Dijkstra Monads for Free

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Abstract

Dijkstra monads are a means by which a dependent type theory can be enhanced with support for reasoning about effectful code. These specification-level monads computing weakest preconditions, and their closely related counterparts, Hoare monads, provide the basis on which verification tools like F*, Hoare Type Theory (HTT), and Ynot are built. In this paper we show that Dijkstra monads can be derived “for free” by applying a continuation-passing style (CPS) translation to the standard monadic definitions of the underlying computational effects.

Automatically deriving Dijkstra monads provides a correct-by-construction and efficient way of reasoning about user-defined effects in dependent type theories. We demonstrate these ideas in EMF⋆, a new dependently typed calculus, validating it both by formal proof and via a prototype implementation within F*. Besides equipping F* with a more uniform and extensible effect system, EMF⋆ enables within F* a mixture of intrinsic and extrinsic proofs that was previously impossible.

1. Introduction

Monads are a versatile concept:

\begin{itemize}
  \item From Moggi (1989), they are used to give semantics to call-by-value reduction.
  \item From Moggi (1989), Wadler (1990, 1992), Filinski (1994, 1999, 2010), Benton et al. (2002) and others, they are used as a way to introduce effects into a functional language.
  \item From Moggi (1989); Flanagan et al. (1993); Wadler (1994) and others, they provide a foundation on which to understand program transformations, notably CPS.
\end{itemize}

This paper brings together this threefold use of monads to improve upon a fourth use of monads, namely “Dijkstra monads”, a recent proposal by Swamy et al. (2013, 2016) and Jacobs (2015), who suggest using monads of predicate transformers (like Dijkstra’s weakest preconditions), to verify effectful programs within a functional programming language.

1.1 Example: A Dijkstra monad for stateful computations

In Dijkstra’s (1975) weakest precondition semantics, stateful computations transform postconditions relating results and final state, to preconditions on input states. This gives rise to a Dijkstra monad:

let \( x = e_1 \) in \( e_2 \) is computed to be `bind_WP_ST wp1 (\( \lambda x. \) wp2)`, where wp1 and wp2 are the WPs of \( e_1 \) and \( e_2 \) respectively.

Based on such a construction, Swamy et al. (2013, 2016) (building on previous work by Nanevski et al. (2008)) devised a type system to compute WPs for higher-order, effectful programs, enabling their verification within a dependently typed logic. However, these constructions require a meta-theoretic argument to establish their soundness with respect to the semantics of effectful programs. This typically requires proofs of various correctness and admissibility conditions, including the monad laws and monotonicity.

1.2 For fun: Deriving Dijkstra monads

Rather than being given manually, we show that these predicate transformers can be automatically derived by CPS-ing purely functional definitions of monadic effects (with answer type Type). For instance, rather than defining WP_ST, one can simply compute it by CPS-ing the familiar ST monad (i.e., state \( \rightarrow \) state) for

\[
WP_ST \ a = ((a \ast state) \rightarrow Type) \rightarrow state \rightarrow Type \quad \text{(unfolded)}
\]

We introduce DM, a simply typed, pure, monadic metalanguage in which one can, in the spirit of Wadler (1992), define a variety of monadic effects, ranging from state and exceptions, to continuations. We define a type-directed CPS translation for this language and show that monads are translated to Dijkstra monads, i.e., monotone, conjunctive, predicate transformer monads.

1.3 For profit: Program verification with user-defined effects

We apply our technique of deriving Dijkstra monads to F* (Swamy et al. 2016), a dependently typed programming language that already has at its core a system of primitive effects specified using Dijkstra monads. Our goal is to make F∗’s effect system easier to configure and extensible beyond the primitive effects it already supports; we proceed as follows.

A core dependent type theory with monadic reflection To formally study our improvements to F*, we define a new dependently typed core calculus, EMF⋆ (for Explicitly Monadic F⋆) that features an extensible effect system. EMF⋆ is loosely based on the Calculus of Constructions (Coquand and Huet 1988) with (among other features): (1) a predicative hierarchy of non-cumulative universes; (2) a weakest-condition calculus for pure programs; (3) refinement types; and (4) a facility for representing user-defined effects using the monadic reflection and reification of Filinski (1994), adapted to the dependently typed setting. New effects can be introduced into the language by defining them in terms of the built-in pure constructs, related to each other via monad morphisms, and each such effect obtains a suitable weakest precondition calculus derived from the underlying pure WPs. We prove the calculus strongly normalizing and the WP calculus sound for total correctness verification, for both pure and effectful programs.
Translating DM to EMF* To extend EMF* with a new effect, one starts by defining a monadic effect (say ST) in DM. Via the CPS transformation, we obtain the Dijkstra variant of that effect (WP\_ST) as a predicate transformer monad in EMF*. A second translation from DM produces expression-level terms to represent monadic computations in EMF*. A logical relations proof shows that monadic computations are correctly specified by their predicate transformers. We show examples of these translations at work for monadic effects including state, exceptions, information-flow control, continuations, and some combinations thereof.

Intrinsic and extrinsic proofs in EMF* Effectful programs in EMF* can be proven correct using one or both of two different reasoning styles. First, using the WP calculus, programs can be proven intrinsically, by decorating their definitions with specifications that must be proven to be at least as strong as their WPs. We refer to this as the intrinsic style, already familiar to users of F*, and other tools like HTT (Nanevski et al. 2008), Dafny (Leino 2010), and Why3 (Filliatre and Paskevich 2013).

Second, through monadic reification, EMF* allows effectful programs to be revealed as their underlying pure implementations. Once reified, one can reason about them via the computational behavior of their definitions. As such, one may define effectful programs with relatively uninformative types, and prove properties about them as needed, via reification. This extrinsic style of proving is familiar to users of systems like Coq or Isabelle, where it is routinely employed to reason about pure functions. In EMF*, this style extends smoothly to terminating effectful programs.

Primitive effects in a call-by-value semantics We see EMF* as a meta-language in which to analyze and describe the semantics of terms in an object language, EMF\_ST, a call-by-value programming language with primitive state. In the spirit of Moggi (1989), we show that EMF* programs that treat their ST effect abstractly soundly model EMF\_ST reductions—technically, we prove a simulation between EMF\_ST and EMF*. As such, our work is a strict improvement on the prior support for primitive effects in F*: despite programming and proving programs in a pure setting, stateful programs can still be compiled to run efficiently in the practically effectful EMF\_ST, while programs with other user-defined effects (e.g., information-flow control) can, unlike before, be executed via their pure encodings.

A prototype implementation for F* We have adapted F* to benefit from the theory developed in this paper, using a subset of F* itself as an implementation of DM, and viewing EMF\_ST as a model of its existing extraction mechanism to OCaml. Programmers can now configure F*’s effect system using simple monadic definitions, use F* to prove these definitions correct, and then use our CPS transformation to derive the Dijkstra monads required to configure F*’s existing type-checker. To benefit from the new extrinsic proving capabilities, we also extended F* with two new typing rules, and changed its normalizer, to handle monadic reflection and reification.

Several examples show how our work allows F* to be easily extended beyond the primitive effects already supported, without compromising its efficient primitive effect compilation strategy; and how the new extrinsic proof style places effectful reasoning in F* on an equal footing with its support for reasoning about pure programs.

1.4 Summary of contributions

The central contribution of our work is designing three closely related lambda calculi, studying their meta-theory and the connections between them, and applying them to provide a formal and practical foundation for a user-extensible effect system for F*. Specifically,

1. EMF*: A new dependent type theory with user-extensible, monadic effects; monadic reflection and reification; WPs; and refinement types. We prove that EMF* is strongly normalizing and that its WPs are sound for total correctness (§3).

2. DM: A simply typed language to define the expression-level monads that we use to extend EMF* with effects. We define a CPS transformation of DM terms to derive Dijkstra monads from expression-level monads, as well as an elaboration of DM terms to EMF*. Moreover, elaborated terms are proven to be in relation with their WPs (§4).

3. EMF\_ST: A call-by-value language with primitive state, whose reductions are simulated by well-typed EMF* terms (§5).

4. An implementation of these ideas within F* (§3.5, §4.6) and several examples of free Dijkstra monads for user-defined effects (§2). We highlight, in particular, the new ability to reason extrinsically about effectful terms.

The auxiliary materials for this paper (https://www.fstar-lang.org/papers/dm4free) contain appendices with complete definitions and proofs for the formal results in §3, §4 (Appendix A below), and §5. The F* source code (https://github.com/FStarLang/FStar) now includes the extensions from §3.5 and §4.6 and the examples from §2 (https://github.com/FStarLang/FStar/tree/master/examples/dm4free).

2. Illustrative examples

We illustrate our main ideas using several examples from F*, contrasting with the state of affairs in F* prior to our work. We start by presenting the core WP calculus for pure programs (§2.1), then illustrate how state can be added to it (§2.2). After showing our basic methodology on state (§2.3 and §2.4), we present a few additional examples, including combining exceptions with state (§2.5), information-flow control (§2.6), and continuations (§2.7).

Notation: The syntax \( \lambda (b_1) \ldots (b_n) \rightarrow t \) introduces a lambda abstraction, where \( b_i \) ranges over binding occurrences \( x \) declaring a variable \( x \) at type \( t \). The type \( b_1 \rightarrow \ldots \rightarrow b_n \rightarrow c \) is the type of a curried function, where \( c \) is a computation type—we emphasize the lack of enclosing parentheses on the \( b_i \). We write just the type in \( b \) when the name is irrelevant, and \( t \rightarrow t' \) for \( t \rightarrow \text{Tot } t' \).

2.1 WPs for pure programs

Reasoning about purely functional programs is a relatively well-understood activity: the type theories underlying systems like Coq, Agda, and F* are already well-suited to the task. Consider proving that pure term \( \text{Pure } x = (\lambda x. t) \rightarrow \text{Pure } c \) always returns a non-negative integer. A natural strategy is an extrinsic proof, which involves giving \( t \) a simple type such as \( \text{int } \rightarrow \text{Tot } \text{int} \), the type of total functions on integers, and then proving a lemma \( \forall x. \text{sr} x \geq 0 \). In the case of \( F* \), the proof of the lemma involves, first, a little computation to turn the goal into \( \forall x. \text{sr} x \geq 0 \), and then reasoning in the theory of integer arithmetic of the Z3 SMT solver (de Moura and Bjørner 2008) to discharge the proof.

An alternative intrinsic proof style in \( F* \) involves giving \( s \) and \( t \) the types \( \text{xint } \rightarrow \text{Pure } \text{int } (\lambda x. \text{post } \rightarrow \text{by } x y \geq 0 \rightarrow \text{post } y) \), a dependent function type of the form \( \text{ct } \rightarrow c \), where the formal parameter \( x \) is in scope in the computation type \( c \) to the right of the arrow. Computation types \( c \) are either \( \text{Tot } t \) (for some type \( t \)) or of the form \( M t \text{wp} \), where \( M \) is an effect label, \( t \) is the result type of the computation, and \( \text{wp} \) is a predicate transformer specifying the semantics of the computation. The computation type \( t \) gives \( s \) the type of the form \( \text{Pure } t \text{wp} \), the type of \( t \)-returning pure computations described by the the predicate transformer \( \text{wp} \) (\( t \rightarrow \text{Type} \) → \( \text{Type} \), a function taking postconditions on the result (predicates of type \( t \rightarrow \text{Type} \)), to preconditions. These predicate transformers form a Dijkstra monad. In this case, the \( \text{wp} \) states that to prove any property \( \text{post } s x \), it suffices to prove
As discussed already in §1, one must define WP,ST t, its return
and bind combinators, proving that these specifications are sound
with respect to the operational semantics of state.

(2) The primitive effectful actions, get and put are assumed to have
the types below—again, these types must be proven sound with respect
to the operational semantics of F*.

(3) Following the rule for sequential composition sketched in §1, F*
uses the specifications of get and put to compute bind _ST_WP wp get (λ x → wp put (x + 1)) as the WP of incr,
which reduces to λ post s0 → post ((i, s0) + 1).

(4) The final step requires proving that the computed WP is at least
as weak as the specified goal, which boils down to showing that
s0 + i > s0, which F* and Z3 handle automatically.

The first two steps above correspond to adding a new effect to F*.
The cost of this is amortized by the much more frequent and
relatively automatic steps 3 and 4. However, adding a new effect
to F* is currently an expert activity, carried out mainly by the language
designers themselves. This is in large part because the first two steps
above are both tedious and highly technical: a dangerous mixture
that can go wrong very easily.

Our primary goal is to simplify those first two steps, allowing
effects to be added to F* more easily and with fewer meta-level
arguments to trust. Besides, although F* supports customization of
its effect system, it only allows programmers to specify refinements of
a fixed set of existing effects inherited from ML, namely, state,
exceptions, and divergence. For example, an F* programmer can
refine the state effect into three sub-effects for reading, writing, and
allocation; but, she cannot add a new effect like alternative combina-

tions of state and exceptions, non-determinism, continuations, etc.
We aim for a more flexible, trustworthy mechanism for extending
F* beyond the primitive effects it currently supports. Furthermore,
we wish to place reasoning about terminating effectful programs on
an equal footing with pure ones, supporting mixtures of intrinsic
and extrinsic proofs for both.

2.3 CPS’ing monads to Dijkstra monads

Instead of manually specifying WP,ST, we program a traditional
ST monad and derive WP,ST using a CPS transform. In §4.1 we
formally present DM, a simply typed language in which to de
monadic effects. DM itself contains a single primitive identity monad
τ, which (as will be explained shortly) is used to control the CPS
transform. We have implemented DM as a subset of F*, and for
the informal presentation here we use the concrete syntax of our
implementation. What follows is an unsurprising definition of a state
monad sta, the type of total functions from s to identity computations
returning a pair (a → s).

let st a = s → τ(a * s)
let return (x:a) : st a = λs₀ → x, s₀
let bind (f:st a) (g:a → st b) : st b = λs₀ → let x,s₁ = f s₀ in g x s₁
let get () : st a = λ s₀ → s₀, s₀
let put (x:s) : st unit = λ a → a, x

This being a subset of F*, we can use it to prove that this definition is
indeed a monad: proofs of the three monad laws for st are discharged
automatically by F* below (eqf is extensional equality on functions,
and assert p requests F* to prove p statically). Other identities
relating combinations of get and put can be proven similarly.

let right_unit_st (fst α) = assert (eqf (bind f return) f)
let left_unit_st (fst α) (fst (α → Type)) is assert (eqf (bind (return x) f) (f x))
let assoc_st (fst α) (fst (α → Type)) (fst (α → Type)) is assert (eqf (bind (f (λ x → (bind g h) i)) (bind (bind f g) h)))

We then follow a two-step recipe to add an effect like st to F*:

**Step 1** To derive the Dijkstra monad variant of st, we apply a
selective CPS transformation called the *-translation (§4.1, first,
on type sta; then, on the various monadic operations. CPS’ing only
those arrows that have τ-computation co-domains, we obtain:

(st a)⁺ = a → (s* a → Type) → Type
return⁺ = λ x s₀ post → post (x, s₀)
bind⁺ = λ f g s₀ post → f s₀ (g x s₁ post)
get⁺ = λ () s₀ post → post (s₀, s₀)
put⁺ = λ x, s₀ post → post ((i, x))

Except for a reordering of arguments, the terms above are identical
to the analogous definitions for WP,ST. We prove that the *-translation
preserves equality: so, having shown the monad laws for st a, we
automatically obtain the monad laws for (st a)⁺. We also prove
that every predicate transformer produced by the *-translation is
monotone (it maps weaker postconditions to weaker preconditions)
and conjunctive (they distribute over conjunctions and universals,
i.e., infinite conjunctions, on the postcondition).

**Step 2** The *-translation yields a predicate transformer semantics
for a new monadic effect, however, we still need a way to extend
F* with the computational behavior of the new effect. For this,
we define a second translation, which elaborates the definitions of
the new monad and its associated actions to Pure computations in F*.
A first rough approximation of what we prove is that for a well-typed
DM computation e : τ t, its elaboration g has type Pure τ e* in EMF*.

The first-order cases are particularly simple: for example,
return = return has type x a → Pure a (return⁺ x) in EMF*; and
get = get has type u unit → Pure s (get⁺ u) in EMF*.
For a higher-

order example, we sketch the elaboration of bind below, writing
st t wp for s₀ₜ s → Pure t (wp s₀):

bind : wpf(st a)⁺ → f g a wpf
→ wpf(a → (st b)⁺) → g(x:st b wpf x)
→ st b (bind⁺ wpf wpf)
= λ wpf f wpf g s₀ → let x, s₁ = f s₀ in g x s₁
Intuitively, a function in DM (like bind) that abstracts over computations (f and g) is elaborated to a function (bind) in EMF* that abstracts both over those computations (f and g again, but at their elaborated types) as well as the WP specifications of those computations (wpf and wp). The result type of bind shows that it returns a computation whose specification matches bind*, i.e., the result of the CPS’ing *-translation.

In other words, the WPs computed by F* for monads implemented as Pure programs corresponds exactly to what one gets by CPS’ing the monads. At first, this struck us as just a happy coincidence, although, of course, we now know that it must be so. We see our proof of this fact as providing a precise characterization of the close connection between and CPS and WPs transformations.

2.4 Reify and reflect, for abstraction and proving

Unlike prior F* formalizations which included primitive exception and state effects, the only primitive monad in EMF* is for Pure computations. Although the translations from DM yield pure definitions of monads in F*, programming directly against those pure implementations is undesirable, since this may break abstractions. For instance, consider an integer-state monad whose state is expected to monotonically increase: revealing its representation as a pure term makes it hard to enforce this invariant. We rely on Filinski’s (1994) monadic reflection for controlling abstraction.

Continuing our example, introducing the state effect in F* produces a new computation type ST (a-Type) (wp: (st a)*) and two coercions

\[
\text{reify} : \text{ST a w} \rightarrow s0 \rightarrow \text{Pure (a \ast s)} \quad (\text{wp s0})
\]

\[
\text{reflect} : (s0 \rightarrow \text{Pure (a \ast s)}) \quad (\text{wp s0}) \rightarrow \text{ST a w}
\]

The reify coercion reveals the representation of an ST computation as a Pure function, while reflect encapsulates a Pure function as a stateful computation. As we will see in subsequent sections, in some cases to preserve abstractions, one or both of these coercions will need to be removed, or restricted in various ways.

To introduce the actions from DM as effectful actions in F*, we reflect the pure terms produced by the elaboration from DM to EMF*, obtaining actions for the newly introduced computation type. For example, after reflection the actions get and put appear within F* at the types below:

\[
\text{get} : \text{unit \rightarrow ST s (get' () )}
\]

\[
\text{put} : s1 \rightarrow \text{ST unit (put' s1)}
\]

As in §2.2, we can still program stateful functionalities and prove them intrinsically, by providing detailed specifications to augment their definitions—of course, the first two steps of the process there are now automatic. However, we now have a means of doing extrinsic proofs by reifying stateful programs, as shown below (taking s=int).

\[
\text{let \text{StNull}} a = \text{ST a (λ s0 \rightarrow \forall x, \text{post } x)}
\]

\[
\text{let \text{incr}} _:_ \text{StNull unit} = \text{let n = get' } \text{in put} (n + 1)
\]

\[
\text{let incr\_increases (s0:s)} = \text{assert (snd (reify (incr' (s0)) s0) = s0 + 1)}
\]

The StNull unit annotation on the second line above gives a weak specification for incr. However, later, when a particular property of incr is required, we can recover it by reasoning extrinsically about the reification of incr' as a pure term.

1 We leave divergence out of scope of the present work as a relatively orthogonal concept. We envisage adding divergence to EMF* and DM as a second primitive effect in the future, with divergent computations interpreted in a partial correctness semantics with only intrinsic proving available. We do not foresee any significant difficulties in doing this, following the treatment of divergence of Swamy et al. (2016). We expect this to provide partial-correctness Dijkstra monads for free.

2.5 Combining monads: state and exceptions, in two ways

To add more effects to F*, one can simply repeat the methodology outlined above. For instance, one can use DM to define exn a = unit \rightarrow (option a) in the obvious way (the unit is necessary, cf. §4.1), our automated two-step recipe extends F* with an effect for terminating programs that may raise exceptions. Of course, we would like to combine the effects to equip stateful programs with exceptions and, here, we come to a familiar fork in the road.

State and exceptions can be combined in two mutually incompati-

ble ways. In DM, we can define both steen a = s \rightarrow τ((option a) \ast s) and exnSt a = s \rightarrow τ(option (a \ast s)). The former is more familiar to most programmers: raising an exception preserves the state; the latter discards the state when an exception is raised, which though less common, is also useful. We focus first on exnSt and then discuss a variant of steen.

Relating st and exnSt Translating st (as before) and exnSt to F* gives us two unrelated effects ST and ExnST. To promote ST computations to ExnST, we define a lift relating st to exnSt, their pure representations in DM, and prove that it is a monad morphism.

\[
\text{let lift (f} st a) : exnSt a = λs0 \rightarrow \text{Some (f} s0)
\]

\[
\text{let lift\_is\_an\_exn\_st\_morphism = assert (λ} x, \text{eq (lift (ST return f)) (ExnST return x))}
\]

Applying our two-step translation to lift, we obtain in F* a computation-type coercion from ST a wp to ExnST a (wp' wp).

Through this coercion, and through F*’s existing inference algorithm (Swamy et al. 2011, 2016), ST computations are implicitly promoted to ExnST computations whenever needed. In particular, the ST actions, get and put, are implicitly available with ExnST. All that remains is to define an additional action, raise = λs \rightarrow None, which gets elaborated and reflected to F* at the type unit \rightarrow ExnST a (λ p \rightarrow p None).

ExnST programs in F* can be verified intrinsically and extrinsically. For an intrinsic proof, we show div_intrinsic below, which raises an exception on a divide-by-zero. To prove it, we make use of an abbreviation ExnSt a p post, which lets us write specifications using pre- and postconditions instead of predicate transformers, which can be more convenient—the F* keywords, requires and ensures are only there for readability and have no semantic content.

\[
\text{let ExnSt a pre post = ExnSt a (λ s0 p \rightarrow pre s0 \land \forall x, \text{post } p x)}
\]

\[
\text{let div\_intrinsic i j : ExnSt int (requires (λ x \rightarrow True)) (ensures (λ x \rightarrow match x with)}
\]

\[
\text{| None \rightarrow j = 0 | Some (z, s1) \rightarrow s1 = s1 \land j < 0 \land z = i/j)}
\]

Alternatively, for an extrinsic proof, we give a weak specification for divextrinsic and verify it by reasoning about its refined definition separately. This time, we add a call to incr in the ST effect in case of a division-by-zero. F*’s type inference lifts incr to ExnST as required by the context. However, as the proof shows, the incr has no effect, since the raise that follows it discards the state.

\[
\text{let ExnStNull a = ExnSt a (λ s0 post \rightarrow \forall x, \text{post } x)}
\]

\[
\text{let div\_extrinsic i j : ExnStNull int if j=0 then incr(); raise () else i/j let lemma\_div\_extrinsic i j = assert (match div\_extrinsic i j with}}
\]

\[
\text{| None \rightarrow j = 0 | Some (z, 0) \rightarrow j < 0 \land z = i/j)
\]

Using reify and reflect we can also build exception handlers, following ideas of Filinski (Filinski 1999). For example, in try_to below, we use a handler and (under-)specify that it never raises an exception.
let try_div i j : ExnSt int
\n(\text{requires} (\lambda x . x \rightarrow \text{True} ))
\n(\text{ensures} (\lambda x \rightarrow \text{Option.isSome } x))
\n= reflect (\lambda s0 \rightarrow \text{match reify } (\text{div}_ \text{intrinic } i \ j) \ s0 \text{ with}
\n| \text{None } \rightarrow \text{Some } (0, s0)
\nx \rightarrow x)\
\nMore systematically, we can first program a Benton and Kennedy (2001) exception handler in \(\text{DM}\), namely, as a term of type
\n\text{exnst } a \rightarrow (\text{unit } \rightarrow \text{exnst } b) \rightarrow (a \rightarrow \text{exnst } b) \rightarrow \text{exnst } b
\nand then translate it to \(\text{F}\), thereby obtaining a weakest precondition rule for it for free. More generally, adapting Plotkin and Pretnar’s algebraic effect handlers (Plotkin and Pretnar 2009) to user-defined monads \(\text{m}\), handlers can be programmed in \(\text{DM}\) as terms of type
\nm \ a \rightarrow (m \ b \rightarrow b) \rightarrow (a \rightarrow b) \rightarrow b
\nand then imported to \(\text{F}\). We leave a more thorough investigation of such effect handlers for Dijkstra monads to the future.

An exception-counting state monad: \text{stexc}\n\nFor another combination of state and exceptions, we define \text{stexc}, which in addition to combining state and exceptions (in the familiar way), also introduces an additional piece of integer state to count the number of exceptions that are raised. In \(\text{DM}\), (omitting the standard return and bind) we write:
\nlet stexc a = (s * int) \rightarrow \tau (\text{option } a * (s * int))
\nlet raise () = λ(s, n) . None, (s, n + 1)
\nlet lift (f : (s * int) \rightarrow \text{stexc } a = λ(s, n) \rightarrow \text{let } x, s1 = f s \text{ in Some } x, (s1, n)
\nNotice that \text{raise} increments a counter. Adding \text{stexc} to \(\text{F}\) proceeds as before. But, we need to be a bit careful with how we use reflection. In particular, an implicit invariant of \text{stexc} is that its second state cell monotonically increases and actually counts the number of raised exceptions. If a programmer is allowed to reflect any \((s * \text{int}) \rightarrow \text{Pure } (a * (s * \text{int}))\) \text{wp} into an \text{ExnSt} computation, then this invariant can be broken. Programmers can rely on \(\text{F}\)’s module system to simply forbid the use of \text{stexc}.\text{reflect} in client modules. Depending on the situation, the module providing the effect may still reveal a restricted version of the reflect operator to a client, e.g., we may only provide \text{reflect.increasing} to clients, which only supports reflecting computations whose exception counter does not decrease. Of course, this only guarantees that the counter over-approximates the number of exceptions raised, which may or may not be acceptable.

let reflect.increasing (f : (s * int) \rightarrow \text{Pure } (a * (s * int)) \text{ wp})
\n: \text{ExnSt } a = λ(s0, n) \text{ post } \rightarrow
\n\text{wp } s0 (λ(s1, n1) \rightarrow \text{post } (s1, n1) \land n1 ≥ n0)
\n= reflect f
\n\n\n\nThe standard combination of state and exceptions (i.e., \text{stexc}) was already provided primitive in \(\text{F}\). The other two combinations shown here were not previously supported, since \(\text{F}\) only allowed primitive effects. In the next two subsections, we present encodings of two other user-defined effects: a dynamic information-flow control monitor (§2.6) and continuations (§2.7).

2.6 Information-flow control

Information-flow control (Sabelfeld and Myers 2006) is a paradigm in which programs are deemed secure when one can prove that its behavior observable to an adversary is independent of the secrets the program may manipulate, i.e., they are non-interferent. Monadic reflection allows us to prove non-interference properties directly, by relating multiple runs of an effectful program (Benton 2004). For example, take the simple stateful program below:

let ifc h = if h then (incr()); let y = get() in decr(); y else get() + 1

It is easy to prove this program non-interferent via the extrinsic, relational proof below, which states that regardless of its secret input \((\text{h0, h1})\), ifc when run in the same public initial state \((s0)\) produces identical public outputs. This generic extrinsic proof style is in contrast to Barthe et al. (2014), whose \text{F}\textsuperscript{*} is a custom extension to \text{F} supporting only intrinsic relational proofs.

let ni.ifc = assert (\forall h0 h1 s0. reify (ifc h0) s0 = reify (ifc h1) s0)

Aside from such relational proofs, with user-defined effects, it is also possible to define monadic, dynamic information-flow control monitors in \(\text{DM}\), deferring non-interference checks to runtime, and to reason about monitored programs in \(\text{F}\). Here’s a simplified example, inspired by the floating label approach of LIO (Stefan et al. 2011). For simplicity, we take the underlying monad to be \text{exnst}, where the state is a security label from a two-point lattice that represents the secrecy of data that a computation may have observed so far.

\n\text{type label } = \text{Low } | \text{High}
\nlet difc a = label \rightarrow \tau (\text{option } a \rightarrow \text{label})
\nOnce added to \(\text{F}\), we can provide two primitive actions to interface with the outside world, where \text{DIFC} is the effect corresponding to \text{difc}. Importantly, writing to a public channel using \text{write} \text{Low} when the current label is \text{High} causes a dynamic failure signaling a potential Leak of secret information.

let join \text{li } l2 = match \text{li } l1 with | \text{High } \rightarrow \text{High } | \text{Low } \rightarrow \text{High} | \text{let reify } (\lambda \text{label } \rightarrow \text{DIFC } \text{bool } (λ \text{lp } \rightarrow \text{if } \text{lp } \text{then } \text{Some } \text{(join } \text{li } \text{l0 })) \text{else } \text{lp} \text{ None})
\nlet flows \text{li } l2 = match \text{li } l1 with | \text{High } \rightarrow \text{false } | \text{true} \text{val write } : \text{label } \rightarrow \text{DIFC } \text{unit } (λ \text{lp } \rightarrow
\n\text{if } \text{lp } \text{then } \text{p } \text{(Some } (λ)) \text{else } \text{p} \text{ None})
\nAs before, it is important to not allow untrusted client code to reflect on \text{DIFC}, since that may allow it to declassify arbitrary secrets. Arguing that \text{DIFC} soundly enforces a form of termination-insensitive non-interference requires a meta-level argument, much like that of Stefan et al. (2011).

We can now write programs like the one below, and rely on the dynamic checks to ensure they are secure.

let b1, b2 = \text{read } \text{Low}, \text{write } \text{Low} (b1 \&\& b2)
\nlet b3 = \text{read } \text{High } \text{in } \text{write } \text{High } (b1 || b3);
\n\text{write } (\text{xor } b3 b3)
\nIn this case, we can also prove that the program fails with a None at the last \text{write} \text{Low}. In contrast to the relational proof sketched earlier, dynamic information-flow control is conservative: even though the last \text{write} reveals no information on the low channel, the monitor raises an error.

2.7 CPS’ing the continuation monad

As a final example before our formal presentation, we ask the irresistible question of whether we can get a Dijkstra monad for free for the continuation monad itself—indeed, we can.

We start by defining the standard continuation monad, \text{cont}, in \(\text{DM}\). Being a subset of \(\text{F}\), we can prove that it is indeed a monad. The equality we need for this proof is an extensional equality at higher order—we use \text{F}’s refinement types to define \text{kont}, a variant of \text{cont} augmented with an extensional equality principle, and (automatically) prove the monad laws for \text{kont}.

let cont a = (a \rightarrow \text{tans} ) \rightarrow \text{tans}
\nlet return x = λk . k x
\nlet bind f g k x = f (λx . g x k)
\n(* kont: continuations with an extensional equality principle *)
\nlet kont a = f (cont a) (λk1 k2 . feq k1 k2 \Rightarrow f k1 = f k2)
\n(* kont is a monad *)
\nlet run (kont a) = assert (feq (bind f return) f);
\nlet \text{unit } (x: a) : (a \rightarrow kont b) = assert (feq (bind (return x) f) (f x))
\nlet assoc (f (kont a) (g a \rightarrow kont b) (h: b \rightarrow kont c)) =
\nassert (feq (bind f (λx . bind (g x) h)) (bind (bind f g) h))
Following our two-step recipe, we derive the Dijkstra variant of \(\text{cont} \), but first we define some abbreviations to keep the notation manageable. The type \( \text{kw}p \) is the type of a predicate transformer specifying a continuation \( a \rightarrow \text{trans} \); and \( k\) is the type of a predicate transformer of the computation that yields the final answer:

\[
\text{kw}p \ a \ = \ a \rightarrow \text{kans} \quad \text{and} \quad \text{kans} \ = \ (\text{ans} \rightarrow \text{Type}) \rightarrow \text{Type} \ = \ (\text{trans})^* \]

Using these abbreviations, we show the \( x \)-translation of \( \text{cont}, \text{return} \) and \( \text{bind} \). Instead of being just a predicate transformer, \((\text{cont})^*\) is a predicate-transformer transformer.

\[
(\text{cont})^* \ = \ \text{kw}p \ a \rightarrow \text{kans} \\
\text{return}^* \ = \ \lambda (x : a) . (\text{wp}f : \text{kw}p \ a \rightarrow \text{wp}k \times) \\
\text{bind}^* \ = \ \lambda f . (\text{wp}f : \text{kw}p \ b \rightarrow f (\lambda (x : a) . g \times \text{wp}k))
\]

For step 2, we show the elaboration of return and bind to \( F^* \), using the abbreviation \( k t \) a \( \text{wp} \) for the type of the elaborated term \( k \), where the \( \text{DM} \) term \( k \) is a continuation of type \( a \rightarrow \text{trans} \) and \( \text{wp}k \). As illustrated in \( \S 2.3 \), elaborating higher-order functions from \( \text{DM} \) to \( F^* \) introduces additional arguments corresponding to the predicate transformers of abstracted computations.

\[k t \ a \ \text{wp} \ = \ x : a \rightarrow \text{Pure} \ \text{ans} \ (\text{wp} x) \]

\[
\text{return} : x : a \rightarrow \text{cont} \ a \ (\text{return}^* \ x) \\
\text{bind} : \text{wp}f (\text{cont} a)^* \ ightarrow \ f (\text{wp}f : \text{kw}p a \rightarrow k t \ a \ \text{wp}k \rightarrow \text{Pure} \ \text{ans} \ (\text{wp} \ \text{wp}f))
\]

In the case of \( \text{return} \), we have one additional argument for the predicate transformer of the continuation \( k \)—the type of the result shows how \( \text{return} \) relates to \( \text{return}^* \). The elaboration bind involves many such additional parameters, but the main point to take away is that its specification is given in terms of \( \text{bind}^* \), using the predicate transformers \( \text{wp}f, \text{wp}g, \text{wp}k \) in place of the \( f, g, k \) computations. In both cases, the definitions of \( \text{return} \) and \( \text{bind} \) match their pre-images in \( \text{DM} \) aside from abstracting over and passing around the additional WP arguments.

To better see the monadic structure in the types of \( \text{return} \) and \( \text{bind} \) we repeat these types, but this time writing \( \text{cont} \) a \( \text{wp} \) for the type \( \text{wp}k \): \( a \rightarrow k t \ a \ \text{wp}k \rightarrow \text{Pure} \ \text{ans} \ (\text{wp} \ \text{wp}k) \):

\[
\text{return} : x : a \rightarrow \text{cont} \ a \ (\text{return}^* \ x) \\
\text{bind} : \text{wpf}(\text{cont} a)^* \rightarrow f (\text{wpf} : \text{cont} a \rightarrow \text{cont} \ a \ \text{wp}g)
\]

3. Explicitly monadic \( F^* \)

We begin our formal development by presenting \( \text{EMF}^* \), an explicitly typed, monadic core calculus intended to serve as a model of \( F^* \). As seen above, the \( F^* \) implementation includes an inference algorithm (Swamy et al. 2016) so that source programs may omit all explicit uses of the monadic \( \text{return}, \text{bind} \) and \( \text{lift} \) operators. We do not revisit that inference algorithm here and leave as future work a formal proof that after inference, \( F^* \) terms can be elaborated into \( \text{EMF}^* \) (along the lines of the elaboration of Swamy et al. (2011)).

### 3.1 Syntax

Figure 1 shows the \( \text{EMF}^* \) syntax. We highlight several key features.

**Expressions, types, WPs, and formulae** are all represented uniformly as terms; however, to evoke their different uses, we often write \( e \) for expressions, \( \tau \) for types, \( \text{wp} \) for WPs, and \( \phi \) for logical formulae. Terms include variables \((x, y, a, b, w \ etc.)\); refinement types \( x : \{ \phi \} \); \( \lambda \) abstractions; dependent products with computation-type co-domains, \( x \rightarrow c \) (with the sugar described in \( \S 2.2 \)); and applications. Constants \( T \) include \( \text{Type}_\ast \), the \( i \)th level from a countable hierarchy of predicative universes.\(^2\) We also include constants for non-dependent pairs and disjoint unions; the former are eliminated using \( \text{fst} \) and \( \text{snd} \) (also constants), while the latter are eliminated using \( \text{case}_{\ast} (e \ as \ y) \). \( x : e_1 \times e_2 \) is the type of a pair of disjoint elements, and \text{pure} \ is a predicate-transformer transformer.

### 3.1.1 Computation types

\[
\begin{align*}
\text{Computation types} \\
c & \ := \ \text{Tot} \ t \ | \ M.\text{bind} \ t \ \text{wp} \ e \ | \ M.\text{return} \ t \ e \ | \ M.\text{lift} \ t \ e \end{align*}
\]

### 3.1.2 Signatures of monadic effects and lifts

\[
\begin{align*}
\text{Signatures of monadic effects and lifts} \\
S & \ := \ D | \ S, D | S, L \\
D & \ := \ F | \ \text{bind} \ \text{return}^* | \ \text{return} \ | \ \text{act}^* | \ \text{act} \\
L & \ := \ \{ \ M.\text{lift} \ M' = e; M.\text{lift}^* M' = \text{wp} \}
\end{align*}
\]

### 3.1.3 Figure 1. Syntax of \( \text{EMF}^* \)

\[
\begin{align*}
\text{Terms} \\
e, t, \text{wp}, \phi & \ := \ x \ | \ T | x : \{ \phi \} | \lambda \ x.t.e | x \rightarrow c | e_1 \ e_2 \\
& \ | \ \text{case}_{\ast}(e \ as \ y) \times e_1 \times e_2 | \text{run} \ e \ | \ \text{pure} \ e \\
& \ | \ \text{reflect} \ e \ M.\text{lift}^* M' t \ \text{wp} e | F.\text{act} \ e \\
M.\text{return} \ t \ e & \ | \ M.\text{bind} \ t \ ? \ \text{wp} e_1 \ e_2 \ x : e_2 \\
\text{Computation types} \\
c & \ := \ \text{Tot} \ t \ | \ M.\text{bind} \ t \ \text{wp} \ e \ | \ M.\text{return} \ t \ e \ | \ M.\text{lift} \ t \ e
\end{align*}
\]

<table>
<thead>
<tr>
<th>Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e, t, \text{wp}, \phi )</td>
</tr>
<tr>
<td>( x, T, x : { \phi }, \lambda x.t.e, x \rightarrow c, e_1 e_2 )</td>
</tr>
<tr>
<td>( \text{case}_{\ast}(e \ as \ y) \times e_1 \times e_2 ), ( \text{run} \ e ), ( \text{pure} \ e )</td>
</tr>
<tr>
<td>( \text{reflect} \ e \ M.\text{lift}^* M' t \ \text{wp} e, F.\text{act} \ e )</td>
</tr>
<tr>
<td>( M.\text{return} \ t \ e ), ( M.\text{bind} \ t \ \text{wp} \ e ), ( M.\text{return} \ t \ e ), ( M.\text{lift} \ t \ e )</td>
</tr>
</tbody>
</table>

---

\( F^* \) is parameterized by a signature \( S \). A user-defined effect \( F \) \( t \ \text{wp} \) is specified using \( D \), the result of translating a \( \text{DM} \) monad. A definition \( D \) is a record containing several fields: \( \text{repr} \) is the type of an \( F \) computation refined as a pure term, \( \text{wp} \) \( \text{type} \) is the type of the \( \text{wp} \) argument to \( F \); \( \text{return}^*, \text{bind}^*, \text{act}^* \) are \( \text{EMF}^* \) expressions, and \( \text{return}^*, \text{bind}^*, \text{act}^* \) are \( \text{EMF}^* \) WPs (\text{act} \( j \) is the \( j \)th action of \( F \)). We use \( S.F.\text{return} \) to denote the lookup of the \( \text{return} \) field from \( F \)’s definition in the signature \( S \), and similar notation for the other fields.

---

\( ^2 \) We have yet to model \( F^* \)’s universe polymorphism, making the universes in \( \text{EMF}^* \) less useful than the ones in \( F^* \). Lacking universes polymorphism, we restrict computation to have results in Type\( _\ast \). A simple remediation would be replicate the monad definitions across the universe levels.

---

\[2016/8/24\]
from the signature for \( M \) to compute the final \( wp \). For example, in the case of the ST monad from §2.3, \( S.T\cdot return^* \Gamma t = \lambda x.t.\lambda s_0.s.\lambda post.post \ (s_0) \). Rule T-\text{ACT} is similar; it looks up the type of the action from the signature, and then behaves like the standard function application rule.

**Monadic reflection and reification.** Rules T-\text{REIFY} and T-\text{REFLECT} are dual, coexisting between a computation type and its underlying pure representation. Rule T-\text{RUN} coheres \( e \) from type \( Pure \) \( wp \) to \( Tot \). However, since the \( Tot \) type is unconditionally total, the second premise of the rule checks that the \( wp \) is satisfiable.

**Refinements, computations types, and proof irrelevance.** \( EMF^* \)'s refinement and computation types include a form of proof irrelevance. In T-\text{REFINER}, the universe of \( \times t \{ \phi \} \) is determined by the universe of \( t \) alone, since a witness for the proposition \( \phi \) is never materialized. Refinement formulas \( \phi \) and \( wp \)s are manipulated using an entailment relation, \( S.\Gamma \vdash \phi \), a proof-irrelevant, classical logic where all the connectives are "squashed" (Nogin 2002), e.g., \( p \wedge q \) and \( p = q \) from §2, are encoded as \( x.\text{unit}(p \wedge q) \) and \( x.\text{unit}(p = q) \), and reside in \( Type_0 \). Similar to T-\text{REFINER}, in C-\text{PURE}, the universe of a computation type is determined only by the result type. Since the \( wp \) is proof irrelevant, the use of \( Type_0 \) in the type of \( wp \) is quite natural, because its proof content is always squashed.

**Subsumption and subtyping judgment.** T-\text{SUB} is a subsumption rule for computations, which makes use of the two judgments \( S.\Gamma \vdash c < c' \) and \( S.\Gamma \vdash t < t' \), shown (selectively) in Figure 3. Rule S-\text{PURE} checks that \( t' < c \), and makes use of the \( S.\Gamma \vdash \phi \) relation to check that \( wp \) is stronger than \( wp' \), i.e., for all postconditions, the precondition computed by \( wp \) implies the precondition computed by \( wp' \).

For user-defined monads \( F \), the subtyping check delegates to their underlying representation \( S.F\cdot repr \). Rule S-\text{PROD} is the standard dependent function subtyping. Rule S-\text{REFINE} drops the refinement from the subtype, and rule S-\text{REFINER} allows subtyping to a refinement type, if we can prove the formula \( \phi \) for an arbitrary \( x \). Finally, rule S-\text{CONV} states that the beta-convertible types are subtypes of each other (\( S.\vdash t \rightarrow t' \) is the small-step evaluation judgment, introduced in the next section).

### 3.2 Static semantics

The expression typing judgment in \( EMF^* \) has the form \( S.\Gamma \vdash e : c \); where \( \Gamma \) is the list of bindings \( x : t \) as usual. Selected rules for the judgment are shown in Figure 2. In the rules, we sometimes write \( S.\Gamma \vdash e : t \) as an abbreviation for \( S.\Gamma \vdash e : Tot t \).

**Monadic returns, binds, lifts, and actions.** Rules T-\text{RETURN}, T-\text{BIND}, and T-\text{LIFT} simply use the corresponding \( wp \) specification.

\( wp \) type \( = \lambda a.s \rightarrow (a \times s \rightarrow Type_0) \rightarrow Type_0 \)

\( return^* = \lambda a. return \)

\( bind^* = \lambda a. bind \)

\( get^* = \lambda e \)

\( put = \lambda e \)

\( put^* = \lambda e \)

The other fields are not defined, since \( Pure \) is handled primarily in the \( EMF^* \) dynamic semantics (§3.3).

### 3.3 \( EMF^* \) dynamic semantics

We now turn to the dynamic semantics of \( EMF^* \), which is formalized as a strong step-by-step refinement relation. Evaluation context are defined as follows:

\[ E ::= \star | \lambda x.t.E | E e | e E | \text{run } E | \text{reify } E | \text{reflect } E \]

We use sans serif font for the actual field values.
This allows us to derive a total correctness property for the Pure monad saying that run-\(ing\) a Pure computation produces a value which satisfies all the postconditions that are consistent with the \(wp\) of the Pure computation.

**Corollary 3 (Total Correctness of Pure).** If \(S; \vdash e : \text{Pure } t\ wp\), then \(\forall p, S; \vdash p : t \rightarrow \text{Type}\_0\) and \(S; \vdash \text{wp } p\), we have \(S; \vdash \text{run } e \rightarrow^{*} v\) such that \(S; \vdash p = v\).

For the user-defined monads \(F\), we can derive their total correctness property by appealing to the total correctness of the Pure monad. For instance, for the \(ST\) monad from §2.3, we can derive the following corollary simply by using the typing of \(\text{refy}\) and Corollary 3.

**Corollary 4 (Total Correctness of \(ST\)).** If \(S; \vdash e : \text{ST } t\ wp\), then \(\forall p, S; \vdash s_0 : s, S; \vdash p : t 	imes s \rightarrow \text{Type}\_0\) and \(S; \vdash \text{wp } s_0 p\), then \(S; \vdash \text{run } ((\text{refy } e) s_0) \rightarrow^{*} v\) such that \(S; \vdash p = v\).

### 3.5 Implementation in \(F^*\)

The implementation of \(F^*\) was relatively easy to adapt to \(EMF^*\). In fact, \(EMF^*\) and \(DM\) and the translation between them were designed to match \(F^*\)'s existing type system, as much as possible. We describe the main changes that were made.

**User-defined non-primitive effects** are, of course, the main new feature. Effect configurations closely match the \(D\) form from Figure 1, the main delta being that non-primitive effects include pure implementations or \(M\_bind\), \(M\_return\), \(M\_lift\_wp\), etc.

**Handling \(\text{refy}\) and \(\text{reflect}\)** in the type-checker involved implementing the two relatively simple rules for them in Figure 2. A more significant change was made to \(F^*\)'s normalization machinery, extending it to support rules that trigger evaluation for \(\text{refy}\) and \(\text{reflect}\) programs. In contrast, before our changes, \(F^*\) would never reduce effect terms. The change to the normalizer is exploited by \(F^*\)'s encoding of proof obligations to an SMT solver—it now encodes the semantics of effectful terms to the solver, after using the normalizer to partially evaluate a \(\text{refy}\) effectful term to its pure form.

**Running programs with user-defined effects** is achieved by extracting it to OCaml, as is usual for \(F^*\), except we now inline the definitions of the underlying pure terms. Effects marked as primitive are extracted as before while making use of primitive effects in OCaml—this is formally justified in §5.

### 4. Dijkstra monads for free

This section formally presents \(DM\), a language for defining effects by giving monads with their actions and lifts between them. Via a pair of translations, we export such definitions to \(EMF^*\) as effect configurations. The first translation of a term \(e\), a CPS, written \(e^*\) produces a predicate-transformer from \(DM\) term; the second one is an \(\alpha\)-translation, \(e_\alpha\) which produces an \(EMF^*\) implementation of a \(DM\) term. The main result shows that for any \(DM\) term the result of the \(\alpha\)-translation is in a suitable logical relation to the elaboration of the term, and thus a valid specification for this elaboration. We also show that the \(\alpha\)-translation always produces monotonic and conjunctive predicates, properties that should always hold for WPs. Finally, we show that the \(\alpha\)-translation preserves all equalities in \(DM\), and thus translates \(DM\) monads into \(EMF^*\) Dijkstra monads.

### 4.1 Source: \(DM\) effect definition language

The source language \(DM\) is a simply-typed lambda calculus augmented with an abstract monad \(\tau\), as in §2.3. The language is essentially that of Filinski (1994) with certain restrictions on allowed types to ensure the correctness of elaboration.

There are two effect symbols: \(n\) (non-effectful) and \(\tau\). The typing judgment is split accordingly, and \(\varepsilon\) ranges over both of them. Every
monadic term needs to be bound via \texttt{bind} to be used.\footnote{In this formalization, \texttt{bind} and \texttt{return} appear explicitly in source programs. When using our implementation, however, the user need not call \texttt{bind} and \texttt{return}; rather, they write programs in a direct style, and \texttt{let}-bindings are turned into \texttt{binds} as needed. §4.6 provides some details on the interpretation and elaboration of concrete F* terms as DM terms.} Functions can only take non-effectful terms as arguments, but may return a monadic result.

The set of DM types is divided into \textit{A} types, \textit{H} types, and \textit{C} types, ranged over by \textit{A}, \textit{C}, and \textit{H}, respectively. They are given by the grammar:

\[
A ::= X \mid b \mid A \to A \mid A + A \mid A \times A
\]

\[
H ::= A \mid C
\]

\[
C ::= H \to A \mid H \to C \mid C \times C
\]

Here \(X\) ranges over type variables (needed to define monads) and \(b\) are base types. The \(\tau\)-arrows represent functions with a monadic result, and our translations will provide WPs for these arrows. \(A\) types are referred to as “\(\tau\)-free”, since they contain no monadic operations. \(C\) types are inherently computational in the sense that they cannot be eliminated into an \(A\) type: every possible elimination will lead to a monadic term. They are referred to as “computational types”. \(H\) types are the union of both, and are called “hypothetical” types, as they represent the types of possible functional arguments. As an example, the state monad is represented as the type \(S \to (X \times S)\), where \(X\) is a type variable and \(S\) is some type representing the state.

DM types do not include “mixed” \(A \times C\) pairs, computational sums \(C + H\), functions of type \(C \to A\), or types with right-nested \(\tau\)-arrows. We do allow nesting \(\tau\)-arrows to the left, providing the generality needed for the continuation monad, and others. These restrictions are crafted to carefully match EMF*. Without them, our translations, would generate ill-typed or logically unrelated \(\tau\)-types, and they do not appear to be severe in practice, as evidenced by the examples in §2.

The syntax for terms (\(\kappa\) standing for constants):

\[
e ::= x \mid e \mid \lambda x.H, e \mid \kappa(e, \ldots, e)\]

\[
| e \mid f | e \mid snd(e) \mid inl(e) | \text{ case } e \mid inl.x.A, e | inr.y.A, e | \text{ return}_t.e | \text{ bind}_t.e \mid \text{ to } \text{ in } e
\]

Typing judgments have the forms \(\Delta \vdash e : H!n\) and \(\Delta \vdash e : A!\tau\), where \(\Delta\) is a finite sequence of type variables and \(\Gamma\) is a normal typing context, whose types only use type variables from \(\Delta\). Here are some example rules:

\[
\Delta \vdash \lambda x.H, e : H \to H'n \quad \Delta \vdash \lambda x.H, e : H \to H'n
\]

\[
\Delta \vdash \text{ return}_t.e : A!\tau
\]

\[
\Delta \vdash \text{ bind}_t.e_1 \text{ to } x \text{ in } e_2 : A!\tau
\]

In these rules we implicitly assume that all appearing types are well-formed with respect to the grammar, e.g., one cannot form a function of type \(C \to A\) by the abstraction rule.

As an example, \texttt{return}_\texttt{S} = \(\lambda x.X\). \texttt{return}_\texttt{S} (\(x, s\)) has type \(X \to S \to (X \times S)\), using these rules.

When defining effects and actions, one deals (at a top level) with non-effectful \(C\) types (\(C!n\)). We present our main results (Theorem 5 and Theorem 6) for terms typed in an empty \(\Gamma\) as these are the most interesting cases; of course to show them, we prove more general results.

4.2 The \(\ast\)-translation

The essence of the \(\ast\)-translation is to translate \texttt{return}_\texttt{\_}\texttt{e} and \texttt{bind}_\texttt{\_}\texttt{e}_1 \texttt{to } \texttt{x} \texttt{in } \texttt{e}_2 to the returns and binds of the continuation

\[
\text{monad. We begin by defining a translation } H^\ast, \text{ that translates any } H \text{ type to the type of its predicates by CPS'ing the } \tau\text{-arrows. First, for any } \tau\text{-free type } A, A^\ast \text{ is essentially the identity, except we replace every arrow } \to \text{ by } \to \rightarrow. \text{ Then, for computation types, we define:}
\]

\[
(H \to A)^* = H^* \to A^*
\]

\[
(C \times C')^* = C^* \times C'^*
\]

\[
(H \to A)^* = H^* \to (A^* \to \text{Type}_0) \to \text{Type}_0
\]

Note that all arrows on the right hand side of the translation have a \(\text{Tot}\) codomain, as per our notational convention.

In essence, the codomains of \(\tau\)-arrows are CPS’d into a WP, which takes as argument a predicate on the result and produces a predicate representing the “precondition”. All other constructs are just translated recursively: the real work is for the \(\tau\)-arrows.

For example, for the state monad \(S \to (X \times S)\), the \(\ast\)-translation produces the EMF* type \(S \to (X \times S) \to \text{Type}_0\) \(\to \text{Type}_0\). It is the type of predicates that map an initial state and a postcondition (on both result and state) into a proposition. Modulo isomorphism (of the order of the arguments and Currying)\footnote{One can tweak our translation to generate WPs that have the usual postcondition to precondition shape. However we found the current shape to be generally easier to work with.} this is exactly the type of WPs in current F*’s state monad (cf. §1, §2.3).

The two main cases for the \(\ast\)-translation for well-typed DM terms are shown below; every other case is simply a homomorphic application of \(\ast\) on the sub-terms.

\[
(\text{return}_t.e)^* = \lambda p.(A^* \to \text{Type}_0) \to p \to e^* \text{ where } \Delta \vdash e : A!n
\]

\[
(\text{bind}_t.e_1 \text{ to } x \text{ in } e_2)^* = \lambda p.(A^* \to \text{Type}_0) \to e_1^* \lambda x.A. e_2^* \text{ where } \Delta \vdash \lambda x.A. e_1^* p
\]

Formally, the \(\ast\)-translation and elaboration are defined over a typing derivation, as one needs more information than what is present on the term. The \(\ast\)-translation of terms and types are related in the following sense, where we define the context \(\Delta\) as \(X_1 : \text{Type}_0, \ldots, X_n : \text{Type}_0\) when \(\Delta = X_1, \ldots, X_n\) (we assume that type variables are also EMF* variables).

After translation, one can abstract over the variables in \(\Delta\) to introduce the needed polymorphism in EMF*. This will also be the case for elaboration.

For example, for the previous definition of return\texttt{\_S} we get the translation \(\lambda x.X. \lambda x:S. \lambda p.(X \times S) \to \text{Type}_0\), \(p(x, s)\), which has the required transformer type: \(S \to (X \times S) \to \text{Type}_0\). It is what one would expect: to prove a postcondition \(p\) about return\texttt{\_S} \(x\), one needs to prove \(p(x, s)\) where \(s\) is the initial state.

**Theorem 5** (well-typing of \(\ast\)-translation).

\[
\Delta \vdash e : C!n \implies \Delta \vdash e^* : C^*.
\]

4.3 Elaboration

Elaboration is merely a massaging of the source term to make it properly typed in EMF*. During elaboration, monadic operations are translated to those of the identity monad in EMF*, namely \texttt{Pure}.

**Elaboration of types** We define two elaboration translations for DM types, which produce the EMF* types of the elaborated expression-level terms. The first translation \(\texttt{A}\) maps an \(A\) type to a simple EMF* type, while the second one \(\texttt{F}_C\ wp\) maps a \(C\) type and a term \(wp\) of type of \(C^*\) (a specification) into an EMF* computational product or pair. The \(\texttt{A}\) translation is the same as the CPS one, i.e., \(\texttt{A} = A^\ast\).

The \(\texttt{F}_C\ wp\) (where \(wp \to C^*\)) definition is by:

\[
(1) \text{ \quad } F_{C \times C'} wp = \text{def } F_C \cdot (\text{fst } wp) \times F_C \cdot (\text{snd } wp)
\]

\[
(2) \text{ \quad } F_{C \to H} wp = \text{def } w' \cdot C' \to F_C \cdot w' \to G_H(wp \cdot w')
\]

\[
(3) \text{ \quad } F_{A \to H} wp = \text{def } x.A \rightarrow G_H(wp \cdot x)
\]
Here we use the notation $G_0^p(wp) = F_C wp$ and $G_1^p(wp) = Pure A wp$.

The main idea is that if an EMF* term $e$ has type $F_C wp$, then $wp$ is a proper specification of the final result. Putting pairs aside for a moment, this means that if one applies enough arguments $e_i$ to $e$ in order to eliminate it into a Pure computation, then $e_1 : Pure A (wp s_1)$, where each $s_i$ is the specification for each $e_i$. This is naturally extended to pairs by the definition, defining the specification for a pair as a pair of proper specifications, as shown by case (1) above.

In case (2), the $w' : C^*$ arguments introduced by $F$ are relevant for the higher-order cases, and serve the following purpose, as illustrated in §2.3 (for the translation of bind for the ST monad, and in §2.7 for the continuation monad): when taking computations as arguments, we first require their specification in order to be able to reason about them at the type level. Taking these specification arguments is also the only way for being WP-polymorphic in EMF*. Note that, according to the dependencies, the $C^*$ argument is only used in the specifications, while we shall see in elaboration that only the $F_C wp$ is used in the expressions. When elaborating terms, we pass this specification as an extra argument where needed.

In case (3), when elaborating functions taking an argument of $A$ type there is no need to take a specification, since the argument is completely non-effectful and can be used at both expression- and type-levels. Informally, a non-effectful term is its own specification.

Returning to our state monad example, the result of $\Gamma \vdash s : Pure (X \times S)$ wp is $s : Pure (X \times S) (wp s)$, i.e., the type of a function $f$ which for any $p$, if one can prove wp $s$ p, then $f$ satisfies $p$.

**Elaboration of terms** is defined in Figure 5 and is, as expected, mostly determined by the translation of types. The translation is formally defined over typing derivations, however, for brevity, we present each translation rule simply with the terms, with the important side-conditions we rely on from the derivation shown in parenthesis. We describe only the most interesting cases.

**Computational abstractions and applications (cases 4 and 10)**

Case (4) translates a function with a computational argument $x : x.C$ to a function that expects two arguments, a WP $x.w' : C^*$ and $x$ itself, related to $x.w$ at a suitably translated type. We track the association between $x$ and $x.w$ using a substitution $s_r$, which maps every computational hypothesis $x : C$ in $s$ to $x.w$ (of type $C^*$) in $s_r$. In case (10), when passing a computation argument $e_2$, we need to eliminate the double abstraction introduced in case (4), passing both $e_2.s_r^t$, i.e., the WP of $e_2$ (substituting free computation variables), and $e_2$ itself.

**Return and bind (cases 14 and 15)**

The last two rules show the translation of return and bind for $\tau$ to return and bind for Pure in EMF*. This is one of the key points: in the elaboration, we interpret the $\tau$ as the identity monad in EMF*, whereas in the $\tau$-translation, we interpret $\tau$ as the continuation monad. Theorem 6, our main theorem, shows that EMF*'s WP computation in the Pure monad for $e$ produces a WP that is logically related to the $\tau$-translation of $e$, i.e., WPs and the CPS coincide formally, at arbitrary order.

**Theorem 6 (Logical relations lemma).** $\Delta \vdash e : C!n$ implies $\Delta \vdash e : F_C e^*$

### 4.4 Monotonicity and conjunctivity

A key property of WPs is monotonicity: weaker postconditions should map to weaker preconditions. This is also an important $F^*$ invariant that allows for logical optimizations of WPs. Similarly, WPs are conjunctive: they distribute over conjunction and universal quantification in the postcondition. We show that any EMF* term obtained from the $\tau$-translation is monotonic and conjunctive, for higher-order generalizations of the usual definitions of these properties (Dijkstra 1997).

We first define a logical relation between EMF* terms $t_1 \preceq \tau t_2$, read "$t_1$ stronger than $t_2$ at type $\tau$" and producing an EMF* formula in Type$_0$, by recursion on the structure of $\tau$: $\xrightarrow{\tau} (X \times S) wp s$.

where $b$ represents any EMF* base type and $X$ any variable.$^6$

For any type $\tau$ that doesn’t mention Type$_0$, the relation reduces to extensional equality. The relation is only defined for the subset of EMF* types that are all-Tot and non-dependent. All types resulting from the $\tau$-translation will be in this subset, so this not a limitation for our purposes.

The $\preceq$ relation is not reflexive. We say that a closed EMF* expression $e$ for which $\vdash e : Tot$ is monotonic when $\vdash e \preceq e$. For the first-order WPs this coincides with the standard definition, and for higher-order predicates it gives a reasonable extension. For a first-order example, let’s take the type of WPs for ST programs: $S \rightarrow (S \rightarrow Type_0) \rightarrow Type_0$:

\[ \frac{f \vdash \varphi \rightarrow \psi}{\varphi \vdash f \preceq \varphi \rightarrow \psi} \]

where $\varphi$ is any EMF* base type and $X$ any variable.$^6$
This is exactly the usual notion of monotonicity for imperative programs (Dijkstra 1997): “if $p_2$ is weaker than $p_1$, then $f \ s \ p_2$ is weaker than $f \ s \ p_1$ for any $s$.”

Since the $\preceq$ relation is trivially preserved by application, monotonicity is too. Also, first-order predicates on base types (such as first-order pre-/postconditions) are always monotonic according to this definition.

We proved that the $\ast$-translation of any well-typed source term $e : C \ n$ gives a monotonic $\ast e$ at the type $C^\ast$. This result is more general than it appears at a first glance: not only does it mean that WPs obtained by any defined return and bind are monotonic, but also for any action or function. Also, lifts between monads and other higher-level computations will preserve this monotonicity. Furthermore, the relation $\ast$ in the conclusion of the theorem is EMF$^\ast$’s validity judgment, i.e., we prove that these properties are actually provable within F$^\ast$ without needing to rely on some meta-level reasoning.

**Theorem 7 (Monotonicity of $\ast$-translation).**
For any $e$ and $C$, $\Delta \vdash e : C \ n$ implies $\Delta \vdash e^\ast \preceq C^\ast \ (e^\ast)$.

We give a similar higher-order definition of conjunctivity, and prove similar results ensuring the $\ast$-translation provides conjunctivity. The definition for conjunctivity is given by the following, where $\alpha$ describes the predicate-free types (including variables).

\[
C_{(\alpha-{\text{Type}}_0)}(\ast \text{Type}_0) \vdash \text{def} \forall \tau. \forall p_1.p_2.w \ s p_1 \ \wedge \ s p_2 = w (\lambda x. p_1 \ x \ x \ x) \ C_w(x), \\
C_{(\ast \text{Type}_1)}(\ast \text{Type}_0) \vdash \text{def} \forall \tau. \forall p_1 \ (\lambda x. p_1 \ x \ x \ x) \ C_{(\ast \text{Type}_1)}(\ast \text{Type}_0),
\]

Again, the relation is not defined on all types, but it does include the image of the type-level $\ast$-translation, so it is enough for our purposes. This relation is also trivially preserved by application. We then prove:

**Theorem 8 (Conjunctivity of $\ast$-translation).**
For any $e$ and $C$, $\Delta \vdash e : C \ n$ implies $\Delta \vdash C_{(\ast \text{Type}_1)}(\ast e)$.

**4.5 The $\ast$-translation preserves equality and monad laws**

We define an equality judgment on source terms that is basically $\beta\eta$-equivalence, augmented with the monad laws for the abstract $\tau$ monad. We show that the $\ast$-translation preserves this equality.

**Theorem 9 (Preservation of equality by CPS).**
If $\Delta \vdash e_1 = e_2 : H \ \tau$ then $\Delta \vdash e_1^\ast = e_2^\ast$.

Since the monad laws are equalities themselves, any source monad will be translated to a specification-level monad of WPs. This also applies to lifts: source monad morphisms are mapped to monad morphisms between Dijkstra monads.

**4.6 Implementing the translations in F$^\ast$**

We devised a prototype implementation of the two translations in F$^\ast$. Users define their monadic effects as F$^\ast$ terms in direct style, as done in §2, and these definitions get automatically rewritten into DM. As explained in §2, instead of $\tau$-arrows ($H \to \tau$), we use a distinguished F$^\ast$ effect $\tau$ to indicate where the CPS should occur. The effect $\tau$ is defined to be an alias for F$^\ast$’s $T$ot effect, which allows the programmer to reason extrinsically about the definitions and prove that they satisfy various properties within F$^\ast$, e.g., the monad laws. Once the definitions have been type-checked in F$^\ast$, another minimalistic type-checker kicks in, which has a twofold role. First, it ensures that the definitions indeed belong to DM, e.g., distinguishing $A$ types from C types. Second, it performs bidirectional inference to distinguish monadic computations from pure computations, starting from top-level annotations, and uses this type information to automatically introduce return, and bind, as needed. For instance, in the st example from §2.3, the type-checker rewrites $x, s0$ into return, $(x, s0)$; and let $x, s1 = f \ s0 \ y$ in ...

5. EMF$^\ast$ with primitive state

As we have seen in §3, EMF$^\ast$ encodes all its effects using pure functions. However, one would like to be able to run F$^\ast$ programs efficiently using primitive effects. In this section, we show how EMF$^\ast$’s pure monads apply to F$^\ast$’s existing compilation strategy, which provides primitive support for state via compilation to OCaml, which, of course, has state natively.7 The main theorem of §5.1 states that well-typed EMF$^\ast$ programs using the state monad abstractly (i.e., not breaking the abstraction of the state monad with arbitrary uses of reify and reflect) are related by a simulation to EMF$^\ast$T programs that execute with a primitive notion of state. This result exposes a basic tension: although very useful for proofs, reify and reflect can break the abstractions needed for efficient compilation. In §5.2, we show how to get the best of both worlds by relying on F$^\ast$’s Ghost effect—by restricting reify and reflect to computationally irrelevant code (e.g., proofs)—we can use them freely for proving and then erase them before compilation for efficient execution.

5.1 EMF$^\ast$T: A sub-language of EMF$^\ast$ with primitive state

The syntax of EMF$^\ast$T corresponds to EMF$^\ast$, except, we configure it to just use the ST monad. Other effects that may be added to EMF$^\ast$ can already be expanded into their encodings in its primitive Pure monad—as such, we think of EMF$^\ast$T as modeling a compiler target for EMF$^\ast$ programs extended with ST implemented primitively, and other arbitrary effects implemented purely. We thus exclude from EMF$^\ast$T the reify and reflect operators and drop type and WP arguments of return, bind and lift operators, since these are no longer relevant here.

The operational semantics of EMF$^\ast$T is a small-step, call-by-value reduction relation between pairs $(s, e)$ of a state $s$ and a term $e$. The relation includes the pure reduction steps of EMF$^\ast$ simply carrying the state along (we only show ST-beta), and three primitive reduction rules for ST, shown below. The only irreducible ST computation is ST.return $v$. The term ST.bind $e \ x.e'$ reduces even without an enclosing reify, since the state is primitive.

\[
\begin{align*}
(s, (\lambda x.t) v) &\to (s, e'[v/x]) & \text{ST-beta} \\
(s, \text{ST.bind (ST.return} v) \ x.e) &\to (s, e'[v/x]) & \text{ST-bind} \\
(s, \text{ST.get}()) &\to (s, \text{ST.return} s) & \text{ST-get} \\
(s, \text{ST.put} s') &\to (s', \text{ST.return} ()) & \text{ST-put}
\end{align*}
\]

**Relating EMF$^\ast$ to EMF$^\ast$T.** We define a (partial) translation from EMF$^\ast$ to EMF$^\ast$T, and show that one or more steps of reduction in EMF$^\ast$T are matched by one or more steps in EMF$^\ast$. This result guarantees that it is sound to verify a program in EMF$^\ast$ and execute it in EMF$^\ast$T, since the verification holds for all EMF$^\ast$ reduction sequences, and EMF$^\ast$T evaluation corresponds to one such reduction.

The main intuition behind our proof is that the reduction of reflect-free EMF$^\ast$ programs maintains terms in a very specific structure—a stateful redex reduces in a context structured like

7F$^\ast$ also compiles exceptions natively to OCaml, however we focus only on state here, leaving a formalization of primitive exceptions to the future—we expect it to be similar to the development here.
a telescope of binds, with the state threaded sequentially as the
telecope evolves. We describe this invariant structure as an EMF*-
context, \( K_s \), parameterized by a state \( s \), where \( E \) is a single-hole,
reify-and-reflect-free EMF* context, a refinement of the evaluation
contexts of \( \text{\S 3} \), to be filled by a reify-and-reflect free EMF* term,
\( f \). Additionally, we separate the \( E \) contexts by their effects into
several sorts: \( \hat{E} \): Tot and \( \hat{E} \): Pure are contexts which when filled
by a suitably typed term produce in EMF* a Tot or Pure term, respectively;
the case \( \hat{E} \): Inert is for an un-refined stateful EMF* term.
The last two cases are the most interesting, representing the base
and inductive case of the telescope of a stateful term “caught in the
act” of reducing—we refer to them as the Active contexts. We
omit the sort of a context when it is irrelevant.

Next, we define a simple translation \( \langle \rangle \) from contexts \( K_s \) to EMF\(_{ST}^* \).

\[
\langle \hat{E} \rangle = \hat{E} \\
\langle \text{Pure.bind} (K_s) p.((\lambda x.\text{reify } f) (\text{fst } p) (\text{snd } p)) \rangle = \text{ST.bind}(K_s) x.f
\]

**Theorem 10** (Simulation). For all well-typed closed, filled contexts
\( K_s f \), either \( K_s \) is Inert, or one of the following is true:

1. \( \exists s', f'. (s.([K_s] f) \longrightarrow^* (s'.([K'_s] f')) \)
   and \( K_s f \longrightarrow^* K'_s s' f' \) and sort \( K_s = \text{sort}(K'_s s') \)
   and if \( K'_s s' \) is not Active then \( s = s'. \)
2. \( K_s \) is Active and \( \exists v'. (s.([K_s] f) \longrightarrow^* (s'.\text{ST.return } v) \)
   and \( K_s f \longrightarrow^* \text{Pure.return } (v,v') \).
3. \( K_s \) is Pure and \( \exists ! v'. ([K_s] f = K_s f = \text{Pure.return } v. \)
4. \( K_s \) is Tot and \( \exists ! v'. ([K_s] f = K_s f = v. \)

5.2 Restoring reify and reflect for extrinsic proofs

Theorem 10 applies only to programs that lack reflect and use reify
only in a very specific manner, as described by the telescoping
invariant. To apply the theorem to \( F^* \), we make use of its facilities
for erase computationally irrelevant code and its module system.
To justify \( F^* \)'s primitive effect implementation, we argue that \( F^*\)
programs using the ST monad are free of reify and reflect, after
erasure of computationally irrelevant code, and hence the post-
erasure programs are candidates for Theorem 10. We briefly describe
how this works, next.

First, as described in \( \text{\S 2.5} \), we use the module system to hide both
the reify and reflect operators from clients of a module FStar.State
defining the ST effect. We expose to clients only ghost_reify, a
function equivalent to reify, but exposed to clients at the signature
shown below. Notice that the function’s co-domain is marked with
the Ghost effect, meaning that it can only be used within specifications
(e.g., WPs and assertions)—any other use will be
flagged as a typing error by \( F^* \).

\[
\text{ghost_reify: ST a wp \rightarrow Ghost (s0:s \rightarrow Pure ((a \times s) (wp s0)))}
\]

Next, within FStar.State, we implement the following combinator,
\text{refine}_ST, a total function that allows a client to augment the
specification of an effectful function \( f \) from some \( wp \) to a weaker
one that additionally records that the value returned by \( f \) on any argu-
ment and input state \( s0 \) corresponds to the computational behavior
of the (ghostly) reification of \( f \).

\[
\text{refine}_ST: f : (x:a \rightarrow ST b (wp x)) \\
\rightarrow \text{Tot } (x:a \rightarrow ST b (\lambda s0 \text{ post } \rightarrow \wp x s0 (\lambda r \rightarrow r \times \text{ghost_reify } (f x) s0 \Rightarrow \text{post } r)))
\]

6. Related work

We have already discussed many elements of related work through-
out the paper. Here we focus on a few themes not covered fully
elsewhere.

**Representing monads** Our work draws a lot from Filinski’s (1994)
monadic reflection methodology, for representing and controlling the
abstraction of monads. In particular, our DM monad definition
language is essentially the language of (Filinski 1994) with some
restrictions on the allowed types. Beyond controlling abstraction,
Filinski shows how monadic reflection enables a universal imple-
mentation of monads using composable continuations and a single
mutable cell. We do not (yet) make use of that aspect of his work,
partly because deploying this technique in practice is challenging,
since it requires compiling programs to a runtime system that
provides composable continuations. Filinski’s (1999) work on repre-
senting layered monads generalizes his technique to the setting of
multiple monads. We also support multiple monads, but instead of
layering monads, we define each monad purely, and relate them via
morphisms. This style is better suited to our purpose, since one of
our primary uses of reification is purification, i.e., revealing the pure
representation of an effectful term for reasoning purposes. With
layering, multiple steps of reification may be necessary, which may
be inconvenient for purification. Finally, Filinski (2010) gives an
operational semantics that is extensible with monadic actions, taking
the view of effects as being primitive, rather than encoded purely.
We take a related, but slightly different view: although effects are
encoded purely in EMF*, we see it as language in which to analyze
and describe the semantics of a primitively effectful object language,
EMF* \(_{ST}^* \), relating the two via a simulation.

**Dependent types and effects** Nanevski et al. developed Hoare
type theory (HTT) (Nanevski et al. 2008) and Ynot (Chipala et al.
2009) as a way of extending Coq with effects. The strategy there is to
provide an axiomatic extension of Coq with a single catch-all monad
in which to encapsulate imperative code. Being axiomatic, their
approach lacks the ability to reason extrinsically about effectful
terms by computation. However, their approach accommodates effects
like non-termination, which EMF* currently lacks. Interestingly, the
internal semantics of HTT is given using predicate transformers,
similar in spirit to EMF*’s WP semantics. It would be interesting to
explore whether or not our free proofs of monotonicity and
concurrency simplify the proof burden on HTT’s semantics.

Zambino (Casinghino et al. 2014) is a dependently typed language
with general recursion, which supports reasoning extrinsically about
potentially divergent code—this approach may be fruitful to apply
to EMF* to extend its extrinsic reasoning to divergent code.

Another point in the spectrum between extrinsic and intrinsic
reasoning is Charguéraud’s (2011) characteristic formulae, which
provide a precise formula in higher-order logic capturing the seman-
tics of a term, similar in spirit to our WPs. However, as opposed to
WPs, characteristic formulae are used interactively to prove pro-
gram properties after definition, although not via computation, but
via logical reasoning. Interesting enough, characteristic formulae
are structured in a way that almost gives the illusion that they are
the terms themselves. CFML is tool in Coq based on these ideas,
providing special tactics to manipulate formulas structured this way.
Brady (2013) encodes algebraic effects with pre- and postcondi-
tions in Idris in the style of Atkey’s (2009) parameterized monads.
Rather than speaking about the computations themselves, the pre-
and postconditions refer to some implicit state of the world, e.g.,
whether or not a file is closed. In contrast, \( F^* \)'s WPs give a full
logical characterization of a computation. Additionally, the WP
style is better suited to computing verification conditions, instead of
explicitly chaining indices in the parameterized monad.
It would be interesting, and possibly clarifying, to link up with recent work on the denotational semantics of effectful languages with dependent types (Ahman et al. 2016); in our case one would investigate the semantics of \( \mathsf{EMF}^* \) and \( \mathsf{EMF}^*_\mathsf{ST} \), which has state, but extended with recursion (and so with nontermination).

**Continuations and predicate transformers**  We are not the first to study the connection between continuations and predicate transformers. For example, Jensen (1978) and Audebaud and Zucca (1999) both derive WPs from a continuation semantics of first-order imperative programs. While they only consider several primitive effects, we allow arbitrary monadic definitions of effects. Also while their work is limited to the first-order case, we formalize the connection between WPs and CPS also for higher-order. The connection between WPs and the continuation monad also appears in Keimel (2015); Keimel and Plotkin (2016).

7. **Looking back, looking ahead**

While our work has yielded the pleasant combination of both a significant simplification and boost in expressiveness for \( \mathsf{F}^* \), we believe it can also provide a useful foundation on which to add user-defined effects to other dependently typed languages. All that is required is the \( \mathsf{Pure} \) monad upon which everything else can be built, mostly for free.

On the practical side, going forward, we hope to make use of the new extrinsic proving capabilities in \( \mathsf{F}^* \) to simplify specifications and proofs in several ongoing program verification efforts that use \( \mathsf{F}^* \). We are particularly interested in furthering the relational verification style, sketched in §2.6. We also hope to scale \( \mathsf{EMF}^* \) to be a definitive semantics of all of \( \mathsf{F}^* \)—the main missing ingredients are recursion and its semantic termination check, inductive types, universe polymorphism, and the extensional treatment of equality.

Along another axis, we have already mentioned our plans to add non-termination (§2.4) to \( \mathsf{DM} \) and \( \mathsf{EMF}^* \), and to investigate translations of effect handlers (§2.5). We also hope to enhance \( \mathsf{DM} \) in other ways, e.g., relaxing the stratification of types and adding inductive types which will allow us to define monads for some forms of nondeterminism and many forms of I/O. Enriching \( \mathsf{DM} \) further, one could also add dependent types, reducing the gap between it and \( \mathsf{F}^* \), and bringing within reach examples like Ahman and Uustalu’s (2013) dependently typed update monads.

**Acknowledgments**  We are grateful to Pierre-Evariste Dagand, Michael Hicks, and Kenji Maillard for interesting discussions. We also thank the anonymous reviewers for their helpful feedback.
A. Appendix

In this appendix we provide proofs and auxiliary results for the theorems that appear in the body of the paper. We also show the full type system for the source language.

A.1 The definitional language DM

In the typing judgment, the metavariable $\Delta$ represents a set of type variables that remains fixed throughout typing. It is used to introduce top-level let-polymorphism on all CPS'd/elaborated terms. A type is well-formed in the context $\Delta$ if all of its variables are in $\Delta$. In rigor, all judgments from here onwards are subject to that constraint, which we do not write down. A context $\Gamma$ is well-formed if both (1) all of its types are well-formed according to $\Delta$ (2) no variable names are repeated. This last condition simplifies reasoning about substitution and does not limit the language in any way.

We assume that every base type in DM is also a base type in EMF* (or that there exists a mapping from them, formally), and that source constants are also present and with the same type (formally, also a mapping for constants that respects the previous one).

The typing judgment for DM is given in Figure 6. We assume that the types appearing in the rules are well-formed. For example, in the (ST-Pair) rule, either both $H$ and $H'$ are in $A$ or both are in $C$ etc.

A.2 CPS translation (WP generation)

The full $\varepsilon$-translation for DM expressions is given in Figure 7. The one for types was previously defined. We define translation on environments in the following way:

$$\Delta = X_1, \ldots, X_n$$
$$\Delta^* = X_1 : Type_0, \ldots, X_m : Type_0$$

$$\Gamma = \lambda H_1 : \ldots, x_n : H_1$$
$$\Gamma^* = \lambda H_1^* : \ldots, x_n : H_1^*$$

One can then prove the following:

Lemma 11 (Well-typing of $\varepsilon$-translation). For any $\Gamma$, $e$, $A$ and $H$:

$$\Delta \vdash e : H \vdash n \Rightarrow \Delta^*, \Gamma^* \vdash e^* : H^*$$
$$\Delta \vdash e : A \vdash \tau \Rightarrow \Delta^*, \Gamma^* \vdash e^* : (A \rightarrow Type_0) \rightarrow Type_0$$

Proof. By induction on the typing derivation. \hfill \Box

In this lemma statement, and in those that follow, when writing $e^*$ we refer to the translation of $e$ using the typing derivation from the premise.

A.3 Elaboration

The definitions of $\Delta$ and the F relation were previously given. For elaboration, we also translate environments, in the following manner:

$$\Delta = X_1, \ldots, X_n$$
$$\Delta^* = X_1 : Type_0, \ldots, X_m : Type_0$$

$$\Gamma = \lambda H_1 : \ldots, x_n : H_1$$
$$\Gamma^* = \lambda H_1^* : \ldots, x_n : H_1^*$$

Note that for any computational variable in the context, we introduce two variables: one for its WP and one for its actual expression. The $\varepsilon^w$ variable, which is assumed to be fresh, is used only at the WP level. Also note that $\Delta^* = \Delta^*$.

For any $\Gamma$, we define the substitution $s^\varepsilon$ as $[\varepsilon^w/x_1, \ldots, \varepsilon^w/x_k]$, for the computational variables $x_1, \ldots, x_k \in \Gamma$.

Similarly to Lemma 11 we show that:

Lemma 12 (Well-typing of $\varepsilon$-translation — elaboration contexts). For any $\Gamma$, $e$, $A$ and $C$ we have:

$$\Delta \vdash e : A \vdash n \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : H^*$$
$$\Delta \vdash e : A \vdash \tau \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : (A^* \rightarrow Type_0) \rightarrow Type_0$$

For expression elaboration we aim to show that:

$$\Delta \vdash e : A \vdash n \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : H^*$$
$$\Delta \vdash e : A \vdash \tau \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : (A^* \rightarrow Type_0) \rightarrow Type_0$$

A.4 Statement of the logical relations lemma

Our main theorem is the fact the $\varepsilon$-translation of a term is properly related, via the F relation, to its elaboration. This means it provides an adequate logical representation of the term.

Theorem 13 (Logical relations lemma). For any $\Delta, \Gamma, e, C, A$:

$$\Delta \vdash e : C \vdash n \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : H^*$$
$$\Delta \vdash e : A \vdash \tau \Rightarrow \Delta^*, \Gamma^* \vdash e^\varepsilon : (A^* \rightarrow Type_0) \rightarrow Type_0$$

Note that in both cases, $e : H \vdash \varepsilon$ implies $e : G_H^\varepsilon(e^\varepsilon)$, as per the definition of $G$ given before.

As a trivial corollary of this theorem, we get that if a computational term is typable in the empty context ($\Delta \vdash e : C \vdash n$) then $\Delta \vdash e : C \vdash e^\varepsilon$, as stated in the paper.

A.5 Proof of the logical relation lemma

Theorem 14. For any $\Delta, \Gamma, \vdash e : A \vdash n \Rightarrow e = e^\varepsilon \varepsilon$. (That is, syntactic equality).

Proof. By induction on the typing derivation. The cases for (ST-RET) and (ST-BIND) do not apply.

1. (ST-Var)
   Our goal is to show $x = x^\varepsilon$. Since the type of $x$ is $A$ the substitution does not affect $x$, thus they’re trivially both $x$.

2. (ST-Const)
   Say $\Delta \vdash e : A \vdash B$. By the induction hypothesis we know that $b_i = b_i^\varepsilon$ for each $i$. We thus trivially get our goal by substitution of the arguments.

3. (ST-Abs)
   Say we concluded $\Delta \vdash e : A \vdash x^\varepsilon : A \vdash n$. Our premise is (note the substitution from the IH does not affect $x$, as it has an $A$-type) the fact that $\varepsilon = e^\varepsilon \varepsilon$. We need to show that:

$$\lambda x : A. e = (\lambda x : A. e)^\varepsilon$$

which is just

$$\lambda x : A. e = \lambda x : A^\varepsilon. e^\varepsilon$$

which is trivial from our hypothesis and since $\Delta = \defeq A^\varepsilon$.

4. (ST-App)
   Say we concluded $\Delta \vdash e : A \vdash n$ by the premises

$$\Delta \vdash f : A \vdash n \Rightarrow \Delta \vdash e : A \vdash n$$

(it cannot be the case that $e$ has some $C$ type, because of the type restrictions). From the inductive hypotheses we have:

$$\varepsilon = f^\varepsilon \varepsilon \Rightarrow f = (f^\varepsilon \varepsilon) \varepsilon \Rightarrow$$

$$\varepsilon = (f^\varepsilon \varepsilon) (e^\varepsilon \varepsilon) \Rightarrow f = (f e)^\varepsilon \varepsilon$$

As required.

5. (ST-Fst), (ST-Snd), (ST-Pair), (ST-Inl), (ST-Inr)
   All of these are trivial by applying the IH. For (ST-Pair) one needs to note that the restrictions will ensure that the type of the pair will be an $A$-type.

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6. (ST-CASE) Say we concluded $\Delta \mid \Gamma \vdash e : A \uparrow n$. As inductive hypothesis we have:

$$
\xi = e^* \sigma \quad e^1 = e^1 \sigma \quad e^2 = e^2 \sigma
$$

($e_1$ and $e_2$ are typed in the context extended with $x$ and $y$ respectively, however since they are $A$-types the substitution is the same)

The goal is:

$$(\text{case}(e) \cdot x. e_1, e_2) \quad \text{when } \Delta \mid \Gamma \vdash e : A \uparrow n)$$

From the IHs, and since $A =_{\text{def}} A^*$ we get our goal.

Theorem 15. If $\Delta \mid \Gamma \vdash e : A \uparrow n$, then $\Delta \mid \Gamma \vdash e : A$.

Proof. By induction on the typing derivation.

1. (ST-VAR)
   We have $\Delta \mid \Gamma \vdash x : A \uparrow n$, with $x \in \Gamma$. By the translation for environments, we have $x : A \downarrow x \in \Gamma$, so this is trivial.

2. (ST-CONST)
   For any constant $\kappa : (b_1, \ldots, b_n) \to b$ say we have $\Delta \mid \Gamma \vdash \kappa(e_1, \ldots, e_n) : b \uparrow n$ by (ST-CONST) (note that $b$ and all the $b_i$ are in $A$). This means that for every $i$ we have as inductive hypothesis:

$$
\Delta \mid \Gamma \vdash e_i : b_i
$$

Since $\kappa$ is also a target constant of the same type, we thus have:

$$
\Delta \mid \Gamma \vdash \kappa : A^\downarrow \kappa \quad A^\downarrow e_1 \ldots e_n : b
$$

Which is exactly our goal as $b = b$.

3. (ST-FST), (ST-NR), (ST-PAIR), (ST-CONS), (ST-INR)
   Trivial by using IH.

4. (ST-CASE)
   Say we concluded $\Delta \mid \Gamma \vdash e : A \uparrow n$. As inductive hypothesis we have:

$$
\xi = e^* \sigma \quad e^1 = e^1 \sigma \quad e^2 = e^2 \sigma \quad (e_1 \text{ and } e_2 \text{ are typed in the context extended with } x \text{ and } y \text{ respectively, however since they are } A\text{-types the substitution is the same})
$$

The goal is:

$$(\text{case}(e) \cdot x. e_1, e_2) \quad \text{when } \Delta \mid \Gamma \vdash e : A \uparrow n)$$

From the IHs, and since $A =_{\text{def}} A^*$ we get our goal.

Before jumping into the logical relation lemma, we will require the following auxiliary lemma, of which we make heavy use.

Lemma 16. Invariance of $F_C w$. If $\Gamma \vdash w_1 = w_2$, then $\Gamma \vdash F_C w_1 \prec : F_C w_2$.

Proof. By induction on $C$.

1. $C \rightarrow A$
   We need to show that

$$
\Gamma \vdash F_{C \rightarrow A} w_1 \prec : F_{C \rightarrow A} w_2
$$

Which is

$\Gamma \vdash x^n : C^n \rightarrow F_C x^n \rightarrow \text{Pure } A (w_2 x^n) \prec : \text{Pure } A (w_2 x^n)$

After two applications of (ST-PROD) (and some trivial reflexivity discharges), the required premise to show is:

$$
\Gamma \vdash x^n : C^n, F_C x^n = P_{\text{Pure }} A (w_2 x^n) \prec : \text{Pure } A (w_2 x^n)
$$
By (S-PURE) we’re required to show that Δ is a subtype of itself (which is trivial by reflexivity of subtyping (S-CONV)) and that w₂ is stronger than w₁, which can be easily proven as they are equal.

(2) A ⊑ A
Very similar to the previous case, but simpler.

(3) C ⊑ C’
We need to show that
\[ \Gamma \vdash F_{C \rightarrow C} \ x \ 1 \triangleleft F_{C \rightarrow C} \ x \ 2 \]
Which is
\[ \Gamma \vdash x^n : C \rightarrow F_{C} x^n \rightarrow F_{C} x^n \triangleleft x^n : C \rightarrow F_{C} x^n \rightarrow F_{C} x^n \]
After two applications of (ST-PROD) (and some trivial reflexivity discharges), the required premise to show is:
\[ \Gamma \vdash x^n : C \rightarrow F_{C} x^n \rightarrow F_{C} x^n \triangleleft F_{C} w_1 x^n \]
As in this context we can show w₁ xⁿ = w₂ xⁿ we apply our IH to the type C’ and are done.

(4) A ⊑ C’
Also very similar to the previous case, but simpler.

(5) C × C’
Trivial by IHs and concluding that fst w₁ = fst w₂, and similarly for snd.

\[ \square \]

Proof of Theorem 13 (Logical relations lemma)

Proof: The two parts are proved by a joint structural induction.

(1) (ST-VAR)
We have \( \Delta \mid \Gamma \vdash x : C \upharpoonright n \), with \( x \in \Gamma \). By the translation for environments, we have \( x^n : C \) and \( x : F_{C} x^n \in \Gamma \). Since \( x \) is covered by the substitution \( s_{r} \), what we need to prove is \( \Delta \Gamma \vdash x : F_{C} x^n \), which is exactly what we have in the environment.

(2) (ST-PAIR)
Suppose we proved \( (e_1, e_2) : C_1 \times C_2 \upharpoonright n \) by (ST-PAIR). We want to show: \( \Delta \vdash (e_1, e_2) : F_{C_1 \times C_2} \ (e_1, e_2)_{s_{r}} \), i.e., that (after reduction inside \( F \)):
\[ \Delta \vdash (e_1, e_2) : F_{C_1 \times C_2} \ (e_1, e_2)_{s_{r}} \]
This is trivial by applying both IHs.

(3) (ST-FST), (ST-SND)
Suppose we proved \( \Delta \mid \Gamma \vdash \text{fst}(e) : C_1 \upharpoonright n \) by (ST-FST). We need to then show
\[ \Delta \vdash \text{fst}(e) : F_{C_1} \text{fst} e_{s_{r}} \]
By our induction hypothesis we have \( \Delta \vdash e : F_{C_1 \times C_2} e_{s_{r}} \), which is
\[ \Delta \vdash e : F_{C_1} \text{fst} e_{s_{r}} \]
It is therefore easy to see that we have our goal.

(4) (ST-ABS)
There are two cases:

- \( A \xrightarrow{\xi} H \)
  Suppose we concluded \( \Delta \mid \Gamma \vdash \lambda x : A. e : A \xrightarrow{\xi} H \upharpoonright n \). Then we have \( \Delta \mid \Gamma, x : A \vdash e : H \upharpoonright n \) and so, by the induction hypothesis, in both cases for \( e \) we have
  \[ \Delta \vdash \lambda x : A. e : F_A \lambda x : A. e_{s_{r}} \]
  And we have to show:
  \[ \Delta \vdash \lambda x : A. e : G_{F_A \lambda x : A. e_{s_{r}}} \]
  Which is
  \[ \Delta \vdash \lambda x : A. e : x : A \rightarrow G_{F_A \lambda x : A. e_{s_{r}}} ((\lambda x : A. e_{s_{r}}) \ x) \]
Since the substitution does not cover \( x \), the argument to \( G \) is just \( e_{s_{r}} \), thus we use our IH to conclude this easily.

- \( C \xrightarrow{\xi} H \)
  Suppose we concluded \( \Delta \mid \Gamma \vdash \lambda x : C. e : C \xrightarrow{\xi} H \upharpoonright n \). Then we have \( \Delta \mid \Gamma, x : C \vdash e : H \upharpoonright n \) and so, by the induction hypothesis we have, in either case for \( e \):
  \[ \Delta \vdash \lambda x : C. e : F_{C} x^n \upharpoonright \varepsilon : G_{F_C} (e_{s_{r}} \ [x^n/x]) \]
  And we have to show:
  \[ \Delta \vdash \lambda x : C. e : F_{C} x^n \upharpoonright \varepsilon : G_{F_C} ((\lambda x : C. e_{s_{r}}) \ x^n) \]
  Which is
  \[ \Delta \vdash \lambda x : C. e : x : C \rightarrow G_{F_C} ((\lambda x : C. e_{s_{r}}) \ x^n) \]
  Using \( T\)-Abs twice we can conclude this via
  \[ \Delta \vdash \lambda x : C. e : F_{C} x^n \upharpoonright \varepsilon : G_{F_C} ((\lambda x : C. e_{s_{r}}) \ x^n) \]
  Since the substitution does not cover \( x \), the argument to \( F \) reduces to \( e_{s_{r}} \ [x^n/x] \), thus we use our IH to conclude this easily.

(5) (ST-APP)
Again, there are two possible cases:

- \( A \xrightarrow{\xi} H \)
  We concluded \( \Delta \mid \Gamma \vdash f e : G ! e \). Our premises are \( \Delta \mid \Gamma \vdash f : A \xrightarrow{\xi} C \upharpoonright n \) and \( \Delta \mid \Gamma \vdash e : A \upharpoonright n \). The IH for \( f \) is, expanding \( F \):
  \[ \Delta \vdash \lambda x : A \rightarrow G_{F} (f_{s_{r}} \ x) \]
  By \( T\)-App, and since \( e : A \), this is just:
  \[ \Delta \vdash \lambda x : A \rightarrow G_{F} (f_{s_{r}} \ x) \]
  Since from a previous theorem we know we have \( e = e_{s_{r}} \) (syntactically), we can conclude:
  \[ \Delta \vdash \lambda x : A \rightarrow G_{F} ((f_{s_{r}}) \ (e_{s_{r}})) \]
  This is exactly:
  \[ \Delta \vdash \lambda x : A \rightarrow G_{F} ((f_{s_{r}}) \ (e_{s_{r}})) \]
  which is our goal, in either the \( C \upharpoonright n \) or the \( A \upharpoonright \tau \) case.

- \( C \xrightarrow{\xi} H \)
  We concluded \( \Delta \mid \Gamma \vdash f e : H \upharpoonright n \). Our premises are \( \Delta \mid \Gamma \vdash f : C \xrightarrow{\xi} H \upharpoonright n \) and \( \Delta \mid \Gamma \vdash e : C \upharpoonright n \). The IHs are, expanding \( F \):
  \[ \Delta \vdash \lambda x : C \rightarrow F_{C} x^n \rightarrow G_{F} ((f_{s_{r}}) \ x^n) \]
  Thus by two uses of \( T\)-App (noting that it’s well typed by our IH for \( e \)), we can conclude:
  \[ \Delta \vdash \lambda x : C \rightarrow F_{C} x^n \rightarrow G_{F} ((f_{s_{r}}) \ (e_{s_{r}})) \]
  This is just, syntactically:
  \[ \Delta \vdash \lambda x : C \rightarrow F_{C} x^n \rightarrow G_{F} ((f_{s_{r}}) \ (e_{s_{r}})) \]
  which is our goal, in either the \( C \upharpoonright n \) or the \( A \upharpoonright \tau \) case.

(6) (ST-CASE)
There are two cases depending on wether we eliminate into \( C \upharpoonright n \) or \( A \upharpoonright \tau \). Both of these cases are quite dull, and deal mostly with the typing judgment on the target. This may be skipped without hindering any of the main ideas.

- \( C \upharpoonright n \)
  Suppose that \( \Delta \mid \Gamma \vdash e : A + A' \upharpoonright n \), \( \Delta \mid \Gamma, x : A \vdash e_1 : C \upharpoonright n \), and \( \Delta \mid \Gamma, y : A' \vdash e_2 : C \upharpoonright n \), so that \( \Delta \mid \Gamma \vdash \text{case} \ e \ \text{in} \ x : A. e_1; \ \text{inr} \ y : A'. e_2 : C \upharpoonright n \). We will go into detail only for \( e_1 \)
as the typing and reasoning for $e_2$ is exactly analogous. As inductive hypothesis for $e_1$ we have

$$\Delta \cdot x : \Delta \vdash e_1 : FC \ e'_1 \ s_1$$

We wish to show:

$$\Delta \vdash case_{FC} \ case(x) \ e_1 \ e'_2 \ s_2 \ (\varepsilon \ as \ z) \ x.\ e_1 \ y.\ e_2$$

$$: FC \ case(e') \ x.\ e'_1 \ y.\ e'_2 \ s_1$$

Which is

$$\Delta \vdash case_{FC} \ case(x) \ e_1 \ e'_2 \ s_2 \ (\varepsilon \ as \ z) \ x.\ e_1 \ y.\ e'_2$$

$$: FC \ case(e') \ x.\ e'_1 \ y.\ e'_2 \ s_1$$

Since $\varepsilon = e' \ s_1$ we will prove this has type $FC \ case(\varepsilon) \ x.\ e'_1 \ s_1 \ y.\ e'_2 \ s_1$.

By T-CaseTot, we should show:

$$\Delta \vdash e : \Delta \vdash e_1 : FC \ case(\text{inl} \ x) \ x.\ e'_1 \ y.\ e'_2 \ s_1$$

(And the one for $e_2$). We get the first one trivially by theorem 15. The second is by reduction equivalent to:

$$\Delta \vdash x : \Delta \vdash e_1 : FC \ e'_1 \ s_1$$

Which is exactly our IH for $e_1$, so we’re done.

- $A \vdash \tau$

Our hypotheses are:

$$\Delta \vdash e : \Delta \vdash e_1 + e_2 : \text{Pure} \ A \ (\varepsilon \ v_2)$$

Applying T-Case (non-dependently) we get:

$$\Delta \vdash case(\varepsilon) \ x.\ e_1 \ y.\ e_2 : \text{Pure} \ A \ (\text{case}(\varepsilon) \ x.\ e'_1 \ y.\ e'_2)$$

Since we know $\varepsilon = e' \ s_1$ and since $\varepsilon = \text{def} \ A^*$ this is exactly:

$$\Delta \vdash case(\varepsilon) \ x.\ e_1 \ y.\ e_2 : \text{Pure} \ A \ (\text{case}(\varepsilon) \ x.\ e'_1 \ y.\ e'_2)$$

Which is our goal.

(7) (ST-RET)

We have $\Delta \vdash \varepsilon : A \ n$. We need to show:

$$\Delta \vdash \text{return}_\varepsilon \ v : \text{Pure} \ A \ (\text{return}_\varepsilon \ e' \ s_1)$$

i.e.

$$\Delta \vdash \text{return}_\varepsilon \ v : \text{Pure} \ A \ (\lambda p : A^* \rightarrow \text{Type}_0 \ (e' \ s_1))$$

This is a trivial consequence of Theorem 14 by using the T-Ret rule of emf*, and the fact that $\Delta \vdash \varepsilon = \text{def} \ A^*$.

(8) (ST-BIND)

Suppose we have $\Delta \vdash e_1 : A \vdash \tau$, and $\Delta \vdash \varepsilon : A \vdash e_2 : A' \vdash \tau$, and so $\Delta \vdash \text{bind}_\varepsilon \ e_1 \ to \ x \ in \ A \ in \ e_2 : A' \vdash \tau$. We have to show:

$$\Delta \vdash \text{bind}_\varepsilon \ e_1 \ to \ x : A \ in \ e_2 : \text{Pure} \ A' \ (\text{bind}_\varepsilon \ e_1 \ to \ x : A \ in \ e_2^* \ s_1)$$

that is

$$\Delta \vdash \text{bind}_\varepsilon \ e_1 \ to \ x : A \ in \ e_2 : \text{Pure} \ A' \ (\text{bind}_\varepsilon \ e_1 \ to \ x : A \ in \ e_2^* \ s_1)$$

By our IHs we have:

$$\Delta \vdash e_1 : \text{Pure} \ A \ (e_1^* \ s_1)$$

and

$$\Delta \vdash e_2 : \text{Pure} \ A \ (e_2^* \ s_1)$$

So we get:

$$\Delta \vdash \lambda x : A \ e_2 : \lambda x : A \text{ to } \text{Pure} \ A' \ (e_2^* \ s_1)$$

By the T-Bind rule we can conclude:

$$\Delta \vdash \text{bind}_\varepsilon \ e_1 \ to \ x : A \ e_2 : \lambda x : A \text{ to } \text{Pure} \ A' \ (e_2^* \ s_1)$$

Since $A = A^*$ and $A' = A'^*$ this is exactly our goal.

Note: in this proof, we didn’t use any specific fact about Pure, except the relation between monad operations and their WPs, so this is all generalizable to another target monad that’s already defined and satisfies the base conditions for return and bind.

A.6 Equality preservation

We want to show that any source monad will give rise to specification-level monads in the target. This will be a consequence on the fact that equality is preserved by the $\varepsilon$-translation. From equality preservation we also get the property that lifts are monad morphisms without further effort.

First we define equality for the source language. It is basically standard $\beta\eta$-equivalence adding the monad laws for the base monad $T$. We keep the type and effect of each equality. It is an invariant that if we can derive an equality, both sides are well-typed at the specified type and effect.

The definition of the equality judgment is in Figure 8. Besides those rules, there is a congruence rule for every source construct, as expected.

We then prove that:

$$\Delta \vdash e_1 = e_2 : H ! \varepsilon$$

$$\Delta \vdash e'_1 = e'_2$$

Where by $\vdash$ it is meant the validity judgment of emf*.

Theorem 17 (Preservation of equality by CPS). If $\Delta \vdash e_1 = e_2 : H ! \varepsilon$ for any $\Delta, \Gamma, e_1, e_2, H, \varepsilon$, then one has $\Delta \vdash e'_1 = e'_2$.

Proof. By induction on the equality derivation. Most of the cases are trivial, since emf* has very similar rules for equality. The interesting cases are the monadic equalities, which we show here:

(1) (EQ-M1)

We concluded

$$\Delta \vdash \text{bind}_\varepsilon \ m \ to \ x \ in \ \text{return}_\varepsilon \ x \ = \ m : A \vdash \tau$$

Thus we need to show that

$$\Delta \vdash \text{bind}_\varepsilon \ m \ to \ x \ in \ \text{return}_\varepsilon \ x \ = \ m : \text{Pure} \ A \ (\text{return}_\varepsilon \ e' \ s_1)$$

That is:

$$\Delta \vdash \text{bind}_\varepsilon \ m \ to \ x \ in \ \text{return}_\varepsilon \ x \ = \ m : \text{Pure} \ A \ (\text{return}_\varepsilon \ e' \ s_1)$$

This is trivially provable by $\beta\eta$-reduction.

(2) (EQ-M2)

We concluded

$$\Delta \vdash \text{bind}_\varepsilon \ (\text{return}_\varepsilon \ e) \ to \ x \ in \ f x = f \ v : A' \vdash \tau$$

Thus we need to show that

$$\Delta \vdash \text{bind}_\varepsilon \ (\text{return}_\varepsilon \ e) \ to \ x \ in \ f x = f \ v : \text{Pure} \ A' \ (\text{return}_\varepsilon \ e^* \ s_1)$$

That is:

$$\Delta \vdash \text{bind}_\varepsilon \ (\text{return}_\varepsilon \ e) \ to \ x \ in \ f x = f \ v : \text{Pure} \ A' \ (\text{return}_\varepsilon \ e^* \ s_1)$$

Note that since $x \notin FV(f)$, $x \notin FV(f^* \ s_1)$, this is easily shown by $\beta\eta$-reduction as well.

(3) (EQ-M3)

We concluded

$$\Delta \vdash \text{bind}_\varepsilon \ (\text{bind}_\varepsilon \ m \ to \ x \ in \ e_1) \ to \ y \ in \ e_2$$

$$\vdash \text{bind}_\varepsilon \ m \ to \ x \ in \ \text{bind}_\varepsilon \ e_1 \ to \ y \ in \ e_2 : A' \vdash \tau$$

Thus we need to show that

$$\Delta \vdash \text{bind}_\varepsilon \ (\text{bind}_\varepsilon \ m \ to \ x \ in \ e_1) \ to \ y \ in \ e_2^* \ s_1$$

$$\vdash \text{bind}_\varepsilon \ m \ to \ x \ in \ \text{bind}_\varepsilon \ e_1 \ to \ y \ in \ e_2^* \ s_1$$

$$\text{bind}_\varepsilon \ m \ to \ x \ in \ \text{bind}_\varepsilon \ e_1 \ to \ y \ in \ e_2^* \ s_1.$$
That is:
\[
\Delta \vdash (\lambda p. (\lambda p'. (m^* \mathcal{S}_{\Upsilon}) (\lambda x. (e_1^2 \mathcal{S}_{\Upsilon}) p'))) (\lambda y. (e_2^2 \mathcal{S}_{\Upsilon}) p))
\]

Note that since \(x \notin \text{TV}(e_2) \implies x \notin \text{TV}(e_2^2)\), this is also easily shown by \(\beta \eta\)-reduction.

The proof above is easy, and that should not be surprising, as we are translating our abstract monadic operations into a concrete monad (continuations), thus our source equalities should be trivially satisfied after translation.

\[\square\]

### A.7 Monotonicity

We’re interested in the monotonicity of WPs. Firstly, we need a higher-order definition for this property. Throughout this section we mostly ignore the image of our \(\star\)-translation and work with a larger subset of EMF language. This gives us a stronger result than strictly necessary.

The types where the translation is defined are those non-depended and monad-free (meaning every arrow in them is a subset of \(A\)).

The proof above is easy, and that should not be surprising, as we are translating our abstract monadic operations into a concrete monad (continuations), thus our source equalities should be trivially satisfied after translation.

**Proof.** We prove these two propositions by induction on the typing derivation for \(e\). Throughout the proof \(\Delta\) plays no special role, so we just drop it from the reasoning, keeping in mind that it has to be there for having well-formd types (but nothing else).

Note that during the proof we treat \(\preceq\) abstractly, so any instantiation with a proper type (not necessarily those where \(\preceq\) reduces to equality) would be OK.

Throughout this proof we sometimes skip the subindices for \(\preceq\) in favor of compactness. Hopefully, they should be clear from the context.

1. (ST-\textsc{Var})
   We need to show \(x^1 \preceq x^2\). This is trivial from the context and by using the (\textsc{V-Assume}) rule.
2. (ST-\textsc{Const})
   The constants only deal with base types, so all inductive hypotheses for the arguments reduce to an equality, as does our goal. Our goal is then trivially provable by applications of (\textsc{V-EqP}).
3. (ST-\textsc{Abs})
   Say we concluded \(\Gamma, x : t \vdash e : s \vdash e\). As IH we have:
   \[
   \Gamma^{12}, t^1 : t^* ; t^2 ; t^* [x^1 \preceq x^2, x^2] \models e^1 [x^1 / x] \preceq e^2 [x^2 / x]
   \]
   Where \(s^*\) is either \(s\) or (\(s^* \rightarrow \text{Typ} (\varnothing)\) → \(\text{Typ} (\varnothing\) depending on \(e\).
   The proof is independent of this. What we need to prove is:
   \[
   \Gamma^{12} \models (\lambda x : t^* ; e^1) \preceq_{s^*} (\lambda x : t^* ; e^2)
   \]
   Which by definition is:
   \[
   \Gamma^{12} \models \forall x^1, x^2 : t^* , x^1 \preceq x^2 \implies (\lambda x : t^* ; e^1) x^1 \preceq_{s^*} (\lambda x : t^* ; e^2) x^2
   \]
   By reduction (\(\textsc{V-EqRed} + \textsc{(V-EqS)}\)), this is equivalent to:
   \[
   \Gamma^{12} \models \forall x^1, x^2 : t^*, x^1 \preceq x^2 \implies e^1 [x^1 / x] \preceq e^2 [x^2 / x]
   \]
   Which we can conclude from our IH and three applications on (\textsc{V-VL}).
4. (ST-\textsc{App})
Say \( \Gamma \vdash e : \xi \rightarrow s \), and \( \Gamma \vdash e : a \rightarrow t \). As inductive hypothesis we get:
\[
\Gamma^{12} \vdash f^1 \leq_{r \cdot a} f^2
\]
Where \( \Gamma' \) is either \( \Gamma' s \) or \( (s' \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 \) depending on \( \epsilon \).
Again, the proof is independent of this. Expanding the definition of \( \leq_{r} \) on the left we get:
\[
\Gamma^{12} \vdash \forall x^1, x^2 : r^1, x^1 \leq_{r} x^2 \implies f^1 x^1 \leq_{r} f^2 x^2
\]
We instantiate (using (V-\( \forall \theta \)) \( x^1, x^2 \) with \( e_1, e_2 \), and apply (V-MP) with our other IH to get:
\[
\Gamma^{12} \vdash f^1 e^1 \leq_{r} f^2 e^2
\]
Which is exactly our goal in any \( (\xi_1, \xi_2) \) case.

(5) (ST-RET)
Say \( \Gamma \vdash \text{return}_e e : t \rightarrow \tau \). Our IH gives us:
\[
\Gamma^{12} \vdash e^1 \leq_{r} e^2
\]
And we need to show that:
\[
\Gamma^{12} \vdash (\lambda p. p \ e^1) \leq_{(\tau \rightarrow \text{Type}_0) \rightarrow \text{Type}_0} (\lambda p. p \ e^2)
\]
That is:
\[
\Gamma^{12} \vdash \forall p^1, p^2, p^1 \leq p^2 \implies (\lambda p. p \ e^1) p^1 \leq_{\text{Type}_0} (\lambda p. p \ e^2) p^2
\]
By reduction:
\[
\Gamma^{12} \vdash \forall p^1, p^2, p^1 \leq p^2 \implies p^1 e^1 \leq_{\text{Type}_0} p^2 e^2
\]
Which is trivially provable by preservation of \( \leq_{r} \) by application and the IH for \( e \).

(6) (ST-BIND)
Say \( \Gamma \vdash m : a ! \rightarrow \tau \) and \( \Gamma, x : a \vdash e : b ! \rightarrow \tau \), so we get \( \Gamma \vdash \text{bind}_e m \text{ to } x \in e : b ! \rightarrow \tau \). Our IHs are:
\[
\Gamma^{12} \vdash m^1 \leq_{(a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0} m^2
\]
\[
\Gamma^{12}, x^1 : a^1, x^2 : a^1, [x^1 \leq_{a^1} x^2] \vdash e^1 x^1/x \leq_{(a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0} e^2 x^2/x
\]
We need to show that:
\[
\Gamma^{12} \vdash (\lambda a. m^1 (\lambda x. e^1 a)) \leq_{(\tau \rightarrow \text{Type}_0) \rightarrow \text{Type}_0} (\lambda a. m^2 (\lambda x. e^2 a))
\]
Which can be simplified to:
\[
\Gamma^{12}, p^1, p^2, [p^1 \leq p^2] \vdash m^1 (\lambda x. e^1 a) \leq_{\text{Type}_0} m^2 (\lambda x. e^2 a)
\]
Since \( m^1 \leq m^2 \) by the IH, this can be concluded by:
\[
\Gamma^{12}, p^1, p^2, [p^1 \leq p^2] \vdash \lambda x. e^1 a \leq_{a \rightarrow \text{Type}_0} \lambda x. e^2 a
\]
Which can be simplified to:
\[
\Gamma^{12}, p^1, p^2, [p^1 \leq p^2] \vdash a^1, x^1 : a^1, [x^1 \leq_{a} x^2] \vdash e^1 x^1/x \leq_{\text{Type}_0} e^2 x^2/x
\]
Weakening the IH for \( e \) we know \( e^1 [x^1/x] \leq e^2 [x^2/x] \) in this context. Since we also have \( p^1 \leq p^2 \) and \( \leq_{r} \) is preserved by application we have our goal.

(7) (ST-PAIR), (ST-FST), (ST-INL)
All trivial from IHs.

(8) (ST-CASE)
By case analysis on the IH for the sum type, and reduction.

Having this proof implies that any well-typed term will be given a monotonic specification. And, as a consequence, functions preserve monotonicity.

---

A.8 Conjunctivity

The definition of conjunctivity on EMF* predicate types was given previously. The full theorem which we prove is this:

**Theorem 19** (Conjunctivity of \( e \)-translation — environments). For any \( \Delta, \Gamma, e, H, A \) one has:

1. \( \Delta \vdash \Gamma \vdash e : C \rightarrow n \quad \implies \quad \Delta \vdash \Gamma \vdash \text{con}(e) \)
2. \( \Delta \vdash \Gamma \vdash e : A ! \tau \quad \implies \quad \Delta \vdash \Gamma \vdash \text{con}(e) \)

Where when \( \Gamma = x_1 : t_1, \ldots \), we define \( \text{con}(e) = e_1 : t_1_1, [\pi_1(e_1)]_1, \ldots \)

This trivially implies the previously stated theorem by taking \( \Gamma = \emptyset \).

**Proof.** By induction on the typing derivations. Once again, \( \Delta \) does not play a big role and we omit it.

(1) (ST-VAR)
Trivial from context, for any type.

(2) (ST-CONST)
Does not apply as no constant gives a type \( C \rightarrow n \) nor \( A ! \tau \)

(3) (ST-ABS)
Say we concluded \( \Gamma, x : t \vdash e : s \rightarrow t \) (where that might be \( C \rightarrow n \) or \( A ! \tau \), we treat both cases uniformly). From the IH we get:
\[
\Gamma \vdash e = \text{con}(e)
\]
Where \( s' = s \) or \( (s' \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 \) according to \( (s, e) \). By applying (V-\( \forall \)) twice we get:
\[
\Gamma \vdash \forall x : t' \exists \Gamma', (\text{con}(e)) = C_{\alpha}(e)
\]
Which is the same, by reduction, as:
\[
\Gamma \vdash \forall x : t' \exists \Gamma', (\text{con}(e)) = C_{\alpha}(\lambda x. e \ x)
\]
Thus by definition of \( C \):
\[
\Gamma \vdash C_{\alpha}(\lambda x. e \ x)
\]
As required for both cases.

(4) (ST-APP)
Trivial by the preservation of \( C \) by application, in both cases (applies (V-MP)).

(5) (ST-RET)
Say we concluded \( \Gamma \vdash \text{return}_e e : A ! \tau \). Our goal is then:
\[
\Gamma \vdash C_{\alpha}(\lambda a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 (\lambda a. \ e)
\]
Which is:
\[
\Gamma \vdash \forall p_1, p_2, (\lambda a. p \ e) p_1 \land (\lambda a. p \ e) p_2 = (\lambda a. p \ e) p_1 \land p_2 x
\]
By reduction that’s equivalent to:
\[
\Gamma \vdash \forall p_1, p_2, p_1 e \land p_2 e = p_1 e \land p_2 e
\]
Which is trivially true (without use of any IH) by (V-REFL).

(6) (ST-BIND)
Say we concluded \( \Gamma \vdash \text{bind}_e e_1 \text{ to } x \in e_2 : A ! \tau \), where \( e_1 : A ! \tau \).
Our IHs are:
\[
\Gamma \vdash C_{\alpha}(\lambda a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 (e_1)
\]
\[
\Gamma \vdash C_{\alpha}(\lambda a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 (e_2)
\]
We need to show:
\[
\Gamma \vdash C_{\alpha}(\lambda a \rightarrow \text{Type}_0) \rightarrow \text{Type}_0 (\lambda a. e_1 e_2 p)
\]
Expanding the definition, this is:
\[
\Gamma \vdash \forall p_1, p_2. (\lambda a. e_1 e_2 p) p_1 \land (\lambda a. e_1 e_2 p) p_2 = (\lambda a. e_1 e_2 p) p_1 \land p_2 x
\]
By reduction, this is equivalent to:
\[
\Gamma \vdash \forall p_1, p_2. e_1 (\lambda a. e_1 e_2 p) \land e_1 (\lambda a. e_1 e_2 p) = e_1 (\lambda a. e_1 e_2 p) (\lambda a. p_1 x \land p_2 x)
\]
By the IH for $e_2$ we know $\forall x. e_2^1 (\lambda x.p_1 x \land p_2 x) = e_2^1 p_1 \land e_2^1 p_2$. By reduction and (V-Ext) this means $(\lambda x.e_2^1 (\lambda x.p_1 x \land p_2 x)) = (\lambda x.e_2^1 p_1 \land e_2^1 p_2)$. Thus we replace on the RHS (via (V-SUBST)) and get:
\[
\Gamma_\eta \vdash \forall p_1, p_2. e_1^1 (\lambda x.e_2^1 p_1) \land e_1^1 (\lambda x.e_2^1 p_2) = e_1^1 (\lambda x.e_2^1 p_1 \land e_2^1 p_2)
\]
By some $\eta$-expansion and the IH for $e_1$ we can turn this to:
\[
\Gamma_\eta \vdash \forall p_1, p_2. e_1^1 (\lambda x.e_2^1 p_1) \land e_1^1 (\lambda x.e_2^1 p_2) = e_1^1 (\lambda x.e_2^1 p_1) \land e_1^1 (\lambda x.e_2^1 p_2)
\]
Which is trivially provable by (V-REFL).
(7) (ST-PAIR), (ST-FST)  
All trivial by IHs.
(8) (ST-INL)  
Does not apply for the cases we consider.
(9) (ST-CASE)  
Trivial by (V-SUMIND) and the IHs.

Thus, any term obtained by the $\ast$-translation (return, bind, actions, lifts, ...) will be conjunctive in this sense, which means they also preserve the property through application.

With a completely analogous definition and proof we get the expected result of conjunctivity over (non-empty) universal quantification. The non-empty requirement is not actually stressed during that proof, but it’s the wanted result as WPs (which can be taken as arguments) might not distribute over empty universals.

References

E. Brady. Programming and reasoning with algebraic effects and dependent types. ICFP, 2013.
A. Charguéraud. Characteristic formulae for the verification of imperative programs. ICFP, 2011.