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Citation for published version:
Cao, P, Thompson, J & Haas, H 2017, 'Constant-Modulus Shaped Beam Synthesis via Convex Relaxation', *IEEE Antennas and Wireless Propagation Letters*. https://doi.org/10.1109/LAWP.2016.2594141

Digital Object Identifier (DOI):
10.1109/LAWP.2016.2594141

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
IEEE Antennas and Wireless Propagation Letters

Constant Modulus Shaped Beam Synthesis via Convex Relaxation

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***Abstract*—Constant modulus shaped beam synthesis is widely employed in multi-input multi-output (MIMO) radar systems and MIMO wireless communication systems to improve the effective power gain by using only phase adjustment. To achieve the maximum beam gain, we formulate a new optimization problem to maximize the main lobe gain and also properly suppress the side lobes. However, this problem is NP-hard because of the constant modulus constraint. In order to efficiently solve this problem, we first relax the constant modulus constraint to a convex constraint, and then propose an alternating optimization algorithm to solve the relaxed problem. Interestingly, numerical results imply that the solutions of the relaxed optimization problem are (almost) constant modulus and thus the convex relaxation is usually tight.**

***Index Terms*—Linear antenna arrays, shaped beam synthesis, constant modulus constraint, analog beamforming, alternating optimization, convex relaxation**

I. Introduction and System Model

MILLIMETER wave (mmWave) communication is one promising technology for future fifth generation (5G) wireless networks due to the substantial bandwidth available at mmWave frequencies. In mmWave communication systems, *analog beam-codebook* based beam training is usually used to estimate the directions of the dominant signal rays, where each analog beam codeword is designed with a desired shape [1].

One popular shaped beam synthesis approach is to directly *match* the beam pattern $G(\theta), \theta \in \Theta$ controlled by the analog beamforming vector $\boldsymbol{w}$, where Θ is defined as the physical angles that cover the entire spatial directions, to a pre-defined shape function $d(\theta) \in \mathbb{R}_+, \forall \theta \in \Theta$, It can be formulated to the following optimization problem or its variants:

$$\min_{\boldsymbol{w}} \max_{\theta \in \Theta} |G(\theta) - d(\theta)| \quad \text{(1a)}$$
$$\text{s. t. Constant modulus constraint for } \boldsymbol{w} \quad \text{(1b)}$$

to minimize the maximum *matching error* between $G(\theta)$ and $d(\theta)$ over Θ. However, the constant modulus constraint makes Problem (1) NP-hard in general [2], and its global optimal solution is still unknown.

Due to the difficulty of meeting the constant modulus constraint, all previous work study shaped beam synthesis by using heuristic approaches. For instance, the authors in [1] utilize the the subarray method [3] and beam-spoiling techniques [4] to generate an approximate "flattened" analog beam pattern. In [5], a desired beam pattern is synthesized while maximally suppressing both the auto-correlation and cross-correlation side lobes at/between given spatial angles. The most recent work [6] applies the semidefinite programming (SDP) and rank-one solution reconstruction to solve the constant-modulus constrained problem. In [7], the sum of matching errors between the beam pattern and the desired shape function over Θ are minimized by using a Lagrange programming neural network (LPNN) based approach [8].

Based on the basic idea of Problem (1), we formulate a new optimization problem with a constant modulus constraint. This work aims to efficiently find a local optimal solution to this NP-hard problem and to achieve a better shaped beam pattern. To make the problem tractable, we first relax the constant modulus constraint to a convex constraint, and apply an efficient alternating optimization algorithm to obtain a locally optimal solution to this relaxed problem. If the obtained solution is not constant modulus, a constant modulus solution needs to be reconstructed. Interestingly, numerical results show that the solutions of the relaxed problem are usually constant modulus or *almost constant modulus* (all the elements are constant modulus except for a few elements with small error), which implies that the relaxation is usually tight and thus the obtained locally optimal solutions of the relaxation problem are usually also locally optimal for the original problem.

II. Problem Formulation and Optimization

Consider the narrow-band and far-field transmission of an M-antenna linear array. The steering vector is expressed as

$$\boldsymbol{a}(\theta) = \left[1, e^{j2\pi r_2 \cos\theta/\lambda}, \cdots, e^{j2\pi r_M \cos\theta/\lambda}\right]^T, \quad (2)$$

where $r_m, m \in \{2, \cdots, M\}$ denotes the distance between the m-th element and the first element, and $\theta \in \Theta \triangleq [-\pi/2, \pi/2]$ denotes the direction of departure/arrival of the signal.[1]

Let $\boldsymbol{w} \triangleq [w_1, w_2, \cdots, w_M]^T \in \mathbb{C}^{M\times 1}$ be a phase shifter for this array, where $w_m, \forall m \in \mathcal{M} \triangleq \{1, 2, \cdots, M\}$ denotes the complex weight for the m-th antenna satisfying $|w_m|^2 = c^2, \forall m$ where c^2 is defined as the fixed power per element. By employing $\boldsymbol{w}$, the beam pattern can be denoted by

$$G(\theta) = |\boldsymbol{w}^H \boldsymbol{a}(\theta)|^2, \quad \theta \in \Theta, \quad (3)$$

which in general consists of main lobe and side lobes corresponding to the angular ranges Θ_m and Θ_s, respectively, where $\Theta_m \cup \Theta_s = \Theta$. We define $|\Theta_m|$ as the main beamwidth.

The work of P. Cao and J. Thompson is financially supported by the UK EPSRC grant EP/L026147/1. The work of H. Haas is financially funded by the UK EPSRC fellowship EP/K008757/1. The authors are with the Institute for Digital Communications, The University of Edinburgh, Edinburgh EH9 3JL, UK (email: {p.cao, john.thompson, h.haas}@ed.ac.uk).

[1] In this model it is equally valid whether θ represents an angle in azimuth where arrays are horizontal or angle in elevation with vertically mounted arrays (one side).

In mmWave systems, it is desired to generate an analog beam pattern with a *flat main lobe* in order to estimate the channel gain after transmit/receive beam alignment [1]. This motivates us to formulate the following optimization problem

$$\max_{\boldsymbol{w}} \min_{\theta\in\Theta_m} |\boldsymbol{w}^H\boldsymbol{a}(\theta)|^2 \qquad \text{s.t. } |w_m|=c, \tag{4}$$

where we only maximize the minimum main lobe gain over Θ_m in order to achieve a flat main lobe with the largest gain. Problem (4) can be rewritten as the following problem

$$\min_{\boldsymbol{w},\epsilon} \quad \epsilon \tag{5a}$$

$$\text{s.t.} \quad ||\boldsymbol{w}^H\boldsymbol{a}(\theta)|-\overline{d}(\theta)| \le \epsilon, \forall\theta\in\Theta_m \tag{5b}$$

$$|w_m|=c, \quad \forall m\in\mathcal{M}, \tag{5c}$$

where $\epsilon \ge 0$ denotes the maximum matching error between $|\boldsymbol{w}^H\boldsymbol{a}(\theta)|$ and $\overline{d}(\theta)$ over Θ_m. Instead of defining a partially achievable shape function as in Problem(1), we stress that $(\overline{d}(\theta))^2 \in \mathbb{R}_+$ in (5b) is defined as an *upper bound* of the main lobe gain such that Problem (5) is equivalent to Problem (4). Problem (5) is NP-hard because of the quadratic constraint in (5b) and the constant modulus constraints in (5c). In order to deal with the constraints (5b), it can be rewritten as

$$\left|\boldsymbol{w}^H\boldsymbol{a}(\theta)-e^{j\psi(\theta)}\overline{d}(\theta)\right| \le \epsilon \text{ and } \psi(\theta)=\angle\boldsymbol{w}^H\boldsymbol{a}(\theta) \tag{6}$$

where $\psi(\theta)\in[-\pi,\pi]$ is introduced as an auxiliary variable. The equivalence between (6) and (5b) is based on

$$||\boldsymbol{w}^H\boldsymbol{a}(\theta)|-\overline{d}(\theta)| = |\boldsymbol{w}^H\boldsymbol{a}(\theta)-e^{j\angle\boldsymbol{w}^H\boldsymbol{a}(\theta)}\overline{d}(\theta)| \tag{7a}$$

$$< ||\boldsymbol{w}^H\boldsymbol{a}(\theta)|-e^{j\psi(\theta)}\overline{d}(\theta)|, \ \forall\psi(\theta)\neq\angle\boldsymbol{w}^H\boldsymbol{a}(\theta). \tag{7b}$$

Given $\psi(\theta)$, (6) becomes a convex constraint for $\boldsymbol{w}$.

However, the non-convex constant modulus constraints (5c) still exist. In order to make the problem tractable, we first *relax* Problem (5) to the following optimization problem

$$\min_{\boldsymbol{w},\epsilon,\psi(\theta)} \quad \epsilon \tag{8a}$$

$$\text{s.t.} \quad \left|\boldsymbol{w}^H\boldsymbol{a}(\theta)-e^{j\psi(\theta)}\overline{d}(\theta)\right| \le \epsilon, \quad \forall\theta\in\Theta_m \tag{8b}$$

$$|\boldsymbol{e}_m^T\boldsymbol{w}| \le c, \quad \forall m\in\mathcal{M}, \tag{8c}$$

where $\boldsymbol{e}_m$ is an $M\times1$ zero-vector except for the m-th element being one such that $\boldsymbol{e}_m^T\boldsymbol{w}=w_m$. The constant modulus constraint (5c) is relaxed to the constraint (8c), which is a convex constraint. The optimal solution $\psi(\theta)$ to Problem (8) is $\psi(\theta)=\angle\boldsymbol{w}^H\boldsymbol{a}(\theta)$ in (6). Except for the constant modulus constraint relaxation, Problem (8) is *equivalent* to Problem (5).

Observe that Problem (8) is still not a jointly convex problem with respect to (w.r.t.) $\{\boldsymbol{w},\epsilon,\psi(\theta)\}$ because of the constraints (8b). However, Problem (8) is jointly convex w.r.t. $\{\boldsymbol{w},\epsilon\}$ when $\psi(\theta)$ is fixed, which can be optimally solved by convex optimization toolboxes, and the optimal solution of $\psi(\theta)$ is known when $\boldsymbol{w}$ is given. Therefore, we can apply the alternating optimization algorithm to optimize $\{\boldsymbol{w},\epsilon\}$ and $\psi(\theta)$ in an iterative fashion. We assume that Θ_m is evenly sampled by K angles, i.e., $\{\theta_k\}_{k=0}^{K-1}$. Then, the alternating optimization algorithm can be described as follows.

1) Convergence and Optimality: In Algorithm 1, $\{\boldsymbol{w},\epsilon\}$ and $\psi(\theta)$ are optimally solved in each iteration, respectively.

Algorithm 1 Alternating Optimization for Problem (8)

Initialization: $i=0$, $\psi^{(0)}(\theta_k)\in[-\pi,\pi],\forall k$ and ϵ_{th}.
repeat
  Given $\psi^{(i)}(\theta_k)$, solve Problem (8) and obtain $\{\boldsymbol{w}^{(i+1)},\epsilon^{(i+1)}\}$;
  Given $\boldsymbol{w}^{(i+1)}$, let $\psi^{(i+1)}(\theta_k)=\angle\boldsymbol{w}^{(i+1),H}\boldsymbol{a}(\theta_k), \ \forall k$.
  $i\leftarrow i+1$.
until $|\epsilon^{(i+1)}-\epsilon^{(i)}|\le\epsilon_{th}$;

Thus, the objective values $\{\epsilon^{(i)}\}$ form a monotonically decreasing sequence and this sequence is lower bounded, which implies that Algorithm 1 always converges to a locally optimal solution of Problem (8). For a non-convex problem with multiple locally optimal solutions, the alternating optimization algorithm might achieve different locally optimal solutions with different initializations. For example, a good initialization could lead to the global optimal solution, although we cannot theoretically prove its global optimality.

2) The Choice of $\overline{d}(\theta)$: Since $(\overline{d}(\theta))^2$ is defined as an upper bound of the beam pattern gain, we set it to be

$$(\overline{d}(\theta))^2 \ge\triangleq \rho(\theta)\arg\max_{\theta} G(\theta)=\rho(\theta)M^2, \ \forall\theta\in\Theta_m \tag{9}$$

where M^2 is the maximum theorectical value of $G(\theta)$ achieved at an angle θ when $\boldsymbol{w}=\boldsymbol{a}(\theta)$, and the scalar $\rho(\theta)\ge1,\forall\theta\in\Theta_m$ can be used to control the shape of the main lobe.

3) Complexity: The main computation in Algorithm 1 is to solve the optimization problem w.r.t. $\{\boldsymbol{w},\epsilon\}$ in each iteration, which is a second order cone program (SOCP) and can be efficiently solved by the CVX solver Sedumi based on the interior-point method. Numerical results also imply that Algorithm 1 only needs a few iterations for convergence. Thus, Algorithm 1 has a reasonably low complexity. We remark that the complexity of Algorithm 1 is not a critical problem, since the analog beam-codebook can be generated *offline*.

III. Solution Analysis and Extension

In this section, we will analyze the solution of Problem (8) by Algorithm 1 for an uniform linear array (ULA) with element-spacing of half a wavelength, and provide sufficient conditions when the constant-modulus relaxation is tight.

A. Solution Analysis

Define $\boldsymbol{w}^\star \triangleq \left[|w_1^\star|e^{j\phi_1^\star},|w_2^\star|e^{j\phi_2^\star},\cdots,|w_M^\star|e^{j\phi_M^\star}\right]^T$ with $|w_m^\star|\in[0,c]$ and $\phi_m^\star\in[-\pi,\pi]$ as a solution to Problem (8) by Algorithm 1. If we have $|w_m^\star|=c,\forall m\in\mathcal{M}$, the constant modulus constraint relaxation is tight, and $\boldsymbol{w}^\star$ will also be locally optimal for Problem (5). Otherwise, we need to reconstruct a constant modulus solution from $\boldsymbol{w}^\star$. One straightforward way is to let

$$\widehat{w}_m^\star \triangleq cw_m^\star/|w_m^\star|, \quad \forall m\in\mathcal{M} \tag{10}$$

be the m-th element of the constant modulus solution $\widehat{\boldsymbol{w}}^\star$. Then, the modulus error can be expressed as $\boldsymbol{\delta}\triangleq\widehat{\boldsymbol{w}}^\star-\boldsymbol{w}^\star$.

The beam pattern gain error caused by the reconstruction (10) for each $\theta\in\Theta$ can be defined as

$$E(\theta) \triangleq ||\widehat{\boldsymbol{w}}^{\star,H}\boldsymbol{a}(\theta)|^2 - |\boldsymbol{w}^{\star,H}\boldsymbol{a}(\theta)|^2| \tag{11a}$$
$$= ||\boldsymbol{\delta}^H\boldsymbol{a}(\theta)|^2 + 2\mathrm{Re}(\boldsymbol{\delta}^H\boldsymbol{a}(\theta)(\boldsymbol{a}(\theta))^H\boldsymbol{w}^\star)|, \tag{11b}$$
$$= \sum_{m=1}^{M}\sum_{n=1}^{M}(c-|w_m^\star|)(c+|w_n^\star|)\times \cos(\phi_m^\star-\phi_n^\star+(n-m)\pi\cos(\theta)), \tag{11c}$$
$$= \sum_{m\in\mathcal{M}_0}(c^2-|w_m^\star|^2) + \sum_{m\in\mathcal{M}_0}(c-|w_m^\star|)\sum_{n\in\mathcal{M}\backslash m}\times (c+|w_n^\star|)\cos(\phi_m^\star-\phi_n^\star+(n-m)\pi\cos(\theta)), \tag{11d}$$

where $\mathcal{M}_0 \subseteq \mathcal{M}$ denotes a set of elements of $|w_m^\star| < c, \forall m \in \mathcal{M}_0$. Interestingly, the results in Fig. 4 show that the solution $\boldsymbol{w}^\star$ to Problem (8) from Algorithm 1 is constant modulus for some main beamwidths (e.g., $|\Theta_m| = 20^o, 30^o, 80^o, 90^o, 100^o$) and *almost* constant modulus (with a small $||\boldsymbol{\delta}||_2$ and only one or two non-constant modulus elements) for $|\Theta_m| = 40^o, 50^o, 60^o, 70^o$. It implies that $E(\theta)$ might be very small in this case. However, without the knowledge of $\{\phi_m^\star\}_{m=1}^M$, it is not possible to exactly estimate the value of $E(\theta)$ based on (11d). For statistical analysis, if we assume the phases $\{\phi_m^\star\}$ are randomly and independently distributed in $[-\pi, \pi]$, the *expectation* of $E(\theta)$ becomes $\sum_{m\in\mathcal{M}_0}(c^2-|w_m^\star|^2)$. Thus, the reconstruction (10) in general will not significantly change the shaped beam pattern since both $\mathcal{M}$ and $||\boldsymbol{\delta}||_2$ are small.

Theorem 1 *For an M-element ULA with with element-spacing of half a wavelength, the optimal solution $\boldsymbol{w}^\star$ of Problem* (8) *will be constant modulus if $\{\phi_m^\star\}$ satisfy*

$$\min_{m\in\mathcal{M}}\left\{\sum_{n=1,n\neq m}^{M}\cos\left(\phi_n^\star-\phi_m^\star+(m-n)\pi\cos(\theta)\right)\right\} > -\frac{1}{2}, \tag{12}$$

for each $\theta \in \Theta_m$. □

Proof: The beam pattern $G(\theta)$ can be rewritten to

$$G(\theta) = \sum_{m=1}^{M}|w_m^\star|^2 + 2\sum_{m=1}^{M}\sum_{n=1,n\neq m}^{M}|w_m^\star||w_n^\star|\times \cos\left(\phi_n^\star-\phi_m^\star+(m-n)\pi\cos(\theta)\right), \tag{13}$$

where $G(\theta)$ is a quadratic function of each $|w_m^\star|$. Thus, if

$$\sum_{n=1,n\neq m}^{M}|w_n^\star|\cos\left(\phi_n^\star-\phi_m^\star+(m-n)\pi\cos(\theta)\right) > -c/2 \tag{14}$$

is fulfilled for $\forall m \in \mathcal{M}$, $G(\theta)$ will be monotonically increasing between $[\frac{c}{2}, c]$. In this case, the maximum of $G(\theta)$ will be surely achieved at $|w_m^\star| = c$. Suppose that $G(\theta)$ is monotonically increasing between $[\frac{c}{2}, c]$ for each $|w_m^\star|$, $\forall m \in \mathcal{M}$, this requires that (12) hold, since each $|w_m^\star|$ will be equal to c. ■

B. Extension to Side Lobe Suppression

Recalling that Problem (8) only maximizes the main lobe, we desire to add a penalty to the objective function such that the side lobes can be also simultaneously suppressed but without a significant influence on the (almost) constant-modulus property of the solution $\boldsymbol{w}^\star$, which is formulated to the following problem

$$\min_{\boldsymbol{w},\epsilon,\psi(\theta)} \quad \epsilon + \mu||\boldsymbol{w}^H\boldsymbol{A}_s||^2 \tag{15a}$$
$$\text{s.t.} \quad \left|\boldsymbol{w}^H\boldsymbol{a}(\theta) - e^{j\psi(\theta)}\overline{d}(\theta)\right| \le \epsilon, \quad \forall\theta \in \Theta_m \tag{15b}$$
$$|w_m| = c, \quad \forall m \in \mathcal{M} \tag{15c}$$

where $\mu \ge 0$ is used to scale the impact of the penalty term on the objective function, and $||\boldsymbol{w}^H\boldsymbol{A}_s||^2$ with $\boldsymbol{A}_s \triangleq [\boldsymbol{a}(\theta_\ell)]_{\theta_\ell\in\Theta_s}$ denotes the sum power of the side lobes. The minimization of $\mu||\boldsymbol{w}^H\boldsymbol{A}_s||^2$ subject to $|w_m| \le c, \forall m$ yields $\boldsymbol{w} \to \boldsymbol{0}$, which conflicts with the main goal of $|w_m| = c, \forall m$. Thus, it implies that the parameter μ should be small such that it does not significantly influence both the main lobe shape and the solution's element-modulus. Problem (8) is a special case of Problem (15) when $\mu = 0$, and thus Problem (15) can be also solved by Algorithm 1.

IV. Numerical Results

In this section, we consider a unit-modulus analog beamforming design example for a 64-element ULA with element-spacing of half a wavelength. The proposed shaped beam synthesis approach is evaluated by solving the general Problem (15) with different choices of $\mu \ge 0$.

First, we evaluate Algorithm 1 with one initialization of $\psi^{(0)}(\theta_k) = 0, \forall k$ and another 99 random initializations all for $\Theta_m = [-20^o, 20^o]$. In Fig. 1, Algorithm 1 with different initializations converges to the same solution for each μ. The solution $\boldsymbol{w}^\star$ is unit-modulus for $\mu = 0.2, 0.4$, and almost unit-modulus for $\mu = 0, 0.6, 0.8$ (only one or two non-unit-modulus elements). The achieved beam patterns by $\widehat{\boldsymbol{w}}^\star$ are shown in Fig. 2, where the beam pattern for $\mu = 0.2, 0.4$ maintains a similar flat main lobe shape but a suppressed side lobe compared with that for $\mu = 0$. However, as μ increases, the main lobe gain will be also reduced (e.g., $\mu = 0.6, 0.8$ in Fig. 2), which is not the desired outcome of maximizing beam gain. Therefore, a small μ enables constant modulus solutions and a more desired beam pattern. These simulations imply that our proposed approach is effective (e.g., more than 30 dB gap between the average main lobe gain and average side lobe gain when $\mu = 0.2$) and the algorithm is also robust to different initializations and has a reasonable low complexity. For instance, Algorithm 1 takes about 10 iterations for $\epsilon_{th} = 0.001$ and about 0.97 second/iteration when it is run by the CVX solver Sedumi in MATLAB2015a/Linux 7.1 environment on a 3-GHz Intel Xeon computer.

In Fig. 3, for $\Theta_m = [-20^o, 20^o]$ we provide a performance comparison with several benchmarks: 1) One is computed by the SDP-based algorithm employed in [6]. By defining $\boldsymbol{W} \triangleq \boldsymbol{w}\boldsymbol{w}^H$, Problem (4) can be rewritten as

$$\max_{\boldsymbol{W}}\ \min_{\theta\in\Theta_m} \quad \mathrm{trace}(\boldsymbol{W}\boldsymbol{a}(\theta)\boldsymbol{a}(\theta)^H) \tag{16a}$$
$$\text{s.t.} \quad \mathrm{trace}(\boldsymbol{W}\boldsymbol{e}_m\boldsymbol{e}_m^H) = c, \quad \forall m \in \mathcal{M} \tag{16b}$$

However, the solution $\boldsymbol{W}^\star$ might not be rank-one and thus we need to extract $\boldsymbol{w}^\star$ from $\boldsymbol{W}^\star$: The "Largest eigen [6]"

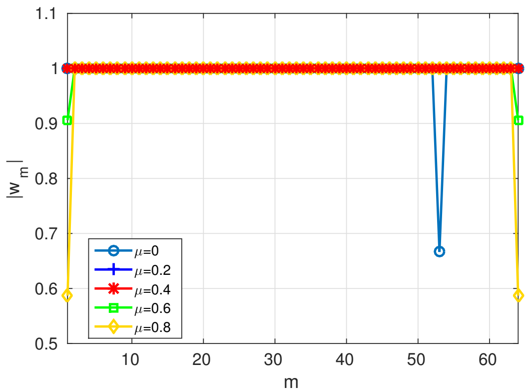

Fig. 1: Solution modulus w.r.t. different μ.

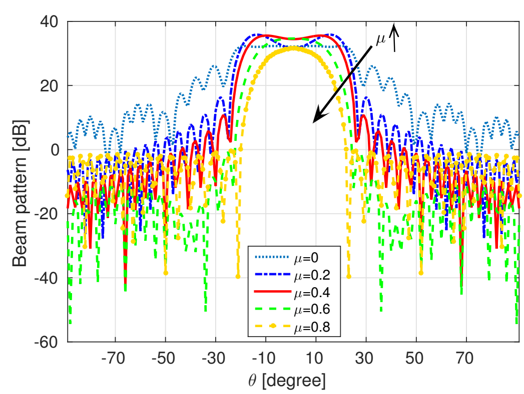

Fig. 2: Shaped beam pattern w.r.t. different μ.

uses the largest eigenvector of $\boldsymbol{W}^{\star}$ to approximate $\boldsymbol{w}$. By the randomization techniques, "max mean rand [6]" and "max min rand [6]" are achieved by the "best" random vectors among one million $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{W}^{\star})$ based on the criteria

$$\max\ \text{mean}_{\theta\in\Theta_m}\ |\widehat{\boldsymbol{\zeta}}^H \boldsymbol{a}(\theta)|, \max\ \min_{\theta\in\Theta_m}\ |\widehat{\boldsymbol{\zeta}}^H \boldsymbol{a}(\theta)|, \quad (17)$$

respectively. 2) The "Subarray beam spoiling [1]" is obtained by employing the subarray beam spoiling techniques used in [1], where the 64-element ULA is divided into four 16-element subarrays and each angle offset is 10^o. 3) The "LPNN based Approach [7]" is based on the LPNN algorithm employed in [8] by minimizing the sum matching errors

$$\min_{\boldsymbol{w}^H\boldsymbol{e}\boldsymbol{e}^H\boldsymbol{w}=1,\forall m} \Big(\sum_{\theta\in\Theta_m} \big(\boldsymbol{w}^H\boldsymbol{a}(\theta)\boldsymbol{a}^H(\theta)\boldsymbol{w} - \overline{d}^2(\theta)\big)^2 + \frac{C_0}{2}\sum_{m=1}^{M}(\boldsymbol{w}^H\boldsymbol{e}_m\boldsymbol{e}_m^H\boldsymbol{w}-1)^2\Big), \quad (18)$$

where $C_0 = 100$ is chosen and $100,000$ iterations are run in the simulations. Observe that the proposed algorithm with $\mu = 0.2$ achieves greater main lobe gain but also lower side lobes compared with the baselines. Second, Algorithm 1 is evaluated for different beamwidths, i.e., Θ_m from $[-10^o, 10^o]$ to $[-50^o, 50^o]$, respectively. The sum modulus error $||\boldsymbol{\delta}||_2$ is shown in Fig. 4, which shows that Algorithm 1 yields the desired (almost) constant modulus solutions.

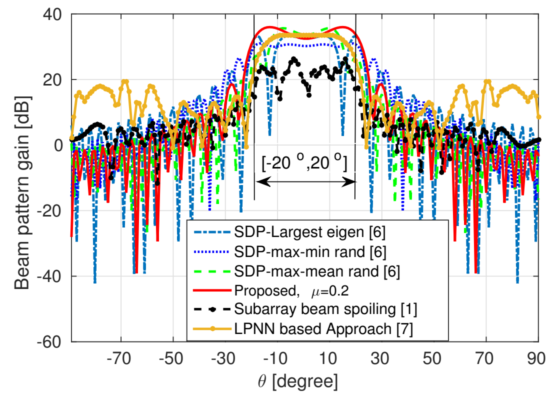

Fig. 3: Shaped beam pattern performance comparison

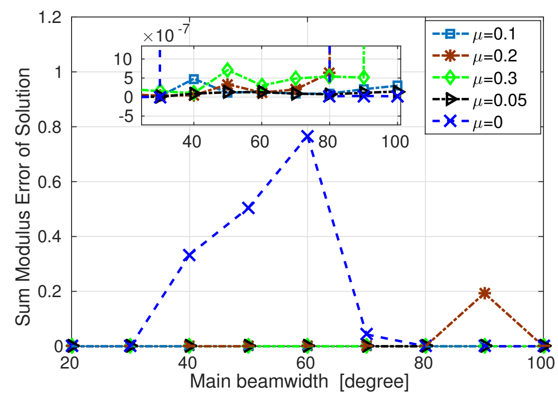

Fig. 4: Sum error $||\boldsymbol{\delta}||_2$ w.r.t. main beamwidth.

V. Conclusion

In this work, numerical results imply that our proposed constant modulus constraint relaxation is (almost) tight, and thus the alternating optimization algorithm usually enables to achieve a locally optimal solution for the constant modulus constrained problem. Furthermore, compared with the benchmarks, the proposed algorithm can generate a better analog beam pattern with a flat main lobe and suppressed side lobes.

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