Lattice Input on the Inclusive Decay Puzzle

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Recent analyses of flavor-breaking hadronic-τ-decay-based sum rules produce values of $|V_{us}| \sim 3\sigma$ low compared to 3-family unitarity expectations. An unresolved systematic issue is the significant variation in $|V_{us}|$ produced by different prescriptions for treating the slowly converging $D = 2$ OPE series. We investigate the reliability of these prescriptions using lattice data for various flavor-breaking correlators and show the fixed-scale prescription is clearly preferred. Preliminary updates of the conventional τ-based, and related mixed τ-electroproduction-data-based, sum rule analyses incorporating B-factory results for low-multiplicity strange τ decay mode distributions are then performed. Use of the preferred FOPT $D = 2$ OPE prescription is shown to significantly reduce the discrepancy between 3-family unitarity expectations and the sum rule results.

The conventional inclusive hadronic τ decay determination of $|V_{us}|$ is obtained by applying the finite energy sum rule (FESR) relation, involving polynomial weight $w(s)$ and kinematic-singularity-free correlator $\Pi(s)$ with spectral function $\rho(s)$,

$$\int_0^{s_0} w(s)\rho(s) \, ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s)\Pi(s) \, ds , \quad (1)$$

to the flavor-breaking (FB) difference $\Delta\Pi_\tau \equiv \left[ \Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)} \right]$, where
\( \Pi^{(J)}_{V/A;ij}(s) \) are the spin \( J = 0,1 \) components of the flavor \( ij \), vector (V) or axial vector (A) current-current 2-point functions. The spectral functions, \( \rho^{(0+1)}_{V/A;ij} \), hence also \( \Delta \rho_{\tau} \), are related to the normalized differential decay distributions, \( dR_{V/A;ij}/ds \), of flavor \( ij \) V- or A-current-induced \( \tau \) decay widths, 
\[
R_{V/A;ij} \equiv \Gamma[\tau^{-} \to \nu_{\tau} \text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^{-} \to e^{-}\bar{\nu}_{e}(\gamma)],
\]
by
\[
dR_{V/A;ij}/ds = 12\pi^{2}|V_{ij}|^{2}S_{EW} \left[w_{\tau}(y_{\tau})\rho^{(0+1)}_{V/A;ij}(s) - w_{L}(y_{\tau})\rho^{(0)}(s)\right]/m_{\tau}^{2}, \tag{2}
\]
with \( y_{\tau} = s/m_{\tau}^{2} \), \( V_{ij} \) the i,j CKM matrix element, \( w_{\tau}(y) = (1-y)^{2}(1+2y) \), \( w_{L}(y) = y(1-y)^{2} \), and \( S_{EW} \) a short-distance electroweak correction factor. The \( J = 0 \) (longitudinal) contributions in (2) are well known phenomenologically and, due to problems with the corresponding \( D = 2 \) OPE series, usually subtracted from \( dR/ds \). The subtracted result, \( dR_{V/A;ij}^{(0+1)}/ds \), allows the construction of \( J = 0 + 1 \) reweighted analogues, 
\[
R_{V/A;ij}^{w}(s_{0}) = \int_{0}^{s_{0}} ds \left[w(s)/w_{\tau}(y_{\tau})\right] dR_{V/A;ij}^{(0+1)}(s)/ds \text{ for any } w(s) \text{ and } s_{0} < m_{\tau}^{2}.
\]
Defining \( \delta R_{V/A;ij}^{w}(s_{0}) = \left[R_{V/A;ij}^{w}(s_{0})/|V_{ud}|^{2}\right] - \left[R_{V/A;ij}^{w}(s_{0})/|V_{us}|^{2}\right] \), one has, for \( s_{0} \) large enough to allow use of the OPE on the RHS of (1),
\[
|V_{us}| = \sqrt{R_{V/A;ij}^{w}(s_{0})/\left[R_{V/A;ij}^{w}(s_{0})/|V_{ud}|^{2}\right] - \delta R_{V/A;ij}^{w,\text{OPE}}(s_{0})}. \tag{3}
\]
This relation has usually been employed in un-reweighted form, with \( w = w_{\tau} \), and the single value \( s_{0} = m_{\tau}^{2} \). This has the advantage that \( R_{V/A;ij}^{w}(s_{0}) \) is determinable from branching fraction information alone, but the disadvantage of precluding tests of the \( s_{0} \)- and \( w(s) \)-independence of the analysis, which could otherwise be used to investigate potential systematic uncertainties (in particular, those associated with the treatment of OPE contributions). Such self-consistency tests were carried out in Refs. [2] [3] [4] and non-trivial \( w(s) \)- and \( s_{0} \)-dependences observed, suggesting shortcomings in the experimental data and/or OPE representation.

The most obvious potential OPE problem lies in the rather slow convergence of the \( D = 2 \) OPE series. In terms of the running \( \overline{MS} \) quantities \( m_{s}(Q^{2}) \) and \( \bar{a} \equiv \alpha_{s}(Q^{2})/\pi \), the \( D = 2 \) series, which is known to 4-loops, is given by
\[
[\Delta \Pi_{\tau}(Q^{2})]_{D=2}^{\text{OPE}} = \frac{3}{2\pi^{2}} \frac{m_{s}^{2}(Q^{2})}{Q^{2}} \sum_{k=0} a_{k} \bar{a}^{k}, \tag{4}
\]
with \( a_{k} = 1, 7/3, 19.93, 208.75 \) for \( k = 0 \cdots 3 \). Since \( \bar{a}(m_{s}^{2}) \simeq 0.10, \bar{a}^{2} > m_{s}^{2} \) at the spacelike point on the contour for all \( s_{0} \leq m_{\tau}^{2} \). The problematic convergence complicates the assessment of \( D = 2 \) truncation errors, and manifests itself, e.g., in the \( \sim 0.0020 \) difference in \( |V_{us}| \) values obtained using two alternate (CIPT or FOPT) versions of the 4-loop-truncated, \( w_{\tau} \)-weighted series.

An alternate determination employs the FB combination \( \Delta \Pi_{\tau-EM} \equiv 9\Pi_{EM} - 5\Pi_{ud}^{(0+1)} + \Pi_{ud,A}^{(0+1)} - \Pi_{us,V+A}^{(0+1)} \) in place of \( \Delta \Pi_{\tau} \). Inclusive electroproduction cross-sections fix the electromagnetic (EM) spectral function. By construction, the \( \Delta \Pi_{\tau-EM} \) \( D = 2 \) series is strongly suppressed, having the form (11), with \( c_{k} \rightarrow \)
\(c_{\tau}^{EM} = 0, -1/3, -4.384, -44.943\) for \(k = 0 \cdots 3\). The \(D = 4\) series is also strongly suppressed. OPE contributions to \(\Delta \Pi_{\tau-EM}\) FESRs, hence also estimated OPE errors, are thus very small\(^6\) and the resulting \(|V_{us}|\) errors essentially entirely experimental. A check of this predicted suppression is of interest.

We investigate the relative merits of the fixed-scale (FOPT-like) and local-scale (\(\mu^2 = Q^2\), i.e., CIPT-like) treatments of the \(\Delta \Pi_{\tau}\) \(D = 2\) series, and the level of \(\Delta \Pi_{\tau - EM}\) suppression, by comparing OPE expectations and lattice data for the two correlator combinations over a range of Euclidean \(Q^2\). Five RBC/UKQCD domain wall fermion ensembles are employed, three, with \(m_\pi = 293, 349, 399\) MeV, having \(1/a = 2.31\) GeV\(^8\) and two, with \(m_\pi = 171, 248\) MeV, having \(1/a = 1.37\) GeV\(^7\).

For technical reasons, conserved-local versions of the flavor \(us\) 2-point functions are numerically challenging and hence, for \(\Delta \Pi_{\tau}\), local-local versions are used. To check that this does not produce residual lattice artifacts which would impact our conclusions, we have also performed the OPE-lattice comparison, using conserved-local data, for the alternate flavor-diagonal FB combination \(\Delta \Pi_{\text{diag}} \equiv \Pi_{V;\ell\ell} - \Pi_{V;ss}\), whose \(D = 2\) series is very similar to that of \(\Delta \Pi_{\tau}\) (\(c_k^{\text{diag}} \rightarrow c_k^{\text{diag}} = 1, 8/3, 24.32, 253.69\) for \(k = 0 \cdots 3\) in \((4)\)). The results confirm those of the local-local study.

Representative OPE-lattice data comparisons for \(\Delta \Pi_{\tau}\) are shown, for the \(1/a = 2.13\) GeV, \(m_\pi = 293\) MeV ensemble, in Fig. 1. The left (right) panel comparison employs the fixed-scale (local-scale) prescription for the \(D = 2\) OPE series. The fixed-scale versions match much better the \(Q^2\) dependence of the lattice results, with the 3-loop-truncated version thereof best matching the overall normalization.

![Graph](image)

**Fig. 1.** OPE and lattice \(\Delta \Pi_{\tau}\) data, \(1/a = 2.31\) GeV, \(m_\pi = 293\) MeV ensemble, \(O(\hat{a}^{1,2,3})\) \(D = 2\) OPE truncation, fixed-scale (left panel) or local-scale (right panel) \(D = 2\) prescription

The comparison of lattice data for \(\Delta \Pi_{\tau}\) and \(\Delta \Pi_{\tau - EM}\) confirms the very strong suppression of \(\Delta \Pi_{\tau - EM}\)\(^4\) (see Ref.\(^4\) for the relevant figure).
We turn to preliminary updates of the $|V_{us}|$ analyses. For the $D = 2$ OPE series, we employ the 3-loop-truncated FOPT prescription favored by lattice data, and for the $ud$ spectral integrals, OPAL data [9] as updated in Ref. [10]. For the $us$ spectral integrals, recent B-factory results are used for the $K\pi$ [11], $K^-\pi^-\pi^+$ [12] and $K_s\pi^-\pi^+$ [13] exclusive mode distributions, and ALEPH results [14], updated for current branching fractions (BFs), for all other modes. Contributions from the latter lie higher in the spectrum, and have much larger errors. The B-factory distributions are unit normalized, and also require current BFs for their overall scales. We work with BFs obtained in a $\pi\mu^2$, $K_{\mu^2}$-constrained HFAG fit, supplemented by the update to $B[\tau^- \rightarrow K^0\pi^-\pi^0\nu_{\tau}]$ produced by the recent Belle result [13]. Other non-trivial shifts in the $us$ BFs also remain possible. To illustrate the changes to $|V_{us}|$ that could result, we consider also an alternate set of $us$ BFs with the recent larger, but not yet finalized, BaBar results [15] for $B[\tau^- \rightarrow K^- n \pi^0\nu_{\tau}]$, $n \leq 3$, used in place of those of the HFAG fit. The first set of $us$ BFs is labelled “$us$ BF set #1” below, the second, alternate set “$us$ BF set #2”. Changes to the $us$ BFs alter the inclusive $us$ spectral distribution, and hence can affect both the magnitude of $|V_{us}|$ and the $s_0$-dependence of the results. The significantly larger preliminary BaBar $K^-\pi^0$ BF is particularly relevant for the FB FESRs considered here, which weight more strongly the low-$s$ part of the spectrum. We consider FESRs employing the weights $w_\tau$ and $w_2(y) = (1 - y)^2$. $w_2$ weights less strongly the higher-$s$, large-error region of the $us$ spectral distribution. Differences between results obtained using the two different weights can thus point to issues with the $us$ spectral distribution.

$|V_{us}|$ results obtained from the $w_\tau$ and $w_2$ versions of the $\Delta\Pi_{\tau}$ FESR are shown, as a function of $s_0$, and also the choice of the input $us$ BF set, in the left panel of Fig. 2. Similar results for the $\Delta\Pi_{\tau-EM}$ FESR are shown in the right panel. $w_2$ results, which are less sensitive to the large-error high-$s$ region, show better $s_0$-stability in both cases. For $w_\tau$, $s_0$-stability is also better for the $\Delta\Pi_{\tau-EM}$ case,
where OPE contributions are suppressed. The convergence of $w_\tau$ results to the more stable $w_2$ ones as $s_0 \to m_\tau^2$, seen for both the $\Delta \Pi_{\tau}$ and $\Delta \Pi_{\tau-EM}$ FESRs, suggests the possibility of residual OPE problems in the $w_\tau$ case, where cancellations on the contour play a larger role. Finally we note that results obtained using the FOPT prescription preferred by the lattice data agree better with 3-family unitarity expectations than do those (not shown here) obtained using CIPT, as do those obtained using $us$ BF set #2 in place of $us$ BF set #1. More details of these analyses will be presented elsewhere.

We close by stressing the preference for FOPT over CIPT for the $D=2$ OPE series. The prescription which underlies CIPT (of summing logarithmic terms to all orders while truncating the series of non-logarithmic terms), though plausible, is motivated by heuristic arguments not generally valid for divergent series\cite{16}, and performs poorly when tested against lattice data for the FB correlators.

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