Painlevé I is used in [8] to show that the sequence $\alpha_n$ is not holonomic (It follows that $u_n$ and $t_n$ in Example 1 are also not holonomic). The proof uses the fact that solutions to Painlevé I have infinitely many singularities and hence cannot satisfy a linear ODE with polynomial coefficients.

In the following we apply the techniques of [14] to prove that (15) holds for any complex constant $\alpha_1$. It is convenient to introduce the formal series

$$u_0(z) = v(z^2) = \sum_{n=2}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} \alpha_n z^{-2n}.$$  

It follows from (16) that $u = u_0(z)$ is a formal solution to the differential equation

$$\frac{1}{4} u'' + \frac{1}{4z} u' - \left(1 + \frac{2\alpha_1}{z^2}\right) u + u^2 + \frac{\alpha_1}{z^2} = 0.$$  

The Stokes lines for this differential equation are the positive and the negative real axes. When the negative real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_1(z) = Ke^{2z} z^{-1/2} \sum_{n=0}^{\infty} c_n z^{-n},$$

in which the Stokes multiplier $K$ is a constant (depending on the constant $\alpha_1$). To determine the coefficients $c_n$ we observe that $u_1$ is a solution of the linear differential equation

$$\frac{1}{4} u_1'' + \frac{1}{4z} u_1' - \left(1 + \frac{2\alpha_1}{z^2} - 2u_0\right) u_1 = 0.$$  

Hence, for the coefficients $c_n$ we can take $c_0 = 1$ and for the others we have

$$nc_n = \frac{1}{4} \left(n - \frac{1}{2}\right)^2 c_{n-1} + 2 \sum_{k=4}^{n+1} b_k c_{n+1-k}, \quad n \geq 1.$$
The first five coefficients are
\[ c_0 = 1, \quad c_1 = \frac{1}{16}, \quad c_2 = \frac{9}{512}, \quad c_3 = \frac{75}{8192} + \frac{2}{3} \alpha_2, \quad c_4 = \frac{3675}{524288} + \frac{13}{24} \alpha_2. \]

In a similar manner it can be shown that when the positive real axis is crossed the Stokes phenomenon switches on a divergent series
\[ u_2(z) = iKe^{-2z}z^{-1/2} \sum_{n=0}^{\infty} (-1)^n c_n z^{-n}. \]

This is all the information that is needed to conclude that
\[ \alpha_n = b_{2n} \sim \frac{K}{\pi} \sum_{k=0}^{\infty} (-1)^k c_k \frac{\Gamma(2n-k-\frac{1}{2})}{2^{2n-k-(1/2)}}, \quad \text{as} \quad n \to \infty. \]

By taking the first 4 terms in this expansion we can verify that (15) holds for any complex constant \( \alpha_1 \).

For more details see [12], [13] and [14]. (It’s best to get the version of the first reference on the website http://www.maths.ed.ac.uk/ adri/public.htm.)

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References


