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Sessions as Propositions

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Recently, Wadler presented a continuation-passing translation from a session-typed functional language, GV, to a process calculus based on classical linear logic, CP. However, this translation is one-way: CP is more expressive than GV. We propose an extension of GV, called HGV, and give translations showing that it is as expressive as CP. The new translations shed light both on the original translation from GV to CP, and on the limitations in expressiveness of GV.

1 Introduction

Linear logic has long been regarded as a potential typing discipline for concurrency. Girard [7] observes that the connectives of linear logic can be interpreted as parallel computation. Abramsky [1] and Bellin and Scott [2] interpret linear logic proofs as $\pi$-calculus processes. While they provide $\pi$-calculus interpretations of all linear logic proofs, they do not provide a proof-theoretic interpretation for arbitrary $\pi$-calculus terms. Caires and Pfenning [3] give a propositions-as-types correspondence between intuitionistic linear logic and session types, interpreting linear logic propositions as session types for a restricted $\pi$-calculus, $\pi$DILL. Of particular importance to this work, they interpret the multiplicative connectives as prefixing, and the exponentials as replicated processes.

Wadler [8] adapts Caires and Pfenning’s work to classical linear logic, interpreting proofs as processes in a restricted $\pi$-calculus, CP. Additionally, Wadler shows that a core session-typed linear functional language, GV, patterned after a similar language due to Gay and Vasconcelos [6], may be translated into CP. However, GV is less expressive than CP: there are proofs which do not correspond to any GV program.

Our primary contribution is HGV (Harmonious GV), a version of GV extended with constructs for session forwarding, replication, and polymorphism. We identify HGV$\pi$, the session-typed fragment of HGV, and give a type-preserving translation from HGV$\pi$ to HGV$\pi^\star$: this translation depends crucially on the new constructs of HGV. We show that HGV is sufficient to express all linear logic proofs by giving type-preserving translations from HGV$\pi$ to CP (\$\llbracket-\rrbracket\$), and from CP to HGV$\pi$ (\$\llparenthesis-\rrparenthesis\$). Factoring the translation of HGV into CP through (\$-\$$)$^\star$ simplifies the presentation, and illuminates regularities that are not apparent in Wadler’s original translation of GV into CP. Finally, we show that HGV, HGV$\pi$, and CP are all equally expressive.

2 The HGV Language

This section describes our session-typed language HGV, contrasting it with Gay and Vasconcelos’s functional language for asynchronous session types [6], which we call LAST, and Wadler’s GV [8]. In designing HGV, we have opted for programming convenience over uniformity, while insisting on a tight correspondence with linear logic. The session types of HGV are given by the following grammar:

$$S ::= !T.S \mid ?T.S \mid \oplus \{l_i : S_i\}_i \mid \& \{l_i : S_i\}_i \mid \text{end}_1 \mid \text{end}_2 \mid X \mid \overline{X} \mid !X].S \mid ?X].S \mid bS \mid \#S$$
Types for input ($T.S$), output ($!T.S$), selection ($\oplus\{l_i : S_i\}_i$) and choice ($\&\{l_i : S_i\}_i$) are standard. Like GV, but unlike LAST, we distinguish output (end₁) and input (end₂) session ends; this matches the situation in linear logic, where there is no conveniently self-dual proposition to represent the end of a session. Variables and their duals ($X, \overline{X}$) and type input ($?[X].S$) and output ($![X].S$), permit definition of polymorphic sessions. We include a notion of replicated sessions, corresponding to exponentials in linear logic: a channel of type $\flat S$ is a “service”, providing any number of channels of type $S$; a channel of type $\flat S$ is the “server” providing such a service. Each session type $S$ has a dual $\overline{S}$ (with the obvious dual for variables $X$):

$\overline{X} = \overline{X} \quad \overline{\overline{X}} = X \quad \overline{X + Y} = \overline{X} + \overline{Y} \quad \overline{X \times Y} = \overline{X} \times \overline{Y} \quad \overline{X^Y} = (\overline{X})^Y \quad \overline{X \rightarrow Y} = \overline{X} \rightarrow \overline{Y}$

Note that dualisation leaves input and output types unchanged. In addition to sessions, HGV’s types include linear pairs, and linear and unlimited functions:

$T, U, V ::= S \mid T \otimes U \mid T \rightarrow U \mid T \rightarrow o U \mid T \rightarrow U$

Every type $T$ is either linear ($\text{lin}(T)$) or unlimited ($\text{un}(T)$); the only unlimited types are services ($\text{un}(\flat S)$), unlimited functions ($\text{un}(T \rightarrow U)$), and end input session types ($\text{un}(\text{end}_2)$). In GV, end₁ is linear. We choose to make it unlimited in HGV because then we can dispense with GV’s explicit terminate construct while maintaining a strong correspondence with CP—end₂ corresponds to $\perp$ in CP, for which weakening and contraction are derivable.

Figure 11 gives the terms and typing rules for HGV: the first block contains the structural rules, the second contains the (standard) rules for lambda terms, and the third contains the session-typed fragment. The fork construct provides session initiation, filling the role of GV’s with...connect...to...structure, but without the asymmetry of the latter. The two are interdefinable, as follows:

$\text{fork } x.M \equiv \text{with } x \text{ connect } M \text{ to } x \quad \text{with } x \text{ connect } M \text{ to } N \equiv \text{let } x = \text{fork } x.M \text{ in } N$

We add a construct link $M N$ to implement channel forwarding; this form is provided in neither GV nor LAST, but is necessary to match the expressive power of CP. (Note that while we could define session forwarding in GV or LAST for any particular session type, it is not possible to do so in a generic fashion.) We add terms sendType $S.M$ and receiveType $X.M$ to provide session polymorphism, and serve $x.M$ and request $M$ for replicated sessions. Note that, as the body $M$ of serve $x.M$ may be arbitrarily replicated, it can only refer to the unlimited portion of the environment. Channels of type $\flat S$ offer arbitrarily many sessions of type $S$; correspondingly, channels of type $\flat S$ must consume arbitrarily many $S$ sessions. The rule for serve $x.M$ parallels that for fork: it defines the server (which replicates $M$) and returns the channel by which it may be used (of type $\overline{S} = \overline{\overline{\overline{S}}}$). As a consequence, there is no rule involving type $\flat S$. We experimented with having such a rule, but found that it was always used immediately inside a fork, while providing no extra expressive power. Hence we opted for the rule presented here.

3 From HGV to HGV$\pi$

The language HGV$\pi$ is the restriction of HGV to session types, that is, HGV without $\rightarrow o$, $\rightarrow$, or $\otimes$. In order to avoid $\otimes$, we disallow plain receive $M$, but do permit it to be fused with a pair elimination let $(x,y) = \text{receive } M \text{ in } N$. We can simulate all non-session types as session types via a translation...
Figure 1: Typing rules for HGV
from HGV to HGV$\pi$. The translation on types is given by the homomorphic extension of the following
equations:

$$(T \to U)^* = !(T)^* . (U)^* \quad (T \to U)^* = \sharp((T)^* . (U)^*) \quad (T \otimes U)^* = ?(T)^* . (U)^*$$

Each target type is the *interface* to the simulated source type. A linear function is simulated by input on a channel; its interface is output on the other end of the channel. An unlimited function is simulated by a server; its interface is the service on the other end of that channel. A tensor is simulated by output on a channel; its interface is input on the other end of that channel. This duality between implementation and interface explains the flipping of types in Wadler’s original CPS translation from GV to CP. The translation on terms is given by the homomorphic extension of the following equations:

$$(\lambda x.M)^* = \text{fork } z . \text{let } (x, z) = \text{receive } z \text{ in link } (M)^* \ z$$

$$(L M)^* = \text{send } (M)^* \ (L)^*$$

$$(M, N)^* = \text{fork } z . \text{link } (\text{send } (M)^* \ z) \ (N)^*$$

$$(\text{let } (x, y) = M \text{ in } N)^* = \text{let } (x, y) = \text{receive } (M)^* \text{ in } (N)^*$$

$$(L : T \to U)^* = \text{serve } z . \text{link } (L)^* \ z$$

$$(L : T \to U)^* = \text{request } (L)^*$$

$$(\text{receive } M)^* = (M)^*$$

Formally, this is a translation on derivations. We write type annotations to indicate $\to$ introduction and elimination. For all other cases, it is unambiguous to give the translation on plain term syntax. Each introduction form translates to an interface in $\lambda M$ of type $S$, where $M : \text{end}$; provides the implementation, with $z : S$ bound in $M$. We can extend the translation on types to a translation on contexts:

$$(x_1 : T_1, \ldots, x_n : T_n)^* = x_1 : (T_1)^*, \ldots, x_n : (T_n)^*$$

It is straightforward to verify that our translation preserves typing.

**Theorem 1** If $\Phi \vdash M : T$ then $(\Phi)^* \vdash (M)^* : (T)^*$.

### 4 From HGV$\pi$ to CP

We present the typing rules of CP in Figure 2. Note that the propositions of CP are exactly those of classical linear logic, as are the cut rules (if we ignore the terms). Thus, CP enjoys all of the standard meta-theoretic properties of classical linear logic, including confluence and weak normalisation. A minor syntactic difference between our presentation and Wadler’s is that our sum ($\oplus$) and choice ($\otimes$) types are $n$-ary, matching the corresponding session types in HGV, whereas he presents binary and nullary versions of sum and choice. Duality on CP types $((\cdot)^\bot)$ is standard:

$$(A \otimes B)^\bot = A^\bot \otimes B^\bot \quad (\oplus \{i : A_i\}_i)^\bot = \& \{i : A_i^\bot\}_i \quad 1^\bot = \bot \quad (\exists X . B)^\bot = \forall X . B^\bot \quad (!A)^\bot = ?A^\bot$$

$$(A \otimes B)^\bot = A^\bot \otimes B^\bot \quad (\& \{i : A_i\}_i)^\bot = \oplus \{i : A_i^\bot\}_i \quad \bot^\bot = 1 \quad (\forall X . B)^\bot = \exists X . B^\bot \quad (?A)^\bot = !A^\bot$$

The semantics of CP terms follows the cut elimination rules in classical linear logic. We interpret the cut relation $\rightarrow$ modulo $\alpha$-equivalence and structural cut equivalence:

$$x \leftrightarrow y \equiv y \leftrightarrow x$$

$$\text{vx.}(P \mid Q) \equiv \text{vx.}(Q \mid P)$$

$$\text{vy.}(\text{vx.}(P \mid Q) \mid R) \equiv \text{vx.}(P \mid \text{vy.}(Q \mid R))$$
The principal cut elimination rules correspond to communication between processes.

\[
\begin{align*}
\forall x.(w \leftrightarrow x \mid P) & \rightarrow P[w/x] \\
\forall x.(x[y].(P \mid Q) \mid x(y).R) & \rightarrow vy.(P \mid \forall x.Q \mid R) \\
\forall x.(x[l_i].P \mid x.\text{case } \{l_i.Q_i\}) & \rightarrow vx.(P \mid Q_i) \\
\forall x.(x[y].P \mid ?x[y]Q) & \rightarrow vy.(P \mid Q) \\
\forall x.(x[y].P \mid Q) & \rightarrow Q, \quad x \notin FV(Q) \\
\forall x.(x[y].P \mid Q[x'/x']) & \rightarrow vx.(x[y].P \mid ?x'(y).P \mid Q) \\
\forall x.(x[A].P \mid x(X).Q) & \rightarrow vx.(P \mid Q[A/X]) \\
\forall x.(x[].0 \mid x().P) & \rightarrow P
\end{align*}
\]

Finally, we provide commuting conversions, moving communication under unrelated prefixes.

\[
\begin{align*}
\forall z.(x[y].(P \mid Q) \mid R) & \rightarrow x[y].(\forall z.(P \mid Q) \mid R), \quad \exists z \in FV(P) \\
\forall z.(x[y].(P \mid Q) \mid R) & \rightarrow x[y].(P \mid \forall z.(Q \mid R)), \quad \exists z \in FV(Q) \\
\forall z.(x(y).P \mid Q) & \rightarrow x(y).\forall z.(P \mid Q) \\
\forall z.(x[l_i].P \mid Q) & \rightarrow x[l_i].\forall z.(P \mid Q) \\
\forall z.(x.\text{case } \{l_i.Q_i\}) \mid R & \rightarrow x.\text{case } \{l_i.Q_i \mid R\} \\
\forall z.(x[y].P \mid Q) & \rightarrow !x(y).\forall z.(P \mid Q) \\
\forall z.(x[y].P \mid Q) & \rightarrow ?x[y].\forall z.(P \mid Q) \\
\forall z.(x[A].P \mid Q) & \rightarrow x[A].\forall z.(P \mid Q) \\
\forall z.(x(X).P \mid Q) & \rightarrow x(X).\forall z.(P \mid Q) \\
\forall z.(x().P \mid Q) & \rightarrow x().\forall z.(P \mid Q)
\end{align*}
\]

A fuller account of CP can be found in Wadler’s work [8].

We now give a translation from HGV\(\pi\) to CP. Post composing this with the embedding of HGV in HGV\(\pi\) yields a semantics for HGV. The translation on session types is as follows:

\[
\begin{align*}
\text{[T]S} & = [T] \perp \otimes [S] \\
\text{[\oplus \{l_i : S_i\}]_i} & = \oplus \{l_i : [S_i]_i\} \\
\text{[\oplus \{l_i : S_i\}]_i} & = \otimes \{l_i : [S_i]_i\} \\
\text{[\oplus S]} & = ![S] \\
\text{[\oplus S]} & = ?![S] \\
\text{[\oplus S]} & = \exists X.[S] \\
\text{[\oplus S]} & = \forall X.[S] \\
\text{[\oplus S]} & = 1 \\
\text{[\oplus S]} & = \bot \\
\text{[X]} & = X \\
\text{[X]} & = X \perp 
\end{align*}
\]
The translation on terms makes use of π. We now present the translation on terms formally specified as a CPS translation on derivations as in Wadler’s presentation. We provide the full translations of weakening and contraction for endγ, as these steps are implicit in the syntax of HGV terms. The other constructs depend only on the immediate syntactic structure, so we abbreviate their translations as mappings on plain terms:

\[
\begin{align*}
\Phi \vdash N : S \\
\Phi, x : \text{end}_\gamma \vdash N : S
\end{align*}
\]

\[
\begin{align*}
\Phi, x : \text{end}_\gamma, x' : \text{end}_\gamma \vdash N[x/x'] : S \\
\Phi, x : \text{end}_\gamma \vdash N[x/x'] : S
\end{align*}
\]

\[
\frac{\frac{\Phi \vdash N : S}{\Phi[x/x'] \vdash N[x/x'] : S}}{\frac{\Phi, x : \text{end}_\gamma \vdash N : S}{\Phi, x : \text{end}_\gamma, x' : \text{end}_\gamma \vdash N[x/x'] : S}}
\]

\[
\begin{align*}
\text{[send] } M N &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[let] } (x,y) &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[case] } M &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[link] } M N &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[sendType] } S M &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[receiveType] } X.M &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[serve] } y.M &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) ) \\
\text{[request] } M &\quad = \nu.\nu.x.\nu.y.([M]y \mid \nu.x.\nu.y.([N]z) )
\end{align*}
\]

Channel z provides a continuation, consuming the output of the process representing the original HGVπ term. The translation on contexts is pointwise.

\[
[x_1 : T_1, \ldots, x_n : T_n] = x_1 : [T_1], \ldots, x_n : [T_n]
\]

As with the translation from HGV to HGVπ, we can show that this translation preserves typing.

**Theorem 2** If \(\Phi \vdash M : S\) then \([M]z \vdash [\Phi], z : [S]^{-}\).

## 5 From CP to HGVπ

We now present the translation \((\_\_\_\_\_\_\_\_\_)\) from CP to HGVπ. The translation on types is as follows:

\[
\begin{align*}
(A \otimes B) &\quad = ![A].(B) \\
(A \otimes B) &\quad = ![A].(B) \\
(A \otimes B) &\quad = ![A].(B) \\
(A \otimes B) &\quad = ![A].(B)
\end{align*}
\]

The translation on terms uses make of let expressions to simplify the presentation; these are expanded to HGVπ as follows:

\[
\text{let } x = M \text{ in } N \equiv ((\lambda x.N)M)^* \equiv M \text{ (fork } z \text{, let } (x,z) = \text{ receive } z \text{ in link } N z)
\]
Together, Theorem 4 and 5 tell us that HGV, HGV

The key soundness property of our translations is that if we translate a term from CP to HGV

and of Gay and Vasconcelos [6]. We have shown that HGV is sufficient to encode arbitrary linear logic

We have proposed a session-typed functional language, HGV, building on similar languages of Wadler [8]

7 Conclusions and Future Work

We have proposed a session-typed functional language, HGV, building on similar languages of Wadler [8]

Dardha et al [4] offers an alternative foundation for session types through a CPS translation of \( \pi \)-calculus with session types into a linear \( \pi \)-calculus. There appear to be strong similarities between their CPS translation and ours. We would like to make the correspondence precise by studying translations between their systems and ours.
In addition we highlight several other areas of future work. First, the semantics of HGV is given only by cut elimination in CP. We would like to give HGV a semantics directly, in terms of reductions of configurations of processes, and then prove a formal correspondence with cut elimination in CP. Second, replication has limited expressive power compared to recursion; in particular, it cannot express services whose behaviour changes over time or in response to client requests. We believe that the study of fixed points in linear logic provides a mechanism to support more expressive recursive behaviour without sacrificing the logical interpretation of HGV. Finally, as classical linear logic proofs, and hence CP processes, enjoy confluence, HGV programs are deterministic. We hope to identify natural extensions of HGV that give rise to non-determinism, and thus allow programs to exhibit more interesting concurrent behaviour, while preserving the underlying connection to linear logic.

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