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Abstract. Non-equilibrium real-space condensation is a phenomenon in which a finite fraction of some conserved quantity (mass, particles, etc.) becomes spatially localized. We review two popular stochastic models of hopping particles that lead to condensation and whose stationary states assume a factorized form: the zero-range process and the misanthrope process, and their various generalizations. We also introduce a new model - a misanthrope process with parallel dynamics - that exhibits condensation and has a pair-factorized stationary state.

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1. Introduction

Real-space condensation is a non-equilibrium phase transition which occurs in various contexts such as granular clustering, traffic jams, wealth condensation or simulations of polydisperse hard spheres [1, 2, 3, 4]. In all of these systems there is some conserved quantity (mass, wealth or volume for example) which is transported through the system. If the global density of this quantity is above a critical value, a finite fraction condenses onto a single lattice site or localized region of space.

Surprisingly, many features of condensation are captured within a simple lattice model known as the zero-range process (ZRP) (for review see [5, 6]). In the simplest one-dimensional asymmetric version of this model, a particle moves from site i to $i+1$ of a one-dimensional periodic lattice with rate $u(m)$, where m is the occupancy (number of particles) of the departure site i. As the rates are totally asymmetric a current always flows and detailed balance cannot be satisfied, thus the stationary state is non-equilibrium. The great advantage of this model is that its non-equilibrium stationary state has a simple factorized form which is amenable to exact analysis. The structure for $P(\{m_i\})$, the probability that each site $1 \le i \le L$ contains mass m_i is

$$P(\{m_i\}) = \frac{1}{Z_L} \prod_{i=1}^{L} f(m_i) \, \delta_{M,\sum_j m_j} \ . \tag{1}$$

Thus the numerator in (1) contains one (non-negative) factor $f(m_i)$ for each site i and $f(m)$ is known as the single-site weight function and depends on $u(m)$ as

$$f(m) = \prod_{i=1}^{m} \frac{1}{u(i)} \quad \text{for} \quad m \ge 1 \quad \text{and} \quad f(0) = 1. \tag{2}$$

The denominator Z_L is the normalization or nonequilibrium partition function

$$Z_L = \sum_{\{m_i=0\}}^{\infty} \prod_{i=1}^{L} f(m_i) \, \delta_{M,\sum_j m_j}. \tag{3}$$

In (1) and (3) the Kronecker delta imposes the constraint that the total mass (or number of particles) M in the system is conserved.

It turns out that the zero-range process is not the only model leading to the factorized steady state (1). In recent years, it has been shown that other models in which the hop rate depends not only on the occupation of the departure site but also on other variables can also exhibit steady states that factorize exactly [7, 8, 9, 10, 11, 12, 13] or approximately [14] over the sites of the system. It has also been discovered that there exist various classes of models in which the steady state factorizes over pairs of sites [15, 16, 17],

$$P(\{m_i\}) = \frac{1}{Z_L} \prod_{i=1}^{L} g(m_i, m_{i+1}) \delta_{M,\sum_j m_j}. \tag{4}$$

In (4) $g(m_i, m_{i+1})$ is the pairwise weight. In the following we will first briefly review some of the models with fully factorized and pair-factorized steady states and discuss

conditions under which condensation occurs in these models. We shall then introduce a new model, a discrete time variant of the misanthrope process [18], with a pair-factorized steady state in which the hopping rate depends on the occupations of the departure and arrival sites. Such discrete time schemes are often used in the simulation of traffic flow and pedestrian dynamics. In the zero-range process it has been shown that a generalisation to discrete time dynamics still results in a factorized stationary state [19]. In this work we show that the misanthrope process may have a pair-factorized stationary state under discrete time dynamics and we establish conditions under which this holds. Further, we present an analysis of condensation in such pair-factorized states.

2. Zero-range process

The zero-range process is specified by the hop rate $u(m)$ which determines the properties of the steady state through the single-site weight $f(m)$ from Eq. (2). It is important to note that any exponential factor Aq^m in $f(m)$ does not change the steady-state properties since it appears in Eq. (1) as a constant prefactor $A^L q^{\sum_i m_i} = A^L q^M$ due to the fixed total mass M and number of sites L. From now on we will generally suppress such exponential factors in $f(m)$.

Condensation in models with factorized stationary states occurs in the limit $L \to \infty$ and fixed density of particles $\rho = M/L$ when the asymptotic (large m) behaviour of $f(m)$ (modulo any exponential factors) is the following:

I $f(m) \sim m^{-\gamma}$ with $\gamma > 2$. The critical mass density, ρ_c, above which condensation occurs is finite but its numerical value depends on the particular form of $f(m)$ and not only on its asymptotic behaviour. The fraction $\rho/\rho_c - 1$ of all particles goes into the condensate. We will refer to this behaviour as standard condensation. ‡

II $f(m)$ increases with m more quickly than exponentially, e.g., as $\sim m!$. This leads to so called strong (or complete) condensation - the critical density $\rho_c = 0$ and a fraction of particles tending to one in the thermodynamic limit is located at one site.

It can be shown that standard condensation (**I**) occurs in the ZRP when the hop rates in the limit of large m asymptotically approach some positive value β as

$$\frac{u(m)}{\beta} \sim 1 + \frac{\gamma}{m} \qquad (5)$$

with $\gamma > 2$, or more slowly than $1/m$. On the other hand strong condensation occurs when $u(m) \to 0$ as $m \to \infty$. For example, $u(m) = 1/m$ yields $f(m) = m!$.

To see why the condensation happens in the two generic cases highlighted above, we shall follow a standard approach [5]. Treating the steady-state probability as the

‡ Actually condensation also occurs if $f(m)$ decays more quickly than $1/m^\gamma$, e.g., as a stretched exponential.

statistical weight of a given configuration, and defining the grand-canonical partition function

$$G(z) = \sum_{\{m_i\}} z^{\sum_i m_i} P(m_1, \ldots, m_L) = \sum_{\{m_i\}} \prod_{i=1}^{L} f(m_i) z^{m_i} = F(z)^L, \qquad (6)$$

where

$$F(z) = \sum_{m=0}^{\infty} f(m) z^m, \qquad (7)$$

we see that the phase transition, signalled by a singularity of $G(z)$ at some z_c, is possible only if the series $F(z)$ has a finite radius of convergence z_c. Moreover, the density calculated as a function of fugacity z from the grand-canonical partition function:

$$\rho(z) = z \frac{F'(z)}{F(z)} \qquad (8)$$

must yield a value $\rho_c < \infty$ as $z \nearrow z_c$ so that the singularity in $G(z)$ and accompanying phase transition occur at finite density. This is only possible if either $f(m)$ decays as a power law in which case we may have a finite ρ_c (case **I**) or $f(m)$ grows very fast with m in which case $z_c = 0$ and $\rho_c = 0$ (case **II**).

Thus the grand canonical ensemble can only realise densities $\rho \leq \rho_c$. When $\rho > \rho_c$ one must work in the canonical ensemble (fixed number of particles) [5, 20]. It turns out that the excess mass $M_{\mathrm{ex}} = M - L\rho_c$ condenses onto a randomly selected lattice site and forms the condensate. The remainder of the system (referred to as the fluid) is described by the grand canonical ensemble at the critical density ρ_c [21, 22, 20, 23, 24].

3. Generalized Class of Models with Factorized Stationary State

So far we have discussed the ZRP as an example of a simple model with a factorized stationary state (1). More generally one can ask, when does a stochastic mass transport model have such a factorized stationary state if the hop rate depends only on the state of the departure site? To this end a class of models was studied in [19] that generalises the ZRP in a number of different ways while maintaining the factorization of the steady state. First, more than one unit of mass can be transferred from site i to $i+1$. Second the dynamics consists of discrete time update and simultaneous transport of mass at different locations is possible at an update. More precisely, in each time step, some number $0 \leq \mu_i \leq m_i$ of particles depart from site i and move to site $i+1$ with probability $\phi(\mu_i|m_i)$ which is known as the chipping kernel. For conservation of probability we require $\sum_{\mu=0}^{m} \phi(\mu|m) = 1$.

It was shown [19] that a necessary and sufficient condition for the stationary state of this class of models to factorize is that the chipping kernel takes the form

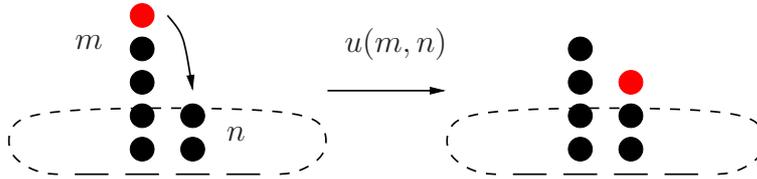$$\phi(\mu|m) = \frac{u(\mu)v(m-\mu)}{f(m)}, \qquad (9)$$

Figure 1. Definition of the Misanthrope process: a particle hops from site with m to site with n particles with rate $u(m, n)$.

where the single site weight $f(m)$ is given by

$$f(m) = \sum_{\mu=0}^{m} u(\mu)v(m-\mu) \ . \tag{10}$$

In these expressions u and v are positive functions. Expression (9) implies a factorization of the chipping kernel into a factor $u(\mu)$ which depends on the mass transferred and a factor $v(m-\mu)$ which depends on the mass which remains. The model can be further generalized to a continuous mass variable [19] but we shall not consider that here. The model was also considered on an arbitrary graph rather than a periodic chain and a condition similar to (9) was shown to be sufficient for factorization [25].

4. Misanthrope process

As already stressed the defining feature of the ZRP and the class of models just discussed is that transition rates or probabilities for the transfer of mass between sites depend only on the departure site and not on the destination site. It is natural to consider more general models in which for example a hop rate takes the form $u(m, n)$ where m is the occupancy of the departure site i and n is the occupancy of the destination site $i + 1$. This type of model is variously referred to as a misanthrope or migration process.

We now define the model that we consider. As in the ZRP case, M particles reside on sites of a 1D closed chain of length L; each site i carries m_i particles, and the conservation of particles requires that $\sum_{i=1}^{L} m_i = M$. The only difference with the ZRP is that a particle hops from site i to site $i + 1$ with rate $u(m_i, m_{i+1})$ which depends on the occupancies of both the departure and the arrival site, see Fig. 1.

This model has a factorized stationary state (1) when certain conditions on $u(m, n)$ are satisfied. Here, we simply quote the constraint on $u(m, n)$ without proof:

$$u(m, n) = u(m + 1, n - 1)\frac{u(1, m)u(n, 0)}{u(m + 1, 0)u(1, n - 1)} + u(m, 0) - u(n, 0). \tag{11}$$

It can be shown that this relation actually reduces to two conditions:

$$u(n, m) = u(m + 1, n - 1)\frac{u(1, m)u(n, 0)}{u(m + 1, 0)u(1, n - 1)}, \tag{12}$$

$$u(n, m) - u(m, n) = u(n, 0) - u(m, 0). \tag{13}$$

These conditions were first written down by Cocozza-Thivent [18]. Various proofs are presented in [26] and [5, 6]. Under these conditions the single site weights $f(m)$ obey a recursion

$$f(n) = f(n-1)\frac{f(1)}{f(0)}\frac{u(1, n-1)}{u(n, 0)} . \tag{14}$$

Conditions (12,13) uniquely define all rates $u(n, m)$ and the single-site weights $f(m)$ in terms of a set of basic hop rates which we denote

$$u(m, 0) = y_m, \tag{15}$$
$$u(1, n) = x_n. \tag{16}$$

Iterating Eq. (14) and using the definitions of y_m, x_n (15,16) we obtain the following expression for the single-site weight $f(n)$:

$$f(n) = f(0)\left(\frac{f(1)}{f(0)}\right)^n \prod_{i=1}^n \frac{x_{i-1}}{y_i}, \tag{17}$$

in which, as noted previously, we may suppress the exponential factor $f(0)\left(\frac{f(1)}{f(0)}\right)^n$.

Equations (12) and (13) can be rewritten as two recursion relations which allow one to find $u(m, n)$ for $m > 2, n > 1$:

$$u(m+1, n-1) = u(n, m)\frac{y_{m+1}}{x_m}\frac{x_{n-1}}{y_n}, \tag{18}$$
$$u(n, m) = u(m, n) - y_m + y_n. \tag{19}$$

By iterating these equations one obtains unique expressions for all $u(m, n)$. However, there is an additional condition that $u(m, n)$ should be non-negative for all m, n for it to be a hopping rate. This imposes some constraints on y_m, x_n which cannot be expressed in a closed form. Therefore it remains an open problem to determine precisely which x_n and y_m lead to a physical model with non negative hopping rates. However, there exists a special case of $u(m, n)$ for which more progress has been made and we shall discuss it now.

5. Factorized hop rate in the misanthrope process

In recent work [27, 26] we considered the special case

$$u(m, n) = w(m)v(n) \tag{20}$$

which corresponds to hopping rates whose dependence on departure and destination site factorizes. One can check that for this form of the hop rate equation (12) is automatically fulfilled. When $v(n) = \text{const}$, equation (13) is also fulfilled and we recover the ZRP with $u(m, n) = w(m)$. When $v(n) \neq \text{const}$, equation (13) leads to the relation between $w(n)$ and $v(n)$:

$$w(n) = C[v(n) - v(0)], \tag{21}$$

with some arbitrary, non-zero constant C. Thus

$$u(m, n) = C[v(m) - v(0)]v(n) \tag{22}$$

is the form of a factorized hop rate that yields a factorized steady state (1) with the single-site weight given by

$$f(m) = \prod_{i=1}^{m} \frac{v(i-1)}{v(i) - v(0)} \ . \tag{23}$$

Before we discuss the condition for condensation in this model, we shall briefly review two simple choices of $v(m)$ that lead to previously studied models.

5.1. Partial Exclusion

Our first simple example of the factorized hopping rate (22) is $C = -1$ and $v(n) = N - n$ with some integer $N > 0$ we have

$$u(m, n) = m(N - n) \ . \tag{24}$$

If $N = 1$ this reduces to the asymmetric simple exclusion process where the occupancy of each site is limited to 1 and $u(1, 1) = 0$. Similarly, the case of general integer $N > 0$ corresponds to 'partial exclusion' [28] where each site of a lattice contains at most N particles. In this context the rate (24) may be understood as each of m particles attempting hops forward to the next site with rate one and the hopping attempt succeeding with probability $N - n$ where n is the occupancy of the destination site.

5.2. Inclusion Process

If we take $C = 1$ and

$$v(n) = n + d \tag{25}$$

in (22) where d is a positive constant, we obtain

$$u(m, n) = m(n + d) \ . \tag{26}$$

This is the hop rate for the so-called Inclusion Process studied in [29]. In the limit $d \to 0$ this model exhibits a distinct form of condensation. (To avoid confusion, we also note that a different model involving coalescence of particles is also referred to as an Inclusion Process [30].)

6. Condensation in the misanthrope process with factorized hop rates

We are interested in a factorized form of $u(m, n)$ from Eq. (22) such that Eq. (14) gives $f(n) \sim n^{-\gamma}$ with $\gamma > 2$ which as we know leads to standard condensation. It turns out that the standard condensation can occur through two contrasting types of dynamics. One mechanism is through the hops rates decaying sufficiently slowly with n, m. This can be achieved through

$$v(m) \cong \beta \left(1 - \frac{\alpha}{m}\right) \tag{27}$$

which leads to

$$u(m,n) \cong \beta(v(0) - \beta) - \frac{\alpha\beta(v(0) - \beta)}{n} + \frac{\alpha\beta^2}{m}, \tag{28}$$

which decays with m in a similar fashion to the ZRP case. The other mechanism is for the hop rates to *increase* with m, and n as

$$u(m,n) \sim (mn)^\gamma \tag{29}$$

with $\gamma > 2$, which is equivalent to

$$v(m) \sim m^\gamma. \tag{30}$$

We refer to the latter case as *explosive* condensation. Explosive condensation exhibits strikingly different dynamical properties to the ZRP-like condensation. In particular, the condensate emerges on a time scale which vanishes with system size L as $\sim (\ln L)^{1-\gamma}$ for $\gamma > 2$ [27]. This is in contrast to the case (27) or the ZRP case for which the time increases with L as $\sim L^2$.

Interestingly, for the misanthrope process the *existence* of condensation depends not only on the asymptotic behaviour of $v(m)$ but also on $v(0)$. This should be contrasted with the ZRP for which it is only the asymptotic decay of the hop rate that determines condensation. To illustrate this point consider the case [26]

$$v(0) < 1, \qquad v(m) = 1 + \frac{1}{m+1}. \tag{31}$$

In this case one can find closed form expressions for the weights $f(n)$ and generating function $F(z)$ and one may determine the critical density given by $z \to 1$:

$$\rho_c = \frac{4(1 - v(0))}{3v(0) - 2}. \tag{32}$$

We see that $\rho_c \to 0$ as $v(0) \to 1$ and $\rho_c \to \infty$ as $v(0) \to 2/3$. Consequently for $v(0) \leq 2/3$ there is no condensation, but for $2/3 < v(0) < 1$ there is standard condensation and for $v(0) = 1$ there is strong condensation, even though $v(m)$ is the same in all cases for $m > 0$.

7. Pair-factorized steady states

So far we have discussed the processes in which the stationary probability $P(m_1, \ldots, m_L)$ factorizes over sites of a 1d closed chain. One can consider generalisations of this structure to, for example, a pair-factorized state in which there is factorization over pairs of adjacent sites in which the stationary probabilities take the form (4) where $g(m,n)$ is the pairwise weight. The factorized stationary state (1) is recovered when $g(m,n)$ factorizes: $g(m,n) = a(m)b(n)$, in which case the single-site weight $f(m) = a(m)b(m)$.

Such pair-factorized stationary states have been considered in models [15, 16] with hopping rates $u(m_{i-1}, m_i, m_{i+1})$ which depend on the state of both (left and right) nearest neighbours:

$$u(m_{i-1}, m_i, m_{i+1}) = \frac{g(m_i - 1, m_{i-1})}{g(m_i, m_{i-1})} \frac{g(m_i - 1, m_{i+1})}{g(m_i, m_{i+1})}, \tag{33}$$
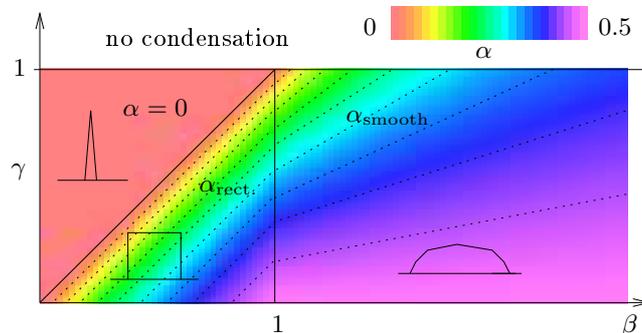
Figure 2. Phase diagram for $K(x) \sim e^{-|x|^{\beta}}$ and $p(m) \sim e^{-m^{\gamma}}$. The spatial extension of the condensate $\sim L^{\alpha}$, where $\alpha_{\text{rect}} = (\beta - \gamma)/(\beta - \gamma + 1)$ for the rectangular and $\alpha_{\text{smooth}} = (\beta - \gamma)/(2\beta - \gamma)$ for the smooth condensate. Dotted lines correspond to constant values of $\alpha = 0.05, 0.1, \ldots, 0.45$. Reproduced from [31].

where $g(m, n)$ is the same two-point weight that appears in the expression for the steady-state probability (4). It has been shown that a pair-factorized stationary state may modify the nature of condensation and allow a condensate spreading over a large, but non extensive number of sites. For example, if

$$g(m, n) = \exp\left[-J|m - n| + (U/2)(\delta_{m,0} + \delta_{n,0})\right], \tag{34}$$

the condensate's shape is a distorted parabola extending to $\sim \sqrt{L}$ sites [15]. References [16, 31] have considered a more general case

$$g(m, n) = K(|m - n|)\sqrt{p(m)p(n)}, \tag{35}$$

where $K(m)$ and $p(m)$ can be arbitrary, sufficiently-fast decaying functions. It turns out that when

$$K(x) \sim e^{-a|x|^{\beta}}, \qquad p(m) \sim e^{-bm^{\gamma}}, \tag{36}$$

for $a, b, \beta, \gamma > 0$, condensation occurs above a certain critical density of particles if $\gamma < 1$. The shape of the condensate changes from a single-site one, through a rectangular condensate, to a parabolic condensate as β increases from zero to one, and from one to infinity, see the phase diagram in Fig. 2. The scaling of the width of the condensate with L depends on the parameters β, γ in a non-trivial way. These results have been recently confirmed numerically [17], with some small discrepancies attributed to finite-size effects.

8. Pair-factorized states for discrete time dynamics

The pair-factorized steady state discussed in the previous section assumes a three-site hop rate (33). A very interesting question to ask is whether a two-point $u(m, n)$ such as the one in the misanthrope process can also lead to a pair-factorized steady state.

One might have hoped that when the hop rate $u(n, m)$ does not satisfy the conditions for a factorized stationary state, there would still be some choices of rates

$u(m, n)$ which yield pair-factorized states. However it was shown in [26] that this is not the case. Therefore the misanthrope process either has a factorized stationary state or the stationary state has some unknown structure.

Here we consider a generalisation of the misanthrope process similar to the generalisation of the zero-range process reviewed in Section 3. That is, we consider a Misanthrope-like model with stochastic discrete-time parallel dynamics where all particles attempt hops at discrete time steps. Such parallel dynamics are often employed in models of vehicular traffic and pedestrian dynamics [32]. Our aim is determine how the structure of the stationary state is modified. It turns out that we need to consider pair factorization for the steady state probability.

The dynamics we consider generalise the misanthrope process in two ways. First masses move at discrete timesteps therefore there can be simultaneous transport of mass at different locations. Second, more than one unit of mass can be transferred from site i to $i + 1$.

More precisely, in each time step, some number μ_i of particles depart from site i and move to site $i + 1$ with probability $\phi(\mu_i|m_i, m_{i+1})$. Clearly, we must have $\phi(\mu|m, n) = 0$ for $\mu > m$ and for conservation of probability we require $\sum_{\mu=0}^{m} \phi(\mu|m, n) = 1$. Following [19] we will refer to $\phi(\mu|m, n)$ as the chipping kernel.

One would like to know whether there is a necessary and sufficient condition on $\phi(\mu|m, n)$ for the steady state $P(m_1, \ldots, m_L)$ to factorize over pairs of neighbouring sites. This means that the following equation must be fulfilled:

$$\prod_{i=1}^{L} g(m_i, m_{i+1}) = \sum_{\mu_1, \ldots, \mu_L} \prod_{i=1}^{L} \phi(\mu_i|m_i + \mu_i - \mu_{i-1}, m_{i+1} + \mu_{i+1} - \mu_i)$$
$$\times g(m_i + \mu_i - \mu_{i-1}, m_{i+1} + \mu_{i+1} - \mu_i) . \qquad (37)$$

The left hand side of this equation gives the weight (unnormalized probability) of some configuration of mass in the system given by $\{m_i\}$. The right hand side gives the sum over the weights of possible configurations at the previous time step multiplied by the transition probabilities to the configuration $\{m_i\}$. The sum over preceding configurations is expressed as a sum over the masses, μ_i, transferred from i to $i + 1$. Thus at the previous time step site i had mass $m_i + \mu_i - \mu_{i-1}$. For a stationary state the two sides of (37) must be equal.

We now claim that a chipping kernel of the form

$$\phi(\mu|m, n) = \frac{u(\mu)a(m - \mu)b(n + \mu)}{g(m, n)}, \qquad (38)$$

where

$$g(m, n) = \sum_{\mu=0}^{m} u(\mu)a(m - \mu)b(n + \mu), \qquad (39)$$

is sufficient for pair factorization provided that some additional constraint is imposed on the functions $u(\mu), a(m), b(n)$. Expression (38) implies a factorization of the chipping kernel into a factor $u(\mu)$ which depends on the mass transferred, a factor $a(m - \mu)$ which

depends on the mass which remains at the departure site and a factor $b(n + \mu)$ which depends on the resulting mass at the destination site.

To demonstrate that the form (38) indeed leads to pair-factorized stationary state, let us insert (38) into the stationarity condition Eq. (37). Rearranging indices we obtain

$$\prod_i g(m_i, m_{i+1}) = \sum_{\{\mu_i\}} \prod_i \left(u(\mu_i) a(m_i - \mu_{i-1}) b(m_{i+1} + \mu_{i+1}) \right)$$

$$= \prod_i \left(\sum_{\mu_i} u(\mu_i) a(m_{i+1} - \mu_i) b(m_i + \mu_i) \right)$$

$$= \prod_i g(m_{i+1}, m_i) . \tag{40}$$

In other words, for pair factorization under the form (38) we require

$$\prod_i g(m_i, m_{i+1}) / g(m_{i+1}, m_i) = 1 . \tag{41}$$

The most general solution to this equation assumes the form (for a proof see [26] Appendix A)

$$g(m, n) = g(n, m) \frac{h(n)}{h(m)}, \tag{42}$$

where $h(m)$ is some function which can be determined by inserting $n = 0$ into the above equation:

$$h(m) = h(0) \frac{g(0, m)}{g(m, 0)} . \tag{43}$$

Finally, from (42) and (43) we obtain

$$g(m, n) g(0, m) g(n, 0) = g(n, m) g(m, 0) g(0, n) . \tag{44}$$

Equation (44), in tandem with the definition (39), is the central result of this section and gives a sufficient condition for the stationary state of the generalised misanthrope process to take a pair-factorized form. It implies conditions on $u(\mu), a(m), b(n)$ which we explore in the next section. It remains an open problem as to whether Eq. (44) is also a necessary condition.

8.1. Single particle hopping

To simplify the discussion we we now focus on the misanthrope-like case when at most one particle can hop from a given site in each timestep. For this purpose we take $u(\mu) = \delta_{\mu,0} + \Delta t \delta_{\mu,1}$, and (39) yields

$$g(m, n) = a(m) b(n) + \Delta t a(m - 1) b(n + 1). \tag{45}$$

At this stage Δt is a parameter but as we shall verify later the limit $\Delta t \to 0$ reduces to the usual continuous time misanthrope process. We will derive now the relation between

$a(m)$ and $b(m)$ for arbitrary $\Delta t > 0$. Inserting (45) in condition (44) leads, after some algebra, to

$$\frac{\tilde{b}(0) - \tilde{b}(m)}{\tilde{a}(m)} - \Delta t \tilde{b}(0)\tilde{b}(m) = \frac{\tilde{b}(0) - \tilde{b}(n)}{\tilde{a}(n)} - \Delta t \tilde{b}(0)\tilde{b}(n), \tag{46}$$

where we have defined

$$\tilde{a}(m) = a(m-1)/a(m), \tag{47}$$
$$\tilde{b}(m) = b(m+1)/b(m). \tag{48}$$

The left and right side of equation (46) are functions of m and n, respectively. In order for (46) to be valid for any m, n, both sides of the above equation must be equal to a constant k. This gives

$$\tilde{b}(m) = \frac{\tilde{b}(0) - \frac{1}{k}\tilde{a}(m)}{1 + \Delta t \tilde{b}(0)\tilde{a}(m)} \tag{49}$$

or, equivalently,

$$\tilde{a}(m) = k\frac{\tilde{b}(0) - \tilde{b}(m)}{1 + k\Delta t \tilde{b}(0)\tilde{b}(m)} \ . \tag{50}$$

Therefore, $\tilde{a}(m)$ is determined by $\tilde{b}(m)$ and vice versa, up to two parameters k and Δt. The chipping probability $\phi(1|m,n)$ can then be expressed as

$$\phi(1|m,n) = k\Delta t \tilde{b}(n)\frac{\tilde{b}(0) - \tilde{b}(m)}{1 + k\Delta t(\tilde{b}(m)\tilde{b}(0) + \tilde{b}(n)\tilde{b}(0) - \tilde{b}(m)\tilde{b}(n))}, \tag{51}$$

and because the hopping probability defines the model, all static and dynamical properties are fully specified by giving $k, \Delta t$ and one of the two functions $\tilde{a}(m)$ or $\tilde{b}(m)$. The pairwise weight function $g(m,n)$ is given by Eq. (45), with $a(m), b(n)$ calculated recursively:

$$a(m) = \prod_{i=1}^{m}\frac{1}{\tilde{a}(i)}, \qquad b(n) = \prod_{i=1}^{n}\tilde{b}(i-1), \tag{52}$$

where we assumed for convenience that $a(0) = b(0) = 1$. This assumption is made without loss of generality as it only rescales $g(m,n)$ by a constant factor.

In the limit $\Delta t \to 0$, we obtain factorization of the steady state, since (45) tends to $g(m,n) = a(m)b(n)$ corresponding to a single-site weight $f(m) = a(m)b(m)$. The chipping probability becomes

$$\phi(1|m,n) = \Delta t\,\tilde{a}(m)\tilde{b}(n) + O(\Delta t^2), \tag{53}$$

and in the continuous time limit $\Delta t \to 0$ this reduces to a hopping rate

$$u(m,n) = \tilde{a}(m)\tilde{b}(n), \tag{54}$$

whereas condition (46) reduces to

$$\frac{\tilde{b}(0) - \tilde{b}(m)}{\tilde{a}(m)} = \frac{\tilde{b}(0) - \tilde{b}(n)}{\tilde{a}(n)}, \tag{55}$$

which is equivalent to the condition (13) for the misanthrope process.

8.2. Condition for condensation

In this section we shall outline how the above discrete-time misanthrope process with factorized hopping probability exhibits condensation above some critical density of particles. We shall learn, perhaps not surprisingly, that the conditions on the hopping probability $\phi(1|m,n)$ are somewhat similar to those for the hopping rate $u(m,n)$ in the continuous time case.

We begin by defining the grand-canonical partition function

$$G(z) = \sum_{\{m_i\}} z^{\sum_i m_i} W(\{m_i\}), \tag{56}$$

where the steady-state weights reads

$$W(\{m_i\}) = \prod_{i=1}^{L} g(m_i, m_{i+1})$$

$$= \prod_{i=1}^{L} [a(m_i)b(m_{i+1}) + \Delta t a(m_i - 1)b(m_{i+1} + 1)]. \tag{57}$$

We now observe that the expression in square brackets in Eq. (57) can be viewed as a product of two vectors:

$$(a(m_i) + \Delta t a(m_i - 1)) \begin{pmatrix} b(m_{i+1}) \\ b(m_{i+1} + 1) \end{pmatrix}, \tag{58}$$

and hence the steady-state weight can be rewritten as

$$W(\{m_i\}) = \prod_{i=1}^{L} (a(m_i) + \Delta t a(m_i - 1)) \begin{pmatrix} b(m_{i+1}) \\ b(m_{i+1} + 1) \end{pmatrix}$$

$$= \mathrm{Tr} \left[\prod_{i=1}^{L} (a(m_i) + \Delta t a(m_i - 1)) \begin{pmatrix} b(m_{i+1}) \\ b(m_{i+1} + 1) \end{pmatrix} \right]$$

$$= \mathrm{Tr} \left[\prod_{i=1}^{L} \begin{pmatrix} b(m_i) \\ b(m_i + 1) \end{pmatrix} (a(m_i) + \Delta t a(m_i - 1)) \right]$$

$$= \mathrm{Tr} \left[\prod_{i=1}^{L} \begin{pmatrix} a(m_i)b(m_i) & \Delta t a(m_i - 1)b(m_i) \\ a(m_i)b(m_i + 1) & \Delta t a(m_i - 1)b(m_i + 1) \end{pmatrix} \right], \tag{59}$$

where in the penultimate step we have cyclically permuted the vectors under the trace and in the last step evaluated the resulting dyadic product. The grand-canonical partition function becomes

$$G(z) = \mathrm{Tr} \left[A(z)^L \right], \tag{60}$$

where $A(z)$ is a 2×2 matrix:

$$A(z) = \begin{pmatrix} \sum_m a(m)b(m)z^m & \Delta t \sum_m a(m-1)b(m)z^m \\ \sum_m a(m)b(m+1)z^m & \Delta t \sum_m a(m-1)b(m+1)z^m \end{pmatrix}. \tag{61}$$

The above matrix has two eigenvalues

$$\lambda_\pm(z) = \frac{1}{2}\left(A_{11}(z) + A_{22}(z) \pm \sqrt{(A_{11}(z) - A_{22}(z))^2 + 4A_{12}(z)A_{21}(z)}\right),(62)$$

where A_{ij} is an element of (61). Since λ_+ is always greater than λ_-, provided that $z > 0$, $\ln G(z) \cong L \ln \lambda_+$ for large system size L, and we obtain that, in the thermodynamic limit, the density-fugacity relation is given by

$$\rho(z) = z\frac{\lambda'_+(z)}{\lambda_+(z)} . \tag{63}$$

Since $\rho(z)$ is an increasing function of z, condensation will occur if $\rho(z)$ tends to a finite value as z approaches its maximum allowed value. This maximum allowed value is z_c – the radius of convergence of $\lambda_+(z)$. From Eqs. (62) and (61) we see that the radius of convergence of $\lambda_+(z)$ is the radius of convergence of any of the four entries of $A(z)$. If we focus, say, on $A_{11}(z)$, we see that condensation criteria reduce to the convergence properties of the sum $\sum_m a(m)b(m)z^m$. Hence, condensation is possible if

$$\tilde{f}(m) = a(m)b(m) \tag{64}$$

behaves in one of two ways described earlier i.e. here $\tilde{f}(m)$ plays the same role as the single-site weight $f(m)$ in section 1 cases **I**, **II**.

9. Discussion

In this paper we have given a short review of condensation in factorized and pair-factorized states. In particular we have discussed the misanthrope process where the hop rates $u(m_i, m_{i+1})$ depend on the occupancy of both the departure site i and destination site $i + 1$. This process provides a new route to the standard condensation scenario for the case of increasing hop rate $u(m, n) \sim m^\gamma n^\gamma$ for $\gamma > 2$.

The dynamics of this new "explosive condensation" has been explored in detail in Refs. [27, 26]. Interestingly, explosive condensation shows some parallels to instantaneous gelation observed in cluster-cluster aggregation models [33, 34, 35, 36], a class of models used to study gelation (see Ref. [37] for a review). We refer the reader to Ref. [26] for a more thorough discussion; here we will only mention it briefly. Instantaneous gelation occurs when clusters of particles of sizes m, n merge irreversibly with rate $\sim m^\mu n^\nu + n^\mu m^\nu$ and when $\mu + \nu > 1$ and the bigger of the exponents μ, ν is greater than one. In this case, gelation i.e. the process of forming a single large cluster occurs instantaneously in the thermodynamic limit. More specifically, the time to gelation decreases to zero with increasing total mass (size) of the system. Instantaneous gelation also occurs in exchange driven growth [36] in which particles are transferred between clusters at a rate $\sim m^\mu n^\nu + n^\mu m^\nu$. Here the condition for instantaneous gelation is that the larger of the two exponents $\gamma = \max(\mu, \nu) > 2$; the time to gelation decreases as $\sim (\ln L)^{-(\gamma-2)}$. This is quantitatively different to explosive condensation, in which the time to condensation scales as $\sim (\ln L)^{-(\gamma-1)}$, despite the fact that the dynamics of both models is qualitatively very similar.

We also have studied a generalisation of the misanthrope process to a parallel discrete time updating scheme in which simultaneous transfer of mass at different locations can occur. We have shown that conditions exist for the stationary state to take a pair-factorized form. These conditions generalize the conditions (12,13) for factorization in the continuous time case. Thus the continuous time factorized stationary state is modified into a pair-factorized state when discrete time dynamics are considered. This contrasts with the zero-range process in which a factorized form is maintained under discrete time updating.

A major open question which remains is the structure of the misanthrope process stationary state for general hopping rates. Also it would be of interest to generalise the conditions for factorization and pair factorization in the misanthrope process to more general geometries than the one-dimensional periodic chain.

Acknowledgments

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