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Citation for published version:

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Publisher's PDF, also known as Version of record

Published In:

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Multi-Agent Only-Knowing Revisited

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Abstract

Levesque introduced the notion of only-knowing to precisely capture the beliefs of a knowledge base. He also showed how only-knowing can be used to formalize non-monotonic behavior within a monotonic logic. Despite its appeal, all attempts to extend only-knowing to the many agent case have undesirable properties. A belief model by Halpern and Lakemeyer, for instance, appeals to proof-theoretic constructs in the semantics and needs to axiomatize validity as part of the logic. It is also not clear how to generalize their ideas to a first-order case. In this paper, we propose a new account of multi-agent only-knowing which, for the first time, has a natural possible-world semantics for a quantified language with equality. We then provide, for the propositional fragment, a sound and complete axiomatization that faithfully lifts Levesque’s proof theory to the many agent case. We also discuss comparisons to the earlier approach by Halpern and Lakemeyer.

Introduction

Levesque’s notion of only-knowing is a single agent monotonic logic that was proposed with the intention of capturing certain types of nonmonotonic reasoning. Levesque (1990) already showed that there is a close connection to Moore’s (1985) autoepistemic logic (AEL). Recently, Lakemeyer and Levesque (2005) showed that only-knowing can be adapted to capture default logic as well. The main benefit of using Levesque’s logic is that, via simple semantic arguments, nonmonotonic conclusions can be reached without the use of meta-logical notions such as fixpoints (Rosati 2000; Levesque and Lakemeyer 2001). Only-knowing is then naturally of interest in a many agent context, since agents capable of non-trivial nonmonotonic behavior should believe other agents to also be equipped with nonmonotonic mechanisms. For instance, if all that Bob knows is that Tweety is a bird and a default that birds typically fly, then Alice, if she knows all that Bob knows, concludes that Bob believes Tweety can fly.\(^1\) Also, the idea of only-knowing a collection of sentences is useful for modeling the beliefs of a knowledge base (KB), since sentences that are not logically entailed by the KB are taken to be precisely those not believed. If many agents are involved, and suppose Alice has some beliefs on Bob’s KB, then she could capitalize on Bob’s knowledge to collaborate on tasks, or plan a strategy against him.

As a logic, Levesque’s construction is unique in the sense that in addition to a classical epistemic operator for belief, he introduces a modality to denote what is at most known. This new modality has a subtle relationship to the belief operator that makes extensions to a many agent case nontrivial. Most extensions so far make use of arbitrary Kripke structures, that already unwittingly discard the simplicity of Levesque’s semantics. They also have some undesirable properties, perhaps invoking some caution in their usage. For instance, in a canonical model (Lakemeyer 1993), certain types of epistemic states cannot be constructed. In another Kripke approach (Halpern 1993), the modalities do not seem to interact in an intuitive manner. Although an approach by Halpern and Lakemeyer (2001) does indeed successfully model multi-agent only-knowing, it forces us to have the semantic notion of validity directly in the language and has proof-theoretic constructs in the semantics via maximally consistent sets. Precisely for this reason, that proposal is not natural, and it is matched with a proof theory that has a set of new axioms to deal with these new notions. It is also not clear how one can extend their semantics to the first-order case. Lastly, an approach by Waaler (2004) avoids such an axiomatization of validity, but the model theory also has problems (Waaler and Solhaug 2005). Technical discussions on their semantics are deferred to later.

The goal of this paper is to show that there is indeed a natural semantics for multi-agent only-knowing for the quantified language with equality. For the propositional subset, there is also a sound and complete axiomatization that faithfully generalizes Levesque’s proof theory.\(^2\) We also differ from Halpern and Lakemeyer in that we do not enrich the language any more than necessary (modal operators for each agent), and we do not make use of canonical Kripke models. And while canonical models, in general, are only workable

\(^1\)We use the terms “knowledge” and “belief” interchangeably in the paper.

\(^2\)The proof theory for a quantified language is well known to be incomplete for the single agent case. It is also known that any complete axiomatization cannot be recursive (Halpern and Lakemeyer 1995; Levesque and Lakemeyer 2001).
semantically and can not be used in practice, our proposal has a computational appeal to it. We also show that if we do enrich the language with a modal operator for validity, but only to establish a common language with (Halpern and Lakemeyer 2001), then we agree on the set of valid sentences. Finally, we obtain a first-order multi-agent generalization of AEL, defined solely using notions of classical logical entailment and theorembhood.

The rest of the paper is organized as follows. We review Levesque’s notions,3 and define a semantics with so-called k-structures. We then compare the framework to earlier attempts. Following that, we introduce a sound and complete axiomatization for the propositional fragment. In the last sections, we sketch the multi-agent (first-order) generalization of AEL, and prove that k-structures and (Halpern and Lakemeyer 2001) agree on valid sentences, for an enriched language. Then, we conclude and end.

The k-structures Approach

The non-modal part of Levesque’s logic4 ONL consists of standard first-order logic with = and a countably infinite set of standard names N.5 To keep matters simple, function symbols are not considered in this language. We call a predicate other than =, applied to first-order variables or standard names, an atomic formula. We write ω to mean that the variable x is substituted in α by a standard name. If all the variables in a formula α are substituted by standard names, then we call it a ground formula. Here, a world is simply a set of ground atoms, and the semantics is defined over the set of all possible worlds W. The standard names are thus rigid designators, and denote precisely the same entities in all worlds. ONL also has two modal operators: L and N. While Lα to be read as "at least α is known", Nα is to be read as "at most ¬α is known". A set of possible worlds is referred to as the agent’s epistemic state e. Defining a model to be the pair (e, w) for w ∈ W, components of ONL’s meaning of truth are:

1. e, w |= p if p ∈ w and p is a ground atom,
2. e, w |= (m = n) iff m and n are identical standard names,
3. e, w |= ¬α iff e, w ¥ α,
4. e, w |= α ∨ β iff e, w |= α or e, w |= β,
5. e, w |= ∀x. α iff e, w |= αx for all standard names x,
6. e, w |= Lo iff for all w’ ∈ e, e, w’ |= α, and
7. e, w |= No iff for all w’ ¥ e, e, w’ |= α.

The main idea is that α is (at least) believed iff it is true at all worlds considered possible, while (at most) α is believed to be false iff it is true at all worlds considered impossible. So,

an agent is said to only-know α, syntactically expressed as Lα ∧ N¬α, when worlds in e are precisely those where α is true. Halpern and Lakemeyer (2001) underline three features of the semantical framework of ONL, the intuitions of which we desire to maintain in the many agent setting:

1. Evaluating Nα does not affect the epistemic possibilities. Formally, in ONL, after evaluating formulas of the form Nα the agent’s epistemic state is still given by e.
2. A union of the agent’s possibilities, that evaluate L, and the impossible worlds that evaluate N, is fixed and independent of e, and is the set of all conceivable states. Formally, in ONL, Lo is evaluated wrt. worlds w ∈ e, and Nα is evaluated wrt. worlds w ∈ W − e; the union of which is W. The intuition is that the exact complement of an agent’s possibilities is used in evaluating N.
3. Given any set of possibilities, there is always a model where precisely this set is the epistemic state. Formally, in ONL, any subset of W can be defined as the epistemic state.

Although these notions seem clear enough in the single agent case, generalizing them to the many agent case is non-trivial (Halpern and Lakemeyer 2001). We shall return to analyze the features shortly. Let us begin by extending the language. Let ONLb be a first-order modal language that enriches the non-modal subset of ONL with modal operators L_i and N_i for i = a, b. For ease of exposition, we only have two agents a (Alice) and b (Bob). Extensions to more agents is straightforward. We freely use O_i, such that O_iα is an abbreviation for L_iα ∧ N_i¬α, and is read as "all that i knows is α". Objective and subjective formulas are understood as follows.

Definition 1. The i-depth of a formula α, denoted |α|_i, is defined inductively as (□_i denotes L_i or N_i):

1. |α|_i = 1 for atoms,
2. |¬α|_i = |α|_i,
3. |∀x. α|_i = |α|_i,
4. |α ∨ β|_i = max(|α|_i, |β|_i),
5. |□_i α|_i = |α|_i,
6. |□_i β|_i = |α|_i + 1, for j ≠ i.

A formula has a depth k if max(α-depth, b-depth) = k. A formula is called i-objective if all epistemic operators which do not occur within the scope of another epistemic operator are of the form □_i for i ≠ j. A formula is called i-subjective if every atom is in the scope of an epistemic operator and all epistemic operators which do not occur within the scope of another epistemic operator are of the form □_i.

For example, a formula of the form L_a L_b L_a p ∨ L_b q has a depth of 4, a a-depth of 3 and a b-depth of 4. L_a b is both b-subjective and a-objective. A formula is called objective if it does not mention any modal operators. A formula is called basic if it does not mention any N_i for i = a, b. We now define a notion of epistemic states using k-structures. The main intuition is that we keep separate the worlds Alice believes from the worlds she considers Bob to believe, to depth k.
Definition 2. A $k$-structure ($k \geq 1$), say $e^k$, for an agent is defined inductively as:
- $e^1 \subseteq W \times \{\{\}\}$,
- $e^k \subseteq W \times \mathbb{E}^{k-1}$, where $\mathbb{E}^m$ is the set of all $m$-structures.

A $e^1$ for Alice, denoted as $e^1_A$, is intended to represent a set of worlds $\{\{w\}\}$. A $e^2$ is of the form $\{(w, e^1_b), (w', e^1_b'), \ldots\}$, and it is to be read as "at $w$, she believes Bob considers worlds from $e^1_b$ possible but at $w'$, she believes Bob to consider worlds from $e^1_b'$ possible". This conveys the idea that Alice has only partial information about Bob, and so at different worlds, her beliefs about what Bob knows differ. We define a $e^k$ for Alice, a $e^k$ for Bob and a world $w \in W$ as a $(k, j)$-model ($e^k_a, e^k_b, w$). Only sentences of a maximal $a$-depth of $k$, and a maximal $b$-depth of $j$ are interpreted wrt. a $(k, j)$-model. The complete semantic definition is:

1. $e^k_a, e^k_b, w \models \varphi$ iff $p \in w$ and $p$ is a ground atom,
2. $e^k_a, e^k_b, w \models (m = n)$ iff $m, n \in N$ and are identical,
3. $e^k_a, e^k_b, w \models \neg \alpha$ iff $e^k_a, e^k_b, w \not\models \alpha$, 
4. $e^k_a, e^k_b, w \models \alpha \lor \beta$ iff $e^k_a, e^k_b, w \models \alpha$ or $e^k_a, e^k_b, w \models \beta$, 
5. $e^k_a, e^k_b, w \models \forall x. \alpha$ iff $e^k_a, e^k_b, w \models \alpha^x$ for all $n \in N$,
6. $e^k_a, e^k_b, w \models L_\alpha$ iff for all $\langle w', e^k_{b-1} \rangle \in e^k_a$, $e^k_{a', b-1}, w' \models \alpha$, 
7. $e^k_a, e^k_b, w \models N_\alpha$ iff for all $\langle w', e^k_{b-1} \rangle \not\in e^k_a$, $e^k_{a', b-1}, w' \models \alpha$

And since $O_\alpha$ syntactically denotes $L_\alpha \land N_\neg \alpha$, it follows from the semantics that

8. $e^k_a, e^k_b, w \models O_\alpha$ iff for all worlds $w'$, for all $e^{k-1}$ for Bob, $\langle w', e^{k-1}_b \rangle \in e^k_a$ iff $e^k_{a', b-1}, w' \models \alpha$.

(The semantics for $L_\alpha$ and $N_\alpha$ are given analogously.) A formula $\alpha$ (of $a$-depth of $k$ and of $b$-depth of $j$) is satisfiable iff there is a $(k, j)$-model such that $e^k_a, e^k_b, w \models \alpha$. The formula is valid (|= $\alpha$) iff $\alpha$ is true at all $(k, j)$-models. Satisfiability is extended to a set of formulas $\Sigma$ (of maximal $a$, $b$-depth of $k$, $j$) in the manner that there is a $(k, j)$-model $e^k_a, e^k_b, w$ such that $e^k_a, e^k_b, w \models \alpha$ for every $\alpha \in \Sigma$. We write $\Sigma \models \alpha$ to mean that for every $(k, j)$-model $e^k_a, e^k_b, w$, if $e^k_a, e^k_b, w \models \alpha$ for all $\alpha \in \Sigma$, then $e^k_a, e^k_b, w \models \alpha$.

Validity is not affected if models of a depth greater than that needed for those used. This is to say, if $\alpha$ is true wrt. all $(k, j)$-models, then $\alpha$ is true wrt. all $(k', j')$-models for $k' \geq k$, $j' \geq j$. We obtain this result by constructing for every $e^k_a'$, a $k$-structure $e_a |_{k', k}$ such that they agree on all formulas of maximal $a$-depth $k$. Analogously for $e^k_b'$.

Definition 3. Given $e^k_a'$, we define $e_a |^k_{k'}$ for $k' \geq k \geq 1$:
1. $e_a |^1_{1} = e_a$,
2. $e_a |^k_{k'} = \{\{w\} \mid \langle w, e^k_{b-1} \rangle \in e^k_a\}$,
3. $e_a |^{k'}_{k} = \{\{w, e^k_{b-1}\} \mid \langle w, e^k_{b-1} \rangle \in e^k_a\}$.

Lemma 4. For all formulas $\alpha$ of maximal $a$, $b$-depth of $k$, $j$, $e^{k'}_{a'}, e^j_b, w \models \alpha$ iff $e_a |^k_k, e_b |^j_j$, $w \models \alpha$, for $k' \geq k$, $j' \geq j$.

Proof. By induction on the depth of formulas. The proof immediately holds for atomic formulas, disjunctions and negations since we have the same world $w$. Assume that the result holds for formulas of $a$, $b$-depth 1. Let $\alpha$ such a formula, and suppose $e^{k'}_{a'}, e^j_b, w \models L_\alpha$ (where $L_\alpha$ has $a$, $b$-depth of 1, 2). Then, for all $\langle w', e^{k'-1}_b \rangle \in e^{k'}_{a'}, e^{k'-1}_b, w' \models \alpha$ (by induction hypothesis) $e_a |^k_k, e_b |^{k'-1}_{k'-1}, w' \models \alpha$ iff $e_a |^k_k, e_b |^{k'-1}_{k'-1}, w \models \alpha$. By construction, we also have $e_a |^k_k, e_b |^{j-1}_j, w \models L_\alpha$. Lastly, since $L_\alpha$ is subjective, $b$'s structure is irrelevant, and thus, $e_a |^k_k, e_b |^{j-1}_j, w \models L_\alpha$.

For the reverse direction, suppose $e_a |^k_k, e_b |^{j-1}_j, w \models L_\alpha$. Then for all $w' \in e_a |^k_k, e_b |^{j-1}_j, \{\} \models \alpha$ (by construction) for all $\langle w', e^{k'-1}_b \rangle \in e^{k'}_{a'}, e^{k'-1}_b, w' \models \alpha$ iff $e_a |^k_k, e_b |^{k'-1}_{k'-1}, w \models \alpha$. Since $b$'s structure is irrelevant, we have $e_a |^k_k, e_b |^{j-1}_j, w \models L_\alpha$. The cases for $L_\alpha$, $L_\alpha$ and $N_\alpha$ are completely symmetric.

Theorem 5. For all formulas $\alpha$ of $a$, $b$-depth of $k$, $j$, if $\alpha$ is true at all $(k, j)$-models, then $\alpha$ is true at all $(k', j')$-models with $k' \geq k$ and $j' \geq j$.

Proof. Suppose $\alpha$ is true at all $(k, j)$-models. Given any $(k', j')$-model, by assumption $e_a |^k_k, e_b |^{j'}_j, w \models \alpha$ and by Lemma 4, $e_a |^{k'}_{k'}, e_b |^{j'}_j, w \models \alpha$.

Knowledge with $k$-structures satisfy weak S5 properties, and the Barcan formula (Hughes and Cresswell 1972).

Lemma 6. If $\alpha$ is a formula, the following are valid wrt. models of appropriate depth ($ \square $ denotes $ L_i $ or $ N_i $):
1. $ \square_i \alpha \land \square_i (\alpha \lor \beta) \equiv \square_i \beta $,
2. $ \square_i \alpha \lor \square_i \alpha $,
3. $ \neg \square_i \alpha \lor \square_i \neg \alpha $,
4. $ \forall x. \square_i \alpha \lor \square_i (\forall x. \alpha) $.

Proof. The proofs are similar. For item 3, wlog let $ \square_i $ be $ L_\alpha $. Suppose $ e^k_a, e^k_b, w \models \neg L_\alpha $, there is some $ \langle w', e^k_{b-1} \rangle \in e^k_b $ such that $ e^k_a, e^k_{b-1}, w' \models \neg \alpha $. Let $ w'' $ be any world such that $ \langle w'', e^k_{b-1} \rangle \in e^k_a $, then $ e^k_a, e^k_{b-1}, w'' \models \neg L_\alpha $. Thus, $ e^k_a, e^k_{b-1}, w \models L_\alpha \neg L_\alpha $. The case of $ N_\alpha $ is analogous.
sentences of some maximal depth \( k \) and they should not be able to conclude anything about what is known at depths higher than \( k \), with one exception. If we were to include a notion of common knowledge (Fagin et al. 1995), then we would get entailments about what is believed at arbitrary depths. With our current model, this cannot be captured, but we are willing to pay that price because in return we get, for the first time, a very simple possible-world style account of only-knowing. Similarly, we have nothing to say about (infinite) knowledge bases with unbounded depth.

**Multi-Agent Only-Knowing**

In this section, we return to the features of only-knowing discussed earlier and verify that the new semantics reasonably extends them to the multi-agent case. We also briefly discuss earlier attempts at capturing these features. Halpern (1993), Lakemeyer (1993), and Halpern and Lakemeyer (2001) independently attempted to extend \( ONC \) to the many agent case.\(^5\) There are some subtle differences in their approaches, but the main restriction is they only allow a propositional language. Henceforth, to make the comparison feasible, we shall also speak of the propositional subset of \( ONC_n \) with the understanding that the semantical framework is now defined for propositions (from an infinite set \( \Phi \)) rather than ground atoms.

The main component in these features is the notion of possibility. In the single agent case, each world represents a possibility. Thus, from a logical viewpoint, a possibility is simply the set of objective formulas true at some world. Further, the set of epistemic possibilities is given by \( \{ \{ \text{objective formulas true at } w \} \mid w \in \epsilon \} \). Halpern and Lakemeyer (2001) correctly argue that the appropriate generalization of the notion of possibility in the many agent case are \( i \)-objective formulas. Intuitively, a possible state of affairs according to \( a \) include the state of the world (objective formulas), as well as what \( b \) is taken to believe. The earlier attempts by Halpern and Lakemeyer use Kripke structures with accessibility relations \( K_i \) for each agent \( i \). Given a Kripke structure \( M \), the notion of possibility is defined as the set of \( i \)-objective formulas true at some Kripke world, and the set of epistemic possibilities is obtained from the \( i \)-objective formulas true at all \( i \)-accessible worlds. Formally, the set of epistemic possibilities true at \( (M, w) \), where \( w \) is a world in \( M \), is defined as \( \{ obj^+ (M, w') \mid w' \in K_i (w) \} \), where \( obj^+ (M, w') \) is a set consisting of \( i \)-objective formulas true at \( (M, w') \).\(^7\) Although intuitive, note that, even for the propositional subset of \( ONC \), a Kripke world is a completely different entity from what Levesque supposes. Perhaps, one consequence is that the semantic proofs in earlier approaches are very involved. In contrast, we define worlds exactly as Levesque supposes. And, our notion of possibility is obtained from the set of \( a \)-objective formulas true at each \( (w, e^{k-1}_a) \) in \( e^k_a \).

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\(^{5}\) For space reasons, we do not review all aspects of these approaches.

\(^{6}\) The superscript \( + \) denotes that the set includes non-basic formulas. Given \( X^+ \), we let \( X = \{ \phi \text{ is basic} \mid \phi \in X^+ \} \).

Definition 7. Suppose \( M = (e^k_a, e^j_a, w) \) is a \((k, j)\)-model.
1. let \( obj^+_a (M) = \{ \text{objective } \phi \mid M \models \phi \} \),
2. let \( Obj^+_a (e^k_a) = obj^+_a (\{ \}, e^1_a, w) \mid (w, e^{k-1}_a) \in e^k_a \),
3. let \( Obj^+_a (e^j_a) = obj^+_a (e^{j-1}_a, \{ \}, w) \mid (w, e^{j-1}_a) \in e^j_a \). All the \( a \)-objective formulas true at a model \( M \), essentially the objective formulas true wrt. \( w \) and the \( b \)-subjective formulas true wrt. \( e^b_j \), are given by \( obj^+_a (M) \). Note that these formulas do not strictly correspond to \( a \)'s possibilities. Rather, we define \( Obj^+_a \) on her epistemic state \( e^k_a \), and this gives us all the \( a \)-objectives formulas that \( a \) considers possible. We shall now argue that the intuition of all of Levesque’s properties is maintained.\(^8\)

Property 1. In the single agent case, this property ensured that an agent’s epistemic possibilities are not affected on evaluating \( N \). This is immediately the case here. Given a model, say \( (e^k_a, e^b_j, w) \), \( a \)'s epistemic possibilities are determined by \( Obj^+_a (e^b_j) \). To evaluate \( N_a, \alpha \), we consider all models \( (e^k_a, e^{j-1}_b, w') \) such that \( (w', e^{j-1}_b) \not\in e^k_a \). Again, \( a \)’s possibilities are given by \( Obj^+_a (e^b_j) \) for all these models, and does not change.

Property 2. In the single agent case, this property ensured that evaluating \( L_a \) and \( N_a \) is always wrt. the set of all possibilities, and completely independent of \( e \). As discussed, in the many agent case, possibilities mean \( i \)-objective formulas and analogously, if \( \alpha \) is a possibility in \( a \)'s view, say an \( a \)-objective formula of maximal \( b \)-depth of \( k \), then we should interpret \( L_a, \alpha \) and \( N_a, \alpha \) wrt. all \( a \)-objective possibilities of max. depth \( k \): the set of \((k + 1)\)-structures. Clearly then, the result is fixed and independent of the corresponding \( e^{k+1} \). The following lemma is a direct consequence of the definition of the semantics.

Lemma 8. Let \( \alpha \) be a \( i \)-objective formula of \( j \)-depth \( k \), for \( j \neq i \). Then, the set of \((k + 1)\)-structures that evaluate \( L_a, \alpha \) and \( N_a, \alpha \) is \( \mathbb{E}^{k+1} \).

Property 3. The third property ensures that one can characterize epistemic states from any set of \( i \)-objective formulas. Intuitively, given such a set, we must have a model where precisely this set is the epistemic state. Earlier attempts at clarifying this property involved constructing a set of maximally \( K45_n \)-consistent sets of basic \( i \)-objective formulas, and showing that there exist an epistemic state that precisely corresponds to this set. But, defining possibilities via \( K45_n \) proof-theoretic machinery inevitably leads to some limitations, as we shall see. We instead proceed semantically, and go beyond basic formulas. Let \( \Omega \) be a satisfiable set of \( i \)-objective formulas, say of maximal \( j \)-depth \( k \), for \( j \neq i \). Let \( \Omega' \) be a set obtained by adding a \( i \)-objective formula \( \gamma \) of maximal \( j \)-depth \( k \) such that \( \Omega' \) is also satisfiable. By considering all \( i \)-objective formulas of maximal \( j \)-depth \( k \), let

\[^8\] It is interesting to note that such a formulation of Levesque’s properties is not straightforward in the first-order case. That is, for the quantified language, it is known that there are epistemic states that can not be characterized using only objective formulas (Levesque and Lakemeyer 2001). Thus, it is left open how one must correctly generalize the features of first-order \( ONC \).

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us construct $\Omega'$, $\Omega''$, . . . by adding formulas iff the resultant set remains satisfiable. When we are done, the resulting $\Omega^*$ is what we shall call a maximally satisfiable $i$-objective set.9 Naturally, there may be many such sets corresponding to $\Omega$. We show that given a set of maximally satisfiable $i$-objective sets, there is a model where precisely this set characterizes the epistemic state.

Theorem 9. Let $S_i$ be a set of maximally satisfiable sets of $i$-objective formulas, and $\sigma$ a satisfiable objective formula. Suppose $S_0$ is of max. $b$-depth $k - 1$ and $S_0$ is of max. $a$-depth $j - 1$. Then there is a model $M^* = \langle e_a^k, e_b^k, w^* \rangle$ such that $M^* \models \sigma, S_a = \text{Obj}_a^+(e^k_b) \text{ and } S_b = \text{Obj}_b^+(e^k_b)$.

Proof. Consider $S_a$. Each $S' \in S_a$ is a maximally satisfiable $a$-objective set, and thus by definition, there is a $k$-structure $\langle w', e_b^{k-1} \rangle$ such that $\{e_b^{k-1}, w' \models S'$. Define such a set of $k$-structures $\{\langle w', e_b^{k-1} \rangle\}$, corresponding to each $S' \in S_a$, and let this be $e_a^k$. It is immediate to verify that $\text{Obj}_a^+(e^k_b) = S_a$. Analogously, for $e_b^k$ using $S_b$. Finally, there is clearly some world $w^*$ where $\sigma$ holds. ■

On Validity

How does the semantics compare to earlier approaches? In particular, we are interested in valid formulas. Lakemeyer (1993) proposes a semantics using K45$_n$-canonical models, but he shows that the formula $\neg\neg\neg O_a O_b p$ for any proposition $p$ is valid. Intuitively, it says that all that Alice knows is that Bob does not only know $p$, and as Lakemeyer argues, the validity of $\neg\neg\neg O_a O_b p$ is unintuitive. After all, Bob could honestly tell Alice that he does not only know $p$. The negation of this formula, on the other hand, is satisfiable in a Kripke structure approach by Halpern (1993), called the i-set approach.10 It is also satisfiable in the $k$-structure semantics. Interestingly, the $i$-set approach and $k$-structures agree on one more notion. The formula $L_a \supset \neg N_a O_b \neg O_b p (\zeta)$ is valid in both, while $\neg \zeta$ is satisfiable wrt. Lakemeyer (1993). (It turns out that the validity of $\zeta$ in our semantic framework is implicitly related to the satisfiability of $O_b \neg O_b p$, so this property is not unreasonable.)

However, we immediately realize that the $i$-set approach and $k$-structures do not share too many similarities beyond those presented above. In fact, the $i$-set approach does not truly satisfy Levesque’s second property. For instance, $N_a \neg O_b p \land L_a \neg O_b p (\lambda)$ is satisfiable in Halpern (1993). Recall that, in this property, the union of models that evaluate $N_1 \alpha$ and $L_1 \alpha$ must lead to all conceivable states. So, the satisfiability of $\lambda$ leaves open the question as to why $O_b p$ is not considered since $\neg O_b p$ is true at all conceivable states. We show that, in contrast, $\lambda$ is not satisfiable in the $k$-structures approach. Lastly, (Halpern and Lakemeyer 2001) involves enriching the language, the intuitions of which are perhaps best explained after reviewing the proof theory, and so we defer discussions to later.11

Theorem 10. The following properties are of the semantics:
1. $O_a \neg O_b p$, for any $p \in \Phi$, is satisfiable.
2. $\models L_a \supset \neg N_a \neg O_b O_a p$.
3. $N_a \neg O_b p \land L_a \neg O_b p$ is not satisfiable.

Proof. Item 1. Let $W_p = \{w \mid w \models p\}$ and let $E$ be all subsets of $W$. It is easy to see that if $e^1_b \in E$, then $\{\}, e^1_k, w \not\models O_b p$, for any world $w$. Now, define $e^2_a$ for $a$ that has all of $W \times E$. Thus, $e^2_a(\{\}, w) \models O_a \neg O_b p$.

Item 2. Suppose $e^2_a(\{\}, w) \models L_a \supset \neg O_b p$, for any $w \in W$. Then, for all $\langle w', e_b^k \rangle \in e^a, e_b^k, e_b^k, e_a^k, w' \models \bot$, and thus, $e^b_a(\{\}, w) \models N_a \neg O_b \neg O_a p$. Then, wrt. all of $\langle w', e_b^k \rangle \not\models e^b_a$, i.e. all of $E^b, \neg O_b \neg O_a p$ must hold. That is, $\neg O_b \neg O_a p$ must be valid. From above, we know this is not the case.

Item 3. Suppose $e^b_a(\{\}, w) \models L_a \neg O_b p$, for any $w$. Then, for all $\langle w', e_b^k \rangle \in e^a, e_a^k, e_b^b, e_a^k, w' \models \neg O_a p$. Since $O_a p$ is satisfiable, there is a $e^k_b$ such that $\{e^k_b, w^* \models O_b p$, and $\langle w^*, e_b^k \rangle \not\models e^b_a$. Then, $e^b_a(\{\}, w) \models \neg N_a \neg O_b p$.

Thus, $k$-structures seem to satisfy our intuitions on the behavior of only-knowing. To understand why, notice that $\neg O_a \neg O_b p$ and $\lambda$ involve the nesting of $N_1$ operators. Lakemeyer (1993) makes an unavoidable technical commitment. A ($i$-objective) possibility is formally a maximally K45$_n$-consistent set of basic $i$-objective formulas. The restriction to basic formulas is an artifact of a semantics based on the canonical model. Unfortunately, there is more to agent $i$’s possibility than just basic formulas. In the case of Halpern (1993), the problem seems to be that $N_1$ and $L_i$ do not interact naturally, and that the full complement of epistemic possibilities is not considered in interpreting $N_1$. In contrast, Theorem 9 shows that we allow non-basic formulas and by using a strictly semantic notion, we avoid problems that arise from the proof-theoretic restrictions. And, since the semantics faithfully complies with the second property, $\lambda$ is not satisfiable.

The natural question is if there are axioms that characterize the semantics. We begin, in the next section, with a proof theory by Lakemeyer (1993) that is known to be sound and complete for all attempts so far, but for a restricted language.

Proof Theory

In the single agent case, ONL’s proof theory consists of axioms of propositional logic, axioms that treat $L$ and $N$ as a classical belief operator in K45, an axiom that allows us to use $N$ and $L$ freely on subjective formulas, modus ponens (MP) and necessitation (NEC) for both $L$ and $N$ as inference rules, and the following axiom:12

11 An approach by (Waaler 2004; Waaler and Solhaug 2005) is also motivated by the proof theory. Discussions are deferred.

12 Strictly speaking, this is not the proof theory introduced in (Levesque 1990), where an axiom replaces the inference rule NEC. Here, we consider an equivalent formulation by Halpern and Lakemeyer (2001).
A5. $N\alpha \supset \neg L\alpha$ if $\neg \alpha$ is a propositionally consistent objective formula.

As we shall see, only the axiom A5 is controversial, since extending any objective $\alpha$ to any $i$-objective $\alpha$ is problematic. Mainly, the soundness of the axiom in the single agent case relies on propositional logic. But in the multi-agent case, since we go beyond propositional formulas establishing this consistency is non-trivial, and even circular. To this end, Lakemeyer (1993) proposes to resolve this consistency by relying on the existing logic $K45_n$. As a consequence, his proof theoretic formulation appropriately generalizes all of Levesque’s axioms, except for A5 where its application is restricted to only basic $i$-objective consistent formulas. We use $\vdash$ to denote provability.

Definition 11. $ONL_n^c$ consists of all formulas $\alpha$ in $ONL_n^c$ such that no $N_j$ may occur in the scope of a $L_i$ or a $N_i$, for $i \neq j$.

The following axioms, along with MP and NEC (for $L_i$ and $N_i$) is an axiomatization that we refer to as $AX_n$. $AX_n$ is sound and complete for the canonical model and the i-set approach for formulas in $ONL_n^c$.

A1$n$. All instances of propositional logic,

A2$n$. $L_i(\alpha \supset \beta) \supset (L_i\alpha \supset L_i\beta)$,

A3$n$. $N_i(\alpha \supset \beta) \supset (N_i\alpha \supset N_i\beta)$,

A4$n$. $\sigma \supset L_i\sigma$ for $i$-subjective $\sigma$,

A5$n$. $N_i\alpha \supset \neg L_i\alpha$ if $\neg \alpha$ is a $K45_n$-consistent objective basic formula.

Observe that, as discussed, the soundness of A5$n$ is built on $K45_n$-consistency. Since our semantics is not based on Kripke structures, proving that every $K45_n$-consistent formula is satisfiable in some $(k, j)$-model is not immediate. We propose a construction called the $(k, j)$-correspondence model. In the following, in order to disambiguate $W$ from Kripke worlds, we shall refer to our worlds as propositional valuations.

Definition 12. The $K45_n$ canonical model $M^c = (W^c, \pi^c, K^c, \sigma^c)$ is defined as follows:

1. $W^c = \{w \mid w$ is a (basic) maximally consistent set \}
2. for all $p \in \Phi$ and worlds $w$, $\pi^c(w)(p) = \text{true}$ iff $p \in w$
3. $(w, w') \in K^c_i$ iff $w \setminus L_i \subseteq w'$, $w \setminus L_i = \{ \alpha \mid L_i\alpha \in w \}$

Definition 13. Given $M^c$, define a set of propositional valuations $W$ such that for each world $w \in W^c$, there is a valuation $[w] \in W$, $[w] = \{p \mid p \in w\}$

Definition 14. Given $M^c$ and a world $w \in W^c$, construct a $(k, j)$-model $(e_{[w]}^k|^c, e_{[w]}^j|^c, [w])$ from valuations $W$ inductively:

1. $e_{[w]}^k|^c = \{([w'], \{\}) \mid w' \in K^c_i(w)\}$
2. $e_{[w]}^j|^c = \{([w'], \{\}, e_{[w']}^k|^c) \mid w' \in K^c_i(w)\}$

Further, $e_{[w]}^j|^c$ is constructed analogously. Let us refer to this model as the $(k, j)$-correspondence model of $(M^c, w)$.

Roughly, Defn. 14 is a construction of a $(k, j)$-model that appeals to the accessibility relations in the canonical model.13 Thus, a $c_\alpha$ for Alice wrt. $w$ has precisely the valuations of Kripke worlds $w' \in K^c_i(w)$. Quite analogously, a $c_\alpha$ for any $i$-objective $\alpha$ is a set \{([w'], e_{[w']}^k|^c) \mid w' \in K^c_i(w)\} as before, but $e_{[w']}^k|^c$ is an epistemic state for Bob and hence refers all worlds $w'' \in K^c_i(w')$. By induction on the depth of a basic formula $\alpha$, we obtain a theorem that $\alpha$ of maximal $a, b$-depth $k, j$ is satisfiable at $(M^c, w)$ iff the $(k, j)$-correspondence model satisfies the formula.

Theorem 15. For all basic formulas $\alpha$ in $ONL_n^c$ and of maximal $a, b$-depth of $k, j$,

$M^c, w \models \alpha$ iff $e_{[w]}^k|^c, e_{[w]}^j|^c, [w] \models \alpha$.

Proof. By definition, the proof holds for propositional formulas, disjunctions and negations. Let us say the result holds for formulas of $a, b$-depth 1. Suppose now $M^c, w \models L_{a\alpha}$, where $L_{a\alpha}$ has $a, b$-depth of 1, 2. Then for all $w' \in K^c_i(w)$, $M^c, w' \models \alpha$ iff (by induction hypothesis) $e_{[w]}^k|^c, e_{[w]}^j|^c, [w'] \models \alpha$ iff $e_{[w]}^k|^c, [w] \models L_{a\alpha}$. By construction, we also have $e_{[w]}^k|^c, [w] \models L_{a\alpha}$. Since $b$’s structure is irrelevant, we get $e_{[w]}^k|^c, e_{[w]}^j|^c, [w] \models L_{a\alpha}$ proving the hypothesis.

For the other direction, suppose $e_{[w]}^k|^c, e_{[w]}^j|^c, [w] \models L_{a\alpha}$. For all $[w'] \in e_{[w]}^k|^c, e_{[w]}^j|^c, [w] \models \alpha$ (by hyp.) $M^c, w' \models \alpha$ for all $w' \in K^c_i(w)$ iff $M^c, w \models L_{a\alpha}$. ■

Lemma 16. Every $K45_n$-consistent basic formula $\alpha$ is satisfiable some $(k, j)$-model.

Proof. It is a property of the canonical model that every $K45_n$-consistent basic formula is satisfiable wrt. the canonical model. Supposing that the formula has a $a, b$-depth of $k, j$ then from Thm 15, we know there is at least the correspondence $(k, j)$-model that also satisfies the formula. ■

Theorem 17. For all $\alpha \in ONL_n^c$, if $AX_n \vdash \alpha$ then $\models \alpha$.

Proof. The soundness is easily shown to hold for A1$n$ – A4$n$. The soundness of A5$n$ is shown by induction on the depth. Suppose $\sigma$ is a propositional formula, and say $\neg \alpha$ is a consistent propositional formula (and hence $K45_n$-consistent). Then there is a world $w^*$ such that $\{\}, w^* \models \neg \alpha$. Given a $e_{\alpha}^k$, if $\langle w^*, e_b^{k-1} \rangle \in e_{\alpha}^k$ for some $e_{\beta}^k$, then $e_{\alpha}^k, \{\}, w \models L_{a\alpha}$ for any world $w$. If not, then $e_{\alpha}^k, \{\}, w \models \neg L_{a\alpha}$. Thus, $e_{\alpha}^k, \{\}, w \models \neg N_{a\alpha}$. Assume the proof holds for $a$-objective formulas of max. $b$-depth $k - 1$. Suppose now, $\alpha$ is such a formula, and $\neg \alpha$ is $K45_n$-consistent. By Lemma 16, there is $(w^*, e_b^{k-1})$, such that $\{\}, e_b^{k-1}, w^* \models \neg \alpha$. Again, if $\langle w^*, e_b^{k-1} \rangle \in e_{\alpha}^k$, then $e_{\alpha}^k, \{\}, w \models \neg L_{a\alpha}$ and if not, then $e_{\alpha}^k, \{\}, w \models \neg N_{a\alpha}$. ■

We proceed with the completeness over the following definition, and lemmas.

13The construction is somewhat similar to the notion of generated submodels of Kripke frames (Hughes and Cresswell 1984).
Definition 18. A formula $\psi$ is said to be independent of the formula $\phi$ wrt. an axiom system $AX$, if neither $AX \vdash \phi \supset \psi$ nor $AX \vdash \phi \supset \lnot \psi$.

Lemma 19 (Halpern and Lakemeyer, 2001). If $\phi_1, \ldots, \phi_m$ are $K45_n$-consistent basic $i$-objective formulas then there exists a basic $i$-objective formula $\psi$ of the form $L_{j \psi} (j \neq i$) that is independent of $\phi_1, \ldots, \phi_m$ wrt. $K45_n$.

Lemma 20. In the lemma above, if $\phi_i$ are i-objective and of maximal j-depth $k$ for $j \neq i$, then there is a $\psi$ of j-depth $2k + 2$.

Lemma 21 (Halpern and Lakemeyer, 2001). If $\phi$ and $\psi$ are i-objective basic formulas, and if $L_{i \phi} \land N_{i \psi}$ is $AX_n$-consistent, then $\phi \lor \psi$ is valid.

Lemma 22 (Halpern and Lakemeyer, 2001). Every formula $\alpha \in \ONL_n$ is provably equivalent to one in the normal form (written below for $n = \{a, b\}$):

$$\forall (\sigma \land L_a \forall a \land \lnot L_a \forall a_1 \land \cdots \land \lnot L_a \forall a_m \land L_b \forall b_0 \land \cdots \land \lnot L_b \forall b_{m_2} \land N_a \forall a_0 \land \cdots \land \lnot N_a \forall a_{m_1} \land N_b \forall b_0 \land \cdots \land \lnot N_b \forall b_{m_2})$$

where $\sigma$ is a propositional formula, and $\forall a_m$ and $\forall b_n$ are i-objective. If $\alpha \in \ONL_n$, $\forall a_m$ and $\forall b_n$ are basic.

Theorem 23. For all formulas $\alpha \in \ONL_n$, if $\models \alpha$ then $AX_n \vdash \alpha$.

Proof. It is sufficient to prove that every $AX_n$-consistent formula $\xi$ is satisfiable wrt. some $(k, j)$-model. If $\xi$ is basic, then by Lemma 16, the statement holds. If $\xi$ is not basic, then wlog, it can be considered in the normal form:

$$\forall (\sigma \land L_a \forall a \land \lnot L_a \forall a_1 \land \cdots \land \lnot L_a \forall a_m \land L_b \forall b_0 \land \cdots \land \lnot L_b \forall b_{m_2} \land N_a \forall a_0 \land \cdots \land \lnot N_a \forall a_{m_1} \land N_b \forall b_0 \land \cdots \land \lnot N_b \forall b_{m_2})$$

where $\sigma$ is a propositional formula, and $\forall a_m$ and $\forall b_n$ are i-objective and basic. Since $\sigma$ is propositional and consistent, there is a clearly a world $w^*$ such that $w^* \models \sigma$. We construct a $k'$-structure such that it satisfies all the $a$-subjective formulas in the normal form above. Following that, a $j$-structure for all the $b$-subjective formulas is constructed identically. The resulting $(k', j)$-model (with $w^*$) satisfies $\xi$.

Let $A$ be all $K45_n$-consistent formulas of the form $\forall a_0 \land \forall a_0 \land \lnot \forall a_j (j \geq 1)$ or the form $\forall a_0 \land \forall a_0 \land \lnot \forall a_j$. Let $\gamma$ be independent of all formulas in $A$, as in Lemma 19 and 20. Note that, while we take $\xi$ itself to be of maximal $a$, $b$-depth of $k, j$, the depth of $\forall a_0 \ldots$ being $a$-objective are of maximal $b$-depth $k - 1$, and hence $\gamma$ is of $b$-depth $2k$ (Lemma 20). Given a consistent set of formulas, the standard Lindembaum construction can be used to construct a maximally consistent set of formulas, all of a maximal $b$-depth $k - 1$.

That is, a formula is considered in the construction only if it has a maximal $b$-depth $k - 1$.

Now, let $S_b$ be a set of all maximally consistent sets of formulas, constructed by only considering formulas of maximal $b$-depth $k - 1$, and containing $\forall a_0 \land \lnot (\forall a_0 \lor (\forall a_0 \land \gamma))$. Since each of these consistent sets are basic and $a$-objective, they are satisfiable by Lemma 16. Thus the sets $S' \subseteq S_b$ are satisfiable wrt. $2k$-structures $(w^*, e_{2k}^b)$. Let $k' = 2k + 1$. By constructing a $k'$-structure for Alice, say $e_{a'}^k$, from each $(w, e_{2k}^b)$ for every $S' \subseteq S_b$, we have that $Obf_a(e_{a'}^k) = S_a$. We shall show that all the $a$-subjective formulas in the normal form are satisfied wrt. $(e_{a'}^k, \{\}, w^*)$.

Since for all $S' \subseteq S_a$, we have $\forall a_0 \land \lnot \forall a_0 \lor (\forall a_0 \land \gamma)$. Then, since $L_a \forall a_0 \land \lnot L_a \forall a_0$ is consistent, it must be that $\forall a_0 \land \lnot \forall a_0$ is consistent. For suppose not, then $\lnot \forall a_0 \lor \forall a_0$ is provable and thus, we have $\forall a_0 \lor \forall a_0$. Then we prove $L_a \forall a_0 \lor \lnot L_a \forall a_0$, and since we have $L_a \forall a_0$, we prove $L_a \forall a_0$, clearly inconsistent with $L_a \forall a_0 \land \lnot L_a \forall a_0$. Now that $\forall a_0 \lor \forall a_0$ is consistent, we have that $\forall a_0 \lor \forall a_0 \lor \forall a_0 \lor \forall a_0$ is consistent. With the former, we also have that $\forall a_0 \lor \forall a_0 \lor \forall a_0 \lor \forall a_0$ is consistent. There are maximally consistent sets that contain one of them, both of which contain $\lnot \forall a_0$. This means that, $e_{a'}^k = \{\}, w^\ast \models \lnot L_a \forall a_0$.

Now, consider some $k'$-structure $(w^*, e_{2k}^b) \not\models e_{a'}^k$. One of the following $a$-objective formulas must hold wrt. this $k'$-structure: (a) $\forall a_0 \land \lnot \forall a_0$, (b) $\forall a_0 \land \lnot \forall a_0$, (c) $\lnot \forall a_0 \land \forall a_0$ or (d) $\forall a_0 \land \lnot \forall a_0$. It can not be (d), since $L_a \forall a_0 \land \lnot L_a \forall a_0$ is consistent, and this implies that $\forall a_0 \lor \forall a_0$ is valid (by Lemma 21). It certainly cannot be (b), for it would be in some $S' \subseteq S_a$. This leaves us with options (c) and (a), both of which have $\lnot \forall a_0$. Since the $k'$-structure was arbitrary, we must have for all $(w, e_{2k}^b) \not\models e_{a'}^k = \{\}, e_{2k}^b$, $w^\ast \models \forall a_0$. Thus, $e_{a'}^k = \{\}, w^\ast \models \lnot N_a \forall a_0$.

Finally, since $N_a \forall a_0 \land \lnot \forall a_0$ is consistent, it must be that $\forall a_0 \lor \lnot \forall a_0$ is consistent. Further, either $\forall a_0 \lor \lnot \forall a_0$ or $\forall a_0 \lor \lnot \forall a_0$ or $\forall a_0 \lor \lnot \forall a_0$ is consistent. If the former, then $\forall a_0 \lor \lnot \forall a_0$ or $\forall a_0 \lor \lnot \forall a_0$ or $\forall a_0 \lor \lnot \forall a_0$ is consistent. Let $\beta$ be that which is consistent. Note that $\lnot \forall a_0 \lor (\forall a_0 \lor \lnot \forall a_0 \lor \forall a_0 \lor \forall a_0)$ is consistent, and hence part of all $S' \subseteq S_a$. This means that $e_{a'}^k = \{\}, w^\ast \models \lnot L_a \forall a_0$. But since $\beta$ itself is consistent, there is a $k$-structure such that $(\{\}, e_{2k}^b)$, $w^\ast \models \forall a_0$. And this $k$-structure can not be in $e_{a'}^k$. This means that $e_{a'}^k = \{\}, w^\ast \models \lnot N_a \forall a_0$. Thus, all the $a$-subjective formulas in the normal form above are satisfiable wrt. $e_{a'}^k$.

Now, observe that, although $L_a \lor \lnot \forall a_0$ is $O_n \lor O_{a,p}$ (c) from Theorem 10 is valid, yet it is not derivable from $AX_n$. In fact, the soundness result is easily extended to the full language $\ONL_n$. Then, the proof theory cannot be complete for the full language since there is $\zeta \in \ONL_n$ such that $\not\models \zeta$ and $\models \zeta$. Similarly, the validity of non-provable formulas $\lnot O_n \lor O_{a,p}$ and $\zeta$ wrt. the canonical model and the i-set approach respectively, show that although $AX_n$ is also sound for the full language in these approaches, it cannot be complete. Mainly, axiom $AS_n$ has to somehow go beyond basic formulas. As Halpern and Lakemeyer (2001) discuss, the problem is one of circularity. We would like the axiom to hold for any $\alpha$ such that it is a consistent i-objective formula, but to deal with consistency we have to clarify what the axiom system looks like.

The approach taken by Halpern and Lakemeyer is to introduce validity (and its dual satisfiability) directly into the language. Formulas in the new language, $\ONL_n$, are shown to be provably equivalent to $\ONL_n$. Some new axioms involving validity and satisfiability are added to the axiom system, and the resultant proof theory $AX_n$ is shown to
be sound and complete for formulas in $\text{ONL}_n^+$, wrt. an extended canonical model. (An extended canonical model follows the spirit of the canonical model construction but by considering maximally $AX^n_k$-consistent sets, and treat $L_i$ and $N_i$ as two independent modal operators.) So, one approach is to show that for formulas in the extended language the set of valid formulas overlap in the extended canonical model and $k$-structures. But then, as we argued, axiomatizing validity is not natural. Also, the proof theory is difficult to use. And in the end, we would still understand the axioms to characterize a semantics bridged on proof-theoretic elements.

Again, what is desired is a generalization of Levesque’s axiom A5, and nothing more. To this end, we propose a new axiom system, that is subtly related to the structure of formulas as are parameters $k$ and $j$. The axiom system has an additional $t$-axioms, and is to correspond to a sequence of languages $\text{ONL}^t_{n+k}$.  

**Definition 24.** Let $\text{ONL}^t_{n+k} = \text{ONL}^t_{n+k}$. Let $\text{ONL}^t_{n+k}$ be all Boolean combinations of formulas of $\text{ONL}^t_n$ and formulas of the form $L_i\alpha$ and $N_i\alpha$ for $\alpha \in \text{ONL}^t_n$.

It is not hard to see that $\text{ONL}^t_{n+k} \supseteq \text{ONL}^t_{n+k}$. Note that $t$ here does not correspond to the depth of formulas. Indeed, a formula of the form $(L_i L_a)^{k+1}p$ is already in $\text{ONL}^t_{n+k}$. Let $AX^t_{n+k}$ be an axiom system consisting of $A_{1n} - A_{4n}$, MP, NEC and $A_{5n} - A_{5n+k}^{t+1}$ defined inductively as:

$A_{5n}^{t+1}, N_i\alpha \supset \neg L_i\alpha$, if $\neg \alpha$ is a K45n-consistent i-objective basic formula.

$A_{5n+k}^{t+1}, N_i\alpha \supset \neg L_i\alpha$, if $\neg \alpha \in \text{ONL}^t_n$, is i-objective, and consistent wrt. $A_{1n} - A_{4n}$, $A_{5n}^{t+1} - A_{5n+k}^{t+1}$.

**Theorem 25.** For all $\alpha \in \text{ONL}^t_n$, if $AX^t_n \vdash \alpha$ then $\models \alpha$.

**Proof.** We prove by induction on $t$. The case of $AX^t_n$ is identical to Theorem 17. So, for the induction hypothesis, let us assume that $AX^t_n$ if $AX^t_n$ if $AX^t_n \vdash \beta$ then $\beta \in \text{ONL}^t_n$ then $\models \beta$. Now, suppose that $\neg \alpha$ is consistent wrt. $AX^t_n$ and is $a$-objective. This implies $\models \neg \alpha$. Thus, there is some $k$-structure $(\sigma, w, e_k^\alpha)$ such that $\{\}, e_k^\alpha, w = \Box \neg \alpha$. Suppose now $(w^+, e_k^\alpha) \in e_k^\alpha+1$ then $e_k^{\alpha+1}, \{\}, w' = \neg L_i\alpha$ and if not then $e_k^{\alpha+1}, \{\}, w' = \neg L_i\alpha$. Thus, $e_k^{\alpha+1}, \{\}, w' = N_i\alpha \supset \neg L_i\alpha$, demonstrating the soundness of $AX^t_{n+k}$.  

We establish completeness in a manner identical to Theorem 23, and thus it necessary to ensure that Lemma 19, 20 and 21 hold for non-basic formulas.

**Lemma 26.** If $\phi_1, \ldots, \phi_m$ are $AX^t_k$-consistent i-objective formulas, then there is a basic formula $\psi$ of the form $L_j\psi$ (j ≠ i) that is independent of $\phi_1, \ldots, \phi_m$ wrt. $AX^t_n$.

**Proof.** Suppose that $\phi_1$ is $a$-objective and of maximal $k$-depth $k$. A formula $\psi$ of the form $(L_i L_a)^{k+1}p$ (where $p \in \Phi$ is in the scope of $k+1 L_i L_a$) is shown to be independent of $\phi_1, \ldots, \phi_m$. Let us suppose we can derive a $\gamma$ of the form $L_i L_a L_a \ldots p$ of maximal depth $k$, to show that neither $\vdash \psi \land \neg \psi$ nor $\vdash \psi \land \neg \psi$. Given any formula, the only axioms in $AX^t_n$ that can introduce $\gamma$ in the scope of modal operators is $A_{4n}$, and $A_{5n}^{t+1}$. Applying $A_{4n}$, gives $L_i \gamma$ or $N_i \gamma$, and then using the axiom again we have $L_i L_i L_a \gamma$ or $L_i N_i \gamma$. It is easy to see that the resulting formulas are clearly independent from $\psi$. Applying $A_{5n}^{t+1}$ on the other hand, allows us to derive $\vdash \gamma \land \neg L_i \gamma$ (γ is consistent wrt. $AX^t_n$ and hence also wrt. $A_{5n}^{t+1}$). Again, we could show $\vdash \gamma \land \neg L_i \gamma$. Continuing this way, it might only be possible to derive $\neg L_i L_a \ldots p$ of depth $2k + 2$, that is indeed independent of $\psi$.

**Lemma 27.** If $\phi$ and $\psi$ are $i$-objective formulas, $\phi, \psi \in \text{ONL}^t_n$, and $L_i \phi \land N_i \psi$ is $AX^t_{n+k}$-consistent then $\models \phi \land \psi$.

**Proof.** Suppose not. Then $\neg \phi \land \neg \psi$ is $AX^t_{n+k}$-consistent, and by $A_{5n}^{t+1}$ we prove $N_i \neg \alpha \supset \neg L_i (\phi \land \psi)$, and thus, $N_i \psi \supset \neg L_i \phi$, and this is not $AX^t_{n+k}$-consistent with $L_i \phi \land N_i \psi$.

**Theorem 28.** For all $\alpha \in \text{ONL}^t_n$, if $\models \alpha$ then $AX^t_n \vdash \alpha$.

**Proof.** Proof by induction on $t$. It is sufficient to show that if a formula $\beta \in \text{ONL}^t_n$ is $AX^t_{n+k}$-consistent then it is satisfiable wrt. some model. We already have the proof for $\text{ONL}^t_n$ (see Theorem 23). Let us assume the proof holds for all formulas $\alpha \in \text{ONL}^t_n$. Particularly, this means that any formula that is $AX^t_{n+k}$-consistent is satisfiable wrt. some $(k', j')$-model. Let $\alpha \in \text{ONL}^t_{n+k}$ (say of maximal $a, b$-depth of $k + 1, j + 1$), and suppose that $\alpha$ is consistent wrt. $AX^t_{n+k}$. It is sufficient to show that $\alpha$ is satisfiable. Wlog, we take it in the normal form:

$\lor \sigma \land L_a \varphi_0 \land \neg L_a \varphi_1 \land \ldots \land \neg L_a \varphi_{am_1} \land L_b \varphi_0 \land \ldots \land \neg L_b \varphi_{bn_2} \land N_a \psi_0 \land \ldots \land \neg N_a \psi_{an_1} \land N_b \psi_0 \land \ldots \land \neg N_b \psi_{bn_2}$.

Note that, by definition, it must be that all of $\varphi_{im}, \psi_{in}$ are at most in $\text{ONL}^t_{n+k}$ (i.e. they may also be in $\text{ONL}^t_{n+k-1}$, and $i$-objective. We proceed as we did for Theorem 23 but without restricting to basic formulas. Let $A$ be all $AX^t_{n+k}$-consistent formulas of the form $\varphi_0 \land \psi_0 \land \neg \varphi_{aj}$ or $\varphi_0 \land \psi_0 \land \neg \psi_{aj}$ (they are of maximal $b$-depth $k$). Let $\gamma$ be independent of all formulas in $A$. Let $S_a$ be the set of all $(AX^t_{n+k})$ maximally consistent sets of formulas, constructed from formulas of maximal $b$-depth $k$, and containing $\varphi_0 \land \neg \psi_0 \lor (\psi_0 \land \gamma)$, and hence by induction hypothesis they are satisfiable in some model. Note that all formulas in $S_a$ are in $\text{ONL}^t_n$. The $b$-depth is maximally $2k + 2$. Letting $k'' = 2k + 2$, we have that for all $S' \in S_a$ there is a $(w, e_k^\gamma)$ such that $\{\}, e_k^\gamma, w = S'$. Let $k'' = k'' + 1$. Letting $e_{k''}$ be all such $k''$-structures $(w, e_k^\gamma)$ for each $S' \in S_a$ makes $Ob_{j''}(e_{k''}) = S_a$ (in contrast, for Theorem 23 we dealt with $Ob_{j''}$). We claim that this $k''$-structure for Alice, a $j''$-structure for Bob constructed similarly, and a world where $\sigma$ holds (there is such a world since $\sigma$ is propositional and consistent) is a model where $\alpha$ is satisfied. The proof proceeds as in Theorem 23. We show the case of $\neg L_a \varphi_{aj}$.  

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1 The idea was also suggested by a reviewer in (Halpern and Lakemeyer 2001) for an axiomatic characterization of the extended canonical model, although its completeness was left open.
Since $L_a \varphi_{a0} \land \neg L_a \varphi_{a3}$ is consistent wrt. $AX_n^{t+1}$, it must be that $\varphi_{a0} \land \neg \varphi_{a3}$ is consistent wrt. $AX_n^{t+1}$. Further, since $\varphi_{a0}, \varphi_{a3} \in ONC_n^t$, they must consist be wrt. $AX_n^t$ (for if not, they cannot by definition be consistent wrt. $AX_n^{t+1}$). This means that either $\varphi_{a0} \land \neg \varphi_{a3} \land \psi_{a0}$ or $\varphi_{a0} \land \neg \varphi_{a3} \land \neg \psi_{a0}$ is consistent. If the former is, then so is $\varphi_{a0} \land \neg \varphi_{a3} \land \psi_{a0} \land \gamma$.

Since $S_a$ consist of all $AX_n^t$-consistent formulas containing $\varphi_{a0} \land (\neg \psi_{a0} \land (\psi_{a0} \land \gamma))$, there is clearly a $S^t \in S_a$ such that $\neg \varphi_{a3} \in S^t$. Consequently, it cannot be that $e_{a}^{k_1}, \{\}, w' \models L_a \varphi_{a3}$. Thus, $e_{a}^{k_1}, \{\}, w' \models \neg L_a \varphi_{a3}$. ■

Thus, we have a sound and complete axiomatization for the propositional fragment of $ON/L_n$. In comparison to Lakemeyer (1993), the axiomatization goes beyond a language that restricts the nesting of $N_i$. In contrast to Halpern and Lakemeyer (2001), the axiomatization does not necessitate the use of semantic notions in the proof theory. A third axiomatization by (Waaler 2004; Waaler and Solhaug 2005) proposes an interesting alternative to deal with the circularity in a generalized $A5$. The idea is to first define consistency by formulating a fragment of the axiom system in the sequent calculus. Quite analogous to having $t$-axioms, they allow us to apply $A5_n$ on $i$-objective formulas of a lower depth, thus avoiding circularity without the need to appeal to satisfiability as in (Halpern and Lakemeyer 2001). Waaler and Solhaug (2005) also define a semantics for multi-agent only-knowing which does not appeal to canonical models. Instead, they define a class of Kripke structures which need to satisfy certain constraints. Unfortunately, these constraints are quite involved and, as the authors admit, the nature of these models “is complex and hard to penetrate.”

To get a feel of the axiomatization, let us consider a well studied example from (Halpern and Lakemeyer 2001) to see where we differ. Suppose Alice assumes the following default: unless I know that Bob knows my secret then he does not know it. If the default is all that she knows, then she nonmonotonically comes to believe that Bob does not know her secret. Let $\gamma$ be a proposition that denotes Alice’s secret, and we want to show that $\vdash O_a(\delta) \supset L_0 \neg L_0 \gamma$, where $\delta = \neg L_0 L_0 \gamma \supset \neg L_0 \gamma$. We write (Def.) to mean $O_a \alpha \equiv L_0 \alpha \land N_0 \neg \alpha$, and we freely reason with propositional logic (PL) or $K45_n$.

1. $O_a(\delta) \supset L_0 \neg L_0 \gamma \neg L_0 \gamma \gamma \gamma$ Def, PL, $A2_n$.
2. $O_a(\delta) \supset N_0 \neg L_0 L_0 \gamma \land N_0 \neg L_0 \gamma \gamma$ Def, PL, $K45_n$.
3. $N_0 \neg L_0 \gamma \neg L_0 \gamma$ $A5_1$.
4. $\neg L_0 \neg L_0 L_0 \gamma \neg L_0 L_0 \gamma$ $A5_n$.
5. $O_a(\delta) \supset L_0 \neg L_0 \gamma \gamma$ 2,3,4, PL.
6. $O_a(\delta) \supset L_0 \neg L_0 \gamma$ 1,5, PL.

We use $A5_1$, and it is applicable because $L_0 \gamma$ is $\alpha$-objective and $K45_n$-consistent. Now, suppose Alice is cautious. She changes her default to assume that if she does not believe Bob to only-know some set of facts $\theta \in \Phi$, then $\theta$ is not all that he knows. We would like to show $\vdash O_a(\neg L_0 O_b \theta \supset \neg O_a \theta) \supset L_0 \neg O_a \theta$. Of course, this default is different from $\delta$ in containing $O_b \theta$ rather than $L_0 \gamma$. The proof is identical, except that we use $A5_1^2$, since $\neg O_a \theta \in ON/L_n^1$ is $\alpha$-objective and $AX_n^1$-consistent. The latter proof requires reasoning with the satisfiability modal operator in Halpern and Lakemeyer (2001), and is not provable with the axioms of Lakemeyer (1993).

### Autoepistemic Logic

Having examined the properties of multi-agent only-knowing, in terms of a semantics for both the first-order and propositional case, and an axiomatization for the propositional case, in the current section we discuss how the semantics also captures autoepistemic logic (AEL). AEL, as originally developed by Moore (1985), intends to allow agents to draw conclusions, by making observations of their own epistemic states. For instance, Alice concludes that she has no brother because if she did have one then she would have known about it, and she does not know about it (Moore 1985). The characterization of such beliefs are defined using fixpoints called stable expansions. In the single agent case, Levesque (1990) showed that the beliefs of an agent who only-knows $\alpha$ is precisely the stable expansion of $\alpha$. Of course, the leverage with the former is that it is specified using regular entailments. In Lakemeyer (1993), and Halpern and Lakemeyer (2001), a many agent generalization of AEL is considered in the sense of a stable expansion for every agent, and relating this to what the agent only-knows. But their generalizations are only for the propositional fragment, while Levesque’s definitions involved first-order entailments. In contrast, we obtain the corresponding quantificational multi-agent generalization of AEL. We state the main theorems below. The proofs are omitted since they follow very closely from the ideas for the single agent case (Levesque and Lakemeyer 2001).

**Definition 29.** Let $A$ be a set of formulas, and $\Gamma$ be the $i$-stable expansion of $A$ iff it the set of first-order implications of $A \cup \{L_i \beta \mid \beta \in \Gamma\} \cup \{\neg L_i \beta \mid \beta \not\in \Gamma\}$.

**Definition 30 (Maximal structure).** If $e_{a}^{k}$ is a $k$-structure, let $e_{a}^{k}$ be a $k$-structure with the addition of all $\langle w', e_{b}^{k-1} \rangle \not\in e_{a}^{k}$ such that for every $\alpha \in ON/L_n^1$ of maximal $a$, b-depth $k$, $k-1$, if $e_{a}^{k}, \{\}, w \models L_0 \alpha$ for any world $w$ then $e_{a}^{k}, e_{b}^{k-1}, \{\}, w' \models \alpha$. Define $\Gamma = \{\beta \mid \beta$ is basic and $e_{a}^{k}, \{\}, w' \models L_0 \alpha\}$ as the belief set of $e_{a}^{k}$.

**Theorem 31.** Let $M = (e_{a}^{k}, e_{b}^{k}, w)$ be a model, where $e_{a}^{k}$ is a maximal structure for $a$. Let $\Gamma$ be the belief set of $e_{a}^{k}$, and suppose $\alpha \in ON/L_n^1$ is of maximal $a$, b-depth $k$, $k-1$. Then, $M \models O_a \alpha$ if $\Gamma$ is the $i$-stable expansion of $\alpha$.

Theorem 31 essentially says that the complete set of basic beliefs at a maximal epistemic state where $\alpha$ is all that $i$ knows, precisely coincides with the $i$-stable expansion of $\alpha$.

### Axiomatizing Validity

Extending the work in (Lakemeyer 1993) and (Halpern 1993), which was only restricted to formulas in $ON/L_n^1$, Halpern and Lakemeyer (2001) proposed a multi-agent only-knowing logic that handles the nesting of $N_i$ operators. But as discussed, there are two undesirable features. The first is a semantics based on canonical models, and the
The second is a proof theory that axiomatizes validity. Although such a construction is far from natural, we show in this section that they do indeed capture the desired properties of only-knowing. This also instructs us that our axiomatization avoids such problems in a reasonable manner.

Recall that the language of (Halpern and Lakemeyer 2001) is $\mathcal{OL}_n^+$, which is $\mathcal{OL}_n$, and a modal operator for validity, $\text{Val}$. A modal operator $\text{Sat}$, for satisfiability, is used freely such that $\text{Val}(\alpha)$ is syntactically equivalent to $\neg\text{Sat}(-\alpha)$. To enable comparisons, we present a variant of our logic, that has all its main features, but has additional notions to handle the extended language. We then show that this logic and (Halpern and Lakemeyer 2001) agree on the set of valid sentences from $\mathcal{OL}_n^+$ (and also $\mathcal{OL}_n$).

The main feature of (Halpern and Lakemeyer 2001) is the proof theory $AX_n^+$, and a semantics that is sound and complete for $AX_n^+$ via the extended canonical modal. $AX_n^+$ consists of A1, ¬A4, MP, NEC and the following:

A5. $\text{Sat}(\alpha) \supset (N_\alpha \supset \neg L_\alpha)$, if $\alpha$ is i-objective.

V1. $\text{Val}(\alpha) \land \text{Val}(\beta) \supset \text{Val}(\beta)$.

V2. $\text{Sat}(p_1 \land \ldots \land p_n)$, if $p_i$’s are literals and $p_1 \land \ldots \land p_n$ is propositionally consistent.

V3. $\text{Sat}(\alpha \land \beta_1) \land \ldots \land \text{Sat}(\alpha \land \beta_k) \land \text{Sat}(\gamma \land \delta_1) \land \ldots \land \text{Sat}(\delta_n) \land \text{Val}(\alpha) \lor \gamma \supset \text{Sat}(\text{Ax} \alpha \land \neg \text{Ax} \beta_1 \land \ldots \land \neg \text{Ax} \delta_1 \ldots \land \neg \text{Ax} \gamma)$, if $\alpha, \beta_1, \beta_k, \gamma, \delta_1$ are i-objective.

V4. $\text{Sat}(\alpha) \land \text{Sat}(\beta) \supset \text{Sat}(\alpha \land \beta)$, if $\alpha$ is i-objective and $\beta$ is $\beta$-subjective.

NEC$\text{Val}$. From $\alpha$ infer $\text{Val}(\alpha)$.

The essence of our new logic, in terms of a notion of depth (with $|\text{Val}(\alpha)| = |\alpha|$) and a semantical account over possible worlds, is as before. The complete semantic definition for formulas in $\mathcal{OL}_n^+$ of maximal $a$, $b$-depth of $k$, $j$ is:

1. -8, as before,

2. $\text{Val}(\alpha) \land \text{Val}(\beta) \supset \text{Val}(\beta)$.

3. $\text{Sat}(p_1 \land \ldots \land p_n)$, if $p_i$’s are literals and $p_1 \land \ldots \land p_n$ is propositionally consistent.

4. $\text{Sat}(\alpha \land \beta_1) \land \ldots \land \text{Sat}(\alpha \land \beta_k) \land \text{Sat}(\gamma \land \delta_1) \land \ldots \land \text{Sat}(\delta_n) \land \text{Val}(\alpha) \lor \gamma \supset \text{Sat}(\text{Ax} \alpha \land \neg \text{Ax} \beta_1 \land \ldots \land \neg \text{Ax} \delta_1 \ldots \land \neg \text{Ax} \gamma)$, if $\alpha, \beta_1, \beta_k, \gamma, \delta_1$ are i-objective.

5. $\text{Sat}(\alpha) \land \text{Sat}(\beta) \supset \text{Sat}(\alpha \land \beta)$, if $\alpha$ is i-objective and $\beta$ is $\beta$-subjective.

NEC$\text{Val}$. From $\alpha$ infer $\text{Val}(\alpha)$.

The proof of this lemma, and those of the following theorems are given in the appendix. We proceed to show that $\text{Sat}(\alpha)$ is provable from $AX_n^+$ iff $\alpha$ is $AX_n^+$-consistent.

Theorem 33. For all $\alpha \in \mathcal{OL}_n^+$, $AX_n^+ \vdash \alpha$ iff $AX_n^+ \vdash \alpha$.

Lemma 35. For all $\alpha \in \mathcal{OL}_n^+$, $\models \alpha$ iff $\alpha$ is valid in (Halpern and Lakemeyer 2001).

Proof. $AX_n^+$ is sound and complete for (Halpern and Lakemeyer 2001), and $AX_n^+$ is sound and complete for $\models$.

Since it can be shown that every $\alpha \in \mathcal{OL}_n^+$ is provably equivalent to some $\alpha' \in \mathcal{OL}_n$ (Halpern and Lakemeyer 2001), we also obtain the following corollary.

Corollary 36. For all $\alpha \in \mathcal{OL}_n^+$, $\models \alpha$ iff $\alpha$ is valid in (Halpern and Lakemeyer 2001).

Conclusions

This paper has the following new results. We have a first-order modal logic for multi-agent only-knowing that we show, for the first time, generalizes Levesque’s semantics. Unlike all attempts so far, we neither make use of proof-theoretic notions of maximal consistency nor Kripke structures (Waaler and Solhaug 2005). The benefit is that the semantic proofs are straightforward, and we understand possible worlds precisely as Levesque meant. We then analyzed a propositional subset, and showed first that the axiom system from Lakemeyer (1993) is sound and complete for a restricted language. We used this result to devise a new proof theory that does not require us axiomatize any semantic notions (Halpern and Lakemeyer 2001). Our axiomatization was shown to be sound and complete for the semantics, and its use is straightforward on formulas involving the nesting of at most operators. In the process, we revisited the features of only-knowing and compared the semantical framework to other approaches. Its behavior seems to coincide with our intuitions, and it also captures a multi-agent generalization of Moore’s AEL. Finally, although the axiomatization of Halpern and Lakemeyer (2001) is not natural, we showed that they essentially capture the desired properties of multi-agent only-knowing, but at much expense.

Acknowledgements

The authors would like to thank the reviewers for helpful suggestions and comments. The first author is supported by a DFG scholarship from the graduate school GK 643.

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Appendix

Lemma 32. For all α ∈ ONL^+_n, AX^+_n ⊢ α iff 1 = α.

Proof. The proof is via induction. Using Theorems 25 and 28 as the base cases in the induction, there is one additional step on the structure of formulas.

Soundness: The base case holds for formulas α ∈ ONL^+_n for AX^+_n. Suppose now if AX^+_n ⊢ α, then AX^+_n ⊢ Val(α).

But if AX^+_n ⊢ α then (by induction hypothesis) at all models e^1_{a_k}, e^1_{b}, w ⊨ Val(α), and so by the definition at all models e^1_{a_k}, e^1_{b}, w ⊨ Val(α) or 1 = Val(α).

Completeness. For the base case, we know that if for all models e^1_{a_k}, e^1_{b}, w ⊩ 1 then AX^+_n ⊢ α. Suppose 1 = Val(α), then by definition, for all models e^1_{a_k}, e^1_{b}, w ⊩ 1 (by hypothesis) AX^+_n ⊢ α. So, AX^+_n ⊢ Val(α).

Theorem 33. For all α ∈ ONL^+_n, AX^+_n ⊢ Sat(α) iff α is AX^+_n-consistent.

Proof. It is helpful to have the following variant of Lemma 27 at hand, and a corollary thereof.

Lemma 37. Suppose φ, ψ ∈ ONL^+_n are i-objective AX^+_n-consistent formulas, and 1 = φ ∨ ψ. Then, L_i(φ ∨ N_iψ) is AX^+_n-consistent.

Proof. Suppose not. Then AX^+_n ⊢ ¬(L_i(φ ∨ N_iψ)), that is AX^+_n ⊢ ¬L_i(φ ∨ N_iψ). Then, by Lemma 32, 1 = ¬L_i(φ ∨ N_iψ). Let W_φ = {w | w ⊩ φ}. Let e^k_{a_i} = W_φ × B^{k−1} be a e^k for Alice. Then clearly, e^k_{a_i} {, w} ⊢ ¬N_iψ. It must be then that e^k_{a_i} {, w} ⊢ ¬N_iψ. Then there is some ⟨w', k^{k−1}⟩ ⊢ e^k_{a_i} such that e^k_{a_i} e^k_{k^{k−1}}, w' ⊢ ¬ψ. And clearly, for all ⟨w', k^{k−1}⟩ ⊢ e^k_{a_i} e^k_{a_k} e^k_{k−1}, w ⊢ ¬φ (by construction). It follows that there is a ⟨w', k^{k−1}⟩ ⊢ e^k_{a_i} such that e^k_{a_i} e^k_{k−1}, w ⊢ ¬(φ ∨ ψ), contradicting the validity of φ ∨ ψ. ■

Corollary 38. Suppose α, β_1, ..., β_k, γ, δ_1, ..., δ_m ∈ ONL^+_n are i-objective AX^+_n-consistent formulas, and 1 = α ∨ γ. Then, L_i(α ∨ ¬L_i β_1 ∨ L_i ¬β_k ∧ N_iγ ∧ ¬N_i δ_1 ∧ ... ∧ ¬N_i δ_m) is AX^+_n-consistent.

Returning to Theorem 33: Proof on the length of the derivational, using induction on t. Let α be a consistent propositional formula. Then, by V2, AX^+_n ⊢ Sat(α). Since it is a consistent propositional formula, it is also AX^+_n-consistent. Assume theorem holds for α ∈ ONL^+_n. Suppose we have Sat(α ∧ β_k), Sat(γ ∧ δ_m), ¬Sat(α ∨ γ)) ∈ ONL^+_n then by V3, AX^+_n ⊢ Sat(L_i(α ∧ ¬L_i β_k ∧ N_iγ ∧ ¬N_i δ_m). By hypothesis α ∧ β_k ∧ γ ∧ δ_m are AX^+_n-consistent. And ¬(α ∨ γ) is not AX^+_n-consistent, and so AX^+_n ⊢ α ∨ γ. By Lemma 32, 1 = α ∨ γ. Clearly, by Corollary 38, L_i(α ∧ ¬L_i β_k ∧ N_iγ ∧ ¬N_i δ_m) is AX^+_n-consistent. Finally, suppose that you have Sat(α) for some i-objective α and Sat(β) for some i-subjective β, then by V4, AX^+_n ⊢ Sat(α ∧ β). By induction hypothesis, α and β are AX^+_n-consistent. By Lemma 32, α is satisfiable and β is satisfiable, and so is α ∧ β. By Lemma 32, α ∧ β is AX^+_n-consistent. The other direction is symmetric. ■

Theorem 34. AX^+_n ⊢ α iff AX^+_n ⊢ α, for α ∈ ONL^+_n.

Proof. Since axioms A1_α − A4_α, MP, NEC, NEC^Val are common to both, their use is not discussed. To show that AX^+_n ⊢ α ⇒ AX^+_n ⊢ α, suppose you had Sat(α) for some i-objective α ∈ ONL^+_n then using A5'_i, one could show that N_iα ⊢ ¬L_iα. From Theorem 33, we also know ¬α is AX^+_n-consistent. Then, we can show N_iα ⊢ ¬L_iα as well using A5'_i. V2, V3, V4 follow immediately from Theorem 33. Assuming now that the proof holds for base cases, using V1, if AX^+_n ⊢ Val(α) and AX^+_n ⊢ Val(α ∧ β) then AX^+_n ⊢ Val(β). Now, by induction hypothesis, AX^+_n ⊢ Val(α) iff by Lemma 32 ⊢ Val(α), and so 1 = Val(α). Similarly, 1 = Val(β). Thus, ⊢ Val(β) by the semantics. By Lemma 32, AX^+_n ⊢ Val(β).

To show that AX^+_n ⊢ α ⇒ AX^+_n ⊢ α, suppose ¬α ∈ ONL^+_n is i-objective and AX^+_n-consistent, then one can prove N_iα ⊢ ¬L_iα. Now, ¬α is also AX^+_n-consistent and by Theorem 33, AX^+_n ⊢ Sat(¬α). Then we can prove N_iα ⊢ ¬L_iα, as desired. ■