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GENERALISED WITT ALGEBRAS AND IDEALIZERS

S. J. SIERRA AND Š. ŠPENKO

Abstract. Let \( k \) be an algebraically closed field of characteristic zero, and let \( \Gamma \) be an additive subgroup of \( k \). Results of Kaplansky-Santharoubane and Su classify intermediate series representations of the generalised Witt algebra \( W_\Gamma \) in terms of three families, one parameterised by \( k^2 \) and two by \( \mathbb{F}^1 \). In this note, we use the first family to construct a homomorphism \( \Phi \) from the enveloping algebra \( U(W_\Gamma) \) to a skew extension \( k[\mathbb{A}^2] \times \Gamma \) of the coordinate ring of \( k^2 \). We show that the image of \( \Phi \) is contained in a (double) idealizer subring of this skew extension and that the representation theory of idealizers explains the three families. We further show that the image of \( U(W_\Gamma) \) under \( \Phi \) is not left or right noetherian, giving a new proof that \( U(W_\Gamma) \) is not noetherian.

We construct \( \Phi \) as an application of a general technique to create ring homomorphisms from shift-invariant families of modules. Let \( G \) be an arbitrary group and let \( A \) be a \( G \)-graded ring. A graded \( A \)-module \( M \) is an intermediate series module if \( M_g \) is one-dimensional for all \( g \in G \). Given a shift-invariant family of intermediate series \( A \)-modules parameterised by a scheme \( X \), we construct a homomorphism \( \Phi \) from \( A \) to a skew extension of \( k[X] \). The kernel of \( \Phi \) consists of those elements which annihilate all modules in \( X \).

1. Introduction

Fix an algebraically closed ground field \( k \) of characteristic zero, and let \( \Gamma \) be a finitely generated additive subgroup of \( k \). The generalised Witt algebra \( W_\Gamma \) is the Lie algebra generated by elements \( e_\gamma : \gamma \in \Gamma \), with \( [e_\gamma, e_\delta] = (\delta - \gamma)e_{\delta + \gamma} \). Recall that an intermediate series representation of \( W_\Gamma \) is an indecomposable representation all of whose \( c_0 \)-eigenspaces are 1-dimensional. It is a theorem of Kaplansky and Santharoubane [KS85] (if \( \Gamma = \mathbb{Z} \)) and of Su [Su94] (for general \( \Gamma \)) that intermediate series representations of \( W_\Gamma \) come in three families (with two modules represented twice): one family parameterised by \( k^2 \) and two parameterised by \( \mathbb{F}^1 \). In this note we use the first family to construct a homomorphism \( \Phi \) from \( U(W_\Gamma) \) to \( T = k[\mathbb{A}^2] \times \Gamma \), and show that the existence of the other two families is a consequence of the fact that the image of \( U(W_\Gamma) \) is a sub-idealizer in \( T \). We further use the homomorphism \( \Phi \) to give a new proof that the enveloping algebra of \( U(W_\Gamma) \) is not noetherian, a fact originally proved in [SW14].

Since our main method is to construct and then analyze a homomorphism from \( U(W_\Gamma) \) to an idealizer in \( T \), we recall some facts about idealizers. We first define \( T \): as a vector space we write \( T = \bigoplus_{\gamma \in \Gamma} k[a, b]t^\gamma \), with \( t^\gamma t^\delta = t^{\gamma + \delta} \) and \( t^\gamma f(a, b) = f(a + \gamma, b) t^\gamma =: f^\gamma t^\gamma \). Note that \( T \) is a bimodule over \( k[a, b] \).

An intermediate series module \( M \) over a \( \Gamma \)-graded ring is an indecomposable \( \Gamma \)-graded module with each \( M_\gamma \) one-dimensional. Such modules are preserved under degree shifting.

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We now consider a subring of \( T \). Fix \( p_0 \in k^2 \), let \( I(p) \) be the ideal \( (a - \alpha, b - \beta) \) of \( k[a, b] \). Let \( V(p) = T/I(p)T \). It is easy to see that the \( V(p) \) are all of the intermediate series right \( T \)-modules; more precisely, the right ideals \( J \) of \( T \) such that \( T/J \) is an intermediate series module are precisely the \( I(p)T \). Likewise, the intermediate series left \( T \)-modules are the \( T/TI(p) \). These families are preserved under degree shifting.

We now consider a subring of \( T \). Fix \( p_0 \in k^2 \), let \( S = S(p_0) = k \oplus I(p_0)T \). The ring \( S \) is an idealizer in \( T \): the largest subalgebra of \( T \) such that \( I(p_0)T \) becomes a two-sided ideal in \( S \). It is known [Rog94] that the representation theory of idealizers involves blowing up. Here for \( p \neq p_0 \), we have that \( V(p) \cong S/(S \cap I(p)T) \) is an intermediate series right \( S \)-module. On the other hand, to define an intermediate series right \( S \)-module at \( p_0 \), we need to consider a point \( q \) infinitely near to \( p_0 \): that is, an ideal \( I(q) \) with \( I(p_0)^2 \subset I(q) \subset I(p_0) \) of \( k[a, b] \) such that \( I(p_0)/I(q) \) is one-dimensional. Such ideals are parameterised by the exceptional \( \mathbb{F}^1 \) in the blowup \( Bl_{p_0}(k^2) \); more specifically, we can write

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\[ I(q) = (y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2) \] for some \([x : y] \in \mathbb{P}^1\). For such \(I(q)\) we have that \(I(p_0) + I(q)T\) is a right ideal of \(S\). Let

\[ P(q) = S/(I(p_0) + I(q)T). \]

Then \(P(q)\) is an intermediate series right \(S\)-module. In fact, we have constructed all right ideals \(J\) of \(S\) such that \(S/J\) is an intermediate series \(S\)-module; they are parameterised by \(\text{Bl}_{p_0}(A^2)\) but it is sometimes more convenient to consider them as parameterised by \(A^2 \setminus \{p_0\}\) together with \(\mathbb{P}^1\).

Left intermediate series \(S\)-modules are also parameterised by \(\text{Bl}_{p_0}(A^2)\). For \(p \in A^2 \setminus \{p_0\}\), the left intermediate series module \(T/TT(p)\) is isomorphic to \(\left( I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a, b]t^\nu \right) / \left( (I(p_0) \cap I(p)) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p) \right)\).

We can extend this construction to a family of modules parameterised by \(\text{Bl}_{p_0}(A^2)\) by adding the \(\mathbb{P}^1\) of points \(q\) infinitely near to \(p_0\):

\[ Q(q) = \frac{I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a, b]t^\nu}{I(q) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p_0)}. \]

Consider now right intermediate series modules over the double idealiser

\[ R = \mathbb{k}[a, b] + (I(p_0)T \cap TI(p_1)) \]

and assume for simplicity that \(p_0, p_1 \in A^2\) have distinct \(\Gamma\)-orbits. These correspond to points of the double blowup \(\text{Bl}_{p_0,p_1}(A^2)\). More precisely, the \(V(p)\) are intermediate series modules for \(p \in A^2 \setminus \{p_0, p_1\}\). From the inclusion \(R \subseteq \mathbb{k} \oplus I(p_0)T\) we obtain a family \(P(q)\) parameterised by the \(\mathbb{P}^1\) of points infinitely near to \(p_0\). Finally, from the inclusion \(R \subseteq \mathbb{k} \oplus TI(p_1)\) we obtain a family \(Q(q)\) of right modules parameterised by the \(\mathbb{P}^1\) of points infinitely near to \(p_1\) and constructed similarly to the construction of the left modules \(Q(q)\) over \(S\).

Let \(\Gamma\) now be an arbitrary group (more generally, a monoid) and let \(A\) be a \(\Gamma\)-graded ring. We give a general result in Theorem 2.2 (respectively, Theorem 2.5) which constructs a ring homomorphism (respectively, an anti-homomorphism) \(\Phi : A \rightarrow \mathbb{k}[X] \times \Gamma\), where \(X\) is a shift-invariant family of right (respectively, left) intermediate series \(A\)-modules; this generalises constructions in \([\text{ATV91}], [\text{RZ08}], [\text{V96}]\).

When we apply this technique to \(U(W_1)\), we show that the image of \(\Phi\) is contained in a double idealizer \(R\) inside the ring \(T\) defined in the second paragraph, and we show in Propositions 3.5, 3.6 that the right intermediate series \(R\)-modules constructed above restrict to precisely the intermediate series representations of \(W_1\). This gives a unified geometric description of what have until now been seen as three distinct families of representations.

We further show in Proposition 4.3 that the image of \(U(W_1)\) under \(\Phi\) is neither right nor left noetherian. For \(\Gamma = \mathbb{Z}\) this was proved in \([\text{SW15}]\) as the main step in proving the non-noetherianity of \(U(W)\). It follows that \(U(W_1)\) is neither right or left noetherian; other proofs are given in \([\text{SW14}], [\text{SW15}]\).

The general behaviour of idealizers leads one to expect that at idealizers in \(T\) at ideals of points in \(\mathbb{k}[a, b]\) will not be noetherian since no points have dense \(\Gamma\)-orbits; see \([\text{Sie11}]\) for a precise statement of a related result for \(N\)-graded rings. However, infinite orbits are dense in \(A^1\). Thus one expects that the factors \(\Phi(U(W_1))|_{b=\beta}\), which live on the \(\Gamma\)-invariant line \((b = \beta)\) in \(A^2\), are noetherian for all \(\beta \in \mathbb{k}\), and we also show in Proposition 4.6 that this is indeed the case.

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## 2. Intermediate series modules and ring homomorphisms

It is well-known that ring homomorphisms can be constructed from shift-invariant families of modules. Let \(A\) be a \((\text{connected } N\text{)}\)-graded ring, generated in degree 1. A **point module over** \(A\) is a cyclic graded \(A\)-module with Hilbert series \(1/(1-t)\). Suppose that \((\text{right }) A\)-point modules are parameterised by a projective scheme \(X\). Let the point module corresponding to \(x \in X\) be \(M^x\). Then the shift functor \(\Psi : M \mapsto M[1]_{\geq 0}\) induces an automorphism \(\sigma\) of \(X\) so that \(\Psi(M^x) \cong M^{\sigma(x)}\).

The following result goes back to \([\text{ATV90}]\) (see also \([\text{V96}]\)), although in this form it is due to Rogalski and Zhang.
Theorem 2.1. ([IZ90] Theorem 4.4) There is an invertible sheaf $\mathcal{L}$ on $X$ so that there is a homomorphism $\phi : A \to B(X, \mathcal{L}, \sigma)$ of graded rings, where $B(X, \mathcal{L}, \sigma)$ is the twisted homogeneous coordinate ring defined in [AV90]. If $A$ is noetherian then $\phi$ is surjective in large degree.

The kernel of $\phi$ is equal in large degree to

$$J = \bigcap \{ \text{Ann}_A(M) \mid M \text{ is a C-point module for some commutative k-algebra C } \}.$$ 

The purpose of this section is to give a version of this theorem for a (not necessarily connected graded) algebra graded by an arbitrary monoid $\Gamma$.

We first need some notation. Let $\Gamma$ be a monoid and let $A$ be a $\Gamma$-graded ring. If $M$ is a $\Gamma$-graded right $A$-module and $\gamma \in \Gamma$, we define the shift $M(\gamma)$ of $M$ by $\gamma$ as:

$$M(\gamma) = \bigoplus_{\delta \in \Gamma} M(\gamma)_\delta,$$

where $M(\gamma)_\delta = M_{\gamma \delta}$. We note that

(2.1) 

$$M(\gamma)_\delta A_\epsilon = M_{\gamma \delta \epsilon} \subseteq M(\gamma)_{\delta \epsilon},$$

so $M(\gamma)$ is again a $\Gamma$-graded right $A$-module. Note that

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\delta \epsilon} = M(\gamma)_\delta$$

and so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\gamma)_\delta$.

If $M$ is a left module we define $M(\gamma)_\delta = M_{\delta \gamma}$. Then (2.1) becomes:

$$A_\gamma M(\gamma)_\delta = A_\epsilon M_{\delta \gamma} \subseteq M(\gamma)_{\epsilon \delta},$$

as needed. We have

$$(M(\gamma))(\delta)_\epsilon = M(\gamma)_{\epsilon \delta} = M(\delta \gamma)_\epsilon$$

so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\delta \gamma)$.

If $A$ is a $\Gamma$-graded ring, an intermediate series module over $A$ is a $\Gamma$-graded left or right $A$-module $M$ so that $\dim M_{\gamma} = 1$ for all $\gamma \in \Gamma$. We will use a shift-invariant family of intermediate series modules to construct a ring homomorphism from $A$ to a $\Gamma$-graded ring, giving a version of Theorem 2.1 in this setting.

Our notation for smash products is that if $\Gamma$ acts on $A$ then $A \times \Gamma = \bigoplus_{\gamma \in \Gamma} A\gamma$, where $\gamma \gamma = \tau \gamma$ and $\gamma \gamma = \tau \gamma$ for all $r \in A, \gamma \in \Gamma$.

Theorem 2.2. Let $\Gamma$ be a monoid with identity $e$ and let $A$ be a $\Gamma$-graded ring. Let $X$ be a reduced affine scheme that parameterises a set of intermediate series right $A$-modules, in the sense that for $x \in X$ there is a module $M^x$ with basis $\{ v^x_\gamma \mid \gamma \in \Gamma \}$, and that there is a $k$-linear function $\phi : A \to k[X]$ so that

$$v^x_{\gamma \gamma} = \phi(r)(x) v^x_\gamma$$

for all $\gamma \in \Gamma, r \in A_{\gamma}$. Further suppose that shifting defines a group antihomomorphism $\sigma : \Gamma \to \text{Aut}_k(A, \gamma \mapsto \sigma \gamma$ so that $M^x(\gamma) \cong M^{x \gamma}(x)$). Here we require that the isomorphism maps $v^x_{\gamma \gamma} \mapsto v^{x \gamma}(x)$.

In this setting the map

$$\Phi : A \to k[X] \times \Gamma, \quad r \in A_{\gamma} \mapsto \phi(r) t^\gamma$$

is a graded homomorphism of algebras. Further,

$$\ker \Phi = \bigcap_{x \in X} \text{Ann}_A M^x.$$ 

Proof. Let $\Gamma$ act on $k[X]$ by $f^{\gamma} = (\sigma \gamma)^*(f)$, so $\sigma$ defines a homomorphism from $\Gamma \to \text{Aut}_k(k[X])$.

Let $r \in A_{\gamma}, s \in A_{\delta}$, and let $\alpha : V^{x}(\gamma) \to V^{x \gamma}(x)$ be the given isomorphism. Then:

$$\alpha(v^x_{\gamma \delta}) = v^x_{\gamma \gamma}(x) s = \phi(s)(\sigma \gamma)(x) v^x_{\delta} = \alpha(\phi(s)(\sigma \gamma)(x)) v^x_{\gamma \delta}.$$

So

(2.2) 

$$v^x_{\gamma \delta} s = \phi(s) \gamma(x) v^x_{\gamma \delta}.$$

Now, using (2.2), we obtain:

$$\phi(rs)(x) v^x_{\gamma \delta} = v^x_{\gamma \delta} r s = \phi(r)(x) v^x_{\gamma \delta} s = \phi(r)(x) \gamma(x) v^x_{\gamma \delta}.$$
and so

\[ (2.3) \]
\[ \phi(rs) = \phi(r)\phi(s)^\gamma. \]

Then by \[ (2.3) \] we have

\[ \Phi(rs) = \phi(rs)t^\gamma = \phi(r)\phi(s)^\gamma t^\gamma = \phi(r)\phi(s)^\gamma t^\delta = \Phi(r)\Phi(s). \]

Since \( \Phi \) is graded, \( \ker \Phi \) is a graded ideal of \( A \). If \( r \in A \) is homogeneous then

\[ \Phi(r) = 0 \iff \phi(r) = 0 \iff v^r_x r = 0 \text{ for all } x \in X. \]

Let \( \gamma \in \Gamma \). Then

\[ v^\gamma_x r = 0 \text{ for all } x \in X \iff v^\gamma_s(x)r = 0 \text{ for all } x \in X \iff v^\gamma_x r = 0 \text{ for all } x \in X, \]

using the isomorphism between \( M^x(\gamma) \) and \( M^{\sigma^\gamma(x)}. \) So

\[ \Phi(r) = 0 \iff v^\gamma_x r = 0 \text{ for all } x \in X, \gamma \in \Gamma \iff r \in \bigcap_{x \in X} \Ann_A M^x. \]

(\text{The reason we require } X \text{ in the theorem statement to be reduced is that we are constructing } \Phi \text{ from the closed points of } X, \text{ and so effectively from the reduced induced structure on } X.)

\textbf{Remark 2.3.} We need the map \( \sigma \) in Theorem 2.2 to be an antihomomorphism because of the equations:

\[ M^{x^\sigma(x)}(\gamma \delta) \cong M^x(\gamma \delta) = (M^x(\gamma))(\delta) \cong M^{\sigma^\gamma(x)}(\delta) \cong M^{\sigma^\delta(\sigma^\gamma(x))}. \]

\textbf{Remark 2.4.} There is a universal module \( M \) for the family \( \{ M^x \mid x \in X \} \), which is isomorphic as a \( k[X] \)-module to \( \bigoplus_{\gamma \in \Gamma} k[X] v^\gamma \). The module structure is given by

\[ (2.4) \]
\[ v^\gamma_s \theta = \phi(s)^\gamma v^\gamma \delta \]

for \( s \in A_\delta \). If we consider the natural right action of \( A \) on \( M = k[X] \rtimes \Gamma \) then we have \( t^\gamma \cdot s = t^\gamma \Phi(s) = t^\gamma \phi(s) t^{-\delta} = \phi(s)^\gamma t^{-\delta} \) for \( s \in A_\delta \). This agrees with \[ (2.4) \] if we set \( v^\gamma = t^\gamma \), and so \( M \cong k[X] \rtimes \Gamma \).

The theorem for left modules is:

\textbf{Theorem 2.5.} Let \( \Gamma \) be a monoid with identity \( e \) and let \( A \) be a \( \Gamma \)-graded ring. Let \( X \) be a reduced affine scheme that parameterises a set of intermediate series left \( A \)-modules, in the sense that the left module \( N^x \) has a basis \( \{ v^\gamma_e \mid \gamma \in \Gamma \} \) and that there is a \( k \)-linear function \( \phi : A \to k[X] \) so that

\[ rv^\gamma_e = \phi(r)(x) v^\gamma_e \]

for all \( \gamma \in \Gamma, r \in A_\gamma \). Further suppose that shifting defines a group homomorphism \( \sigma : \Gamma \to \Aut_k(X), \gamma \mapsto \sigma^\gamma \)

so that \( N^x(\gamma) \cong N^{\sigma^\gamma(x)} \). Here we require that the isomorphism maps \( v^\gamma_e \mapsto v_e^{\sigma^\gamma(x)}. \)

In this setting the map

\[ \Phi : A \to k[X] \rtimes \Gamma^{\text{op}} \quad r \in A_\gamma \mapsto \phi(r)t^\gamma \]

is a graded antihomomorphism of algebras. Further,

\[ \ker \Phi = \bigcap_{x \in X} \Ann_A N^x. \]

\textbf{Proof:} We repeat the proof above to ensure that the change of notation from right to left is handled correctly. Again, let \( f^\gamma = (\sigma^\gamma)^* f \), so \( \sigma \) defines a homomorphism from \( \Gamma^{\text{op}} \to \Aut_k k[X] \). Let \( r \in A_\gamma, s \in A_\delta \), and let \( \alpha : V^x(\delta) \to V^{\sigma^\delta(x)} \) be the given isomorphism. Then:

\[ \alpha(r v^\gamma_s) = r v^{\sigma^\delta(x)}_e = \phi(r)(\sigma^\delta(x)) v^{\sigma^\delta(x)}_e = \alpha(\phi(r)(\sigma^\delta(x)) v^{\sigma^\delta(x)}_e). \]

So

\[ (2.5) \]
\[ rv^\gamma_\delta = \phi(r)(\sigma^\delta(x)) v^{\sigma^\delta(x)}_\delta. \]

Now, using \[ (2.5) \], we obtain:

\[ \phi(rs)(x) v^{\sigma^\delta(x)}_\delta = r sv^{\sigma^\delta(x)}_e = \phi(s)(x) rv^\gamma_\delta = \phi(s)(x) \phi(r)(\sigma^\delta(x)) v^{\sigma^\delta(x)}_\delta \]
and so
\[(2.6) \quad \phi(rs) = \phi(s)\phi(r)\delta.\]
Then by (2.6) we have
\[\Phi(rs) = \phi(s)\phi(r)^{\delta}t^{\gamma} = \phi(s)\phi(r)^{\delta_\alpha\nu\gamma} = \phi(s)\phi(r)^{t^{\gamma}} = \Phi(s)\Phi(r).\]

The proof of the last statement is identical to the proof in Theorem 2.2. \qed

Remark 2.6. We need the map \(\sigma\) in Theorem 2.3 to be a homomorphism because:
\[N^{\sigma\gamma}(x) = N^{\gamma}(\sigma\delta) = (N^{\sigma\gamma}(\delta))(\gamma) = N^{\sigma\gamma}(x)(\gamma) = N^{\sigma\gamma}(\sigma^{\gamma}(x)).\]
Note also that a graded anti-homomorphism from a \(\Gamma\)-graded algebra should map to a \(\Gamma^{op}\)-graded algebra, as we indeed have.

Remark 2.7. We likewise obtain the universal left module for the \(N^x\) from \(\Phi\). Set \(N = k[X] \times \Gamma^{op}\). The left action induced by \(\Phi\) is \(r \cdot \delta = \delta\Phi(r)\) because \(\Phi\) is an anti-homomorphism, so we get
\[r \cdot t^{\delta} = t^{\delta}\Phi(r) = t^{\delta}\phi(r)t^{\gamma} = \phi(r)^{t^{\delta}t^{\delta_{\alpha\nu}\gamma}} = \phi(r)^{t^{\gamma}}\]
for \(r \in A_\gamma\), which is the structure we expect.

Remark 2.8. Let \(\text{Bir}(X)\) be the group of birational self-maps of \(X\). In the settings above, suppose that shifting defines elements of \(\text{Bir}(X)\), in the sense that \(\sigma\) maps \(\Gamma\) to \(\text{Bir}(X)\). We get a generalization of Theorems 2.2 and 2.3 by replacing \(k[X]\) and \(\text{Aut}(k[X])\) with \(k(X)\) and \(\text{Bir}(X)\), respectively.

3. Intermediate series modules over higher rank Witt algebras

Let \(\Gamma\) be a rank \(n\) \(\mathbb{Z}\)-submodule of \(k\). The rank \(n\) Witt algebra \(W_\Gamma\) (or higher rank Witt algebra if \(n \geq 2\), sometimes called the centerless higher rank Virasoro algebra) is the Lie algebra with \(k\)-basis \(\{e_\nu \mid \nu \in \Gamma\}\) and bracket
\[[e_\mu, e_\nu] = (\nu - \mu)e_{\nu + \mu}\]
for \(\nu, \mu \in \Gamma\). The rank one Witt algebra is the “usual” Witt algebra, which we denote by \(W\).

As \(U(W_\Gamma)\) is \(\Gamma\)-graded one can consider intermediate series modules as in Section 2. They are the standard intermediate series modules of Lie algebras, called also Harish-Chandra modules over \((W_\Gamma, k\nu)=0\); i.e., modules of the form \(\bigoplus_{\gamma \in \Gamma} V_\gamma\), where \(V_\gamma\) is the \(\gamma\)-eigenspace for \(e_0\) and has dimension 1.

The intermediate series \(W_\Gamma\)-modules have been classified in [Su94] Theorem 2.1, generalizing the classification [KSS5] for the Witt algebra. There are three families of indecomposable intermediate series \(W_\Gamma\)-modules:
\[\begin{align*}
V_{(\alpha, \beta)} &= \bigoplus_{\nu \in \Gamma} k\nu, \quad e_\mu v_\nu = (\alpha + \beta \mu + \nu)v_{\mu + \nu}, \\
A_{(\alpha, \beta)} &= \bigoplus_{\nu \in \Gamma} k\nu, \quad e_\mu v_\nu = \begin{cases} 
\nu v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\
(\alpha + \beta \mu)v_\mu & \nu = 0, \\
0 & \mu + \nu = 0,
\end{cases} \\
B_{(\alpha, \beta)} &= \bigoplus_{\nu \in \Gamma} k\nu, \quad e_\mu v_\nu = \begin{cases} 
(\mu + \nu)v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\
0 & \nu = 0, \\
(\alpha + \beta \mu)v_0 & \mu + \nu = 0,
\end{cases}
\end{align*}\]

where \((\alpha, \beta) \in k^2\). Note that \(A_{(\alpha, \beta)}, B_{(\alpha, \beta)}\) are only defined where \((\alpha, \beta) \neq (0, 0)\) and depend up to isomorphism (rescaling of \(v_0\)) only on \([\alpha : \beta] \in \mathbb{P}^1\). We will therefore denote them by \(A_{[\alpha : \beta]}, B_{[\alpha : \beta]}\). Note also that we have \(A_{[1 : 0]} \cong V_{(1, 0)}\) (by \(v_0 \mapsto v_0\) and \(v_\nu \mapsto \nu v_\nu\) when \(\nu \neq 0\)) and \(B_{[1 : 0]} \cong V_{(0, 0)}\) (by \(v_0 \mapsto v_0\) and \(v_\nu \mapsto v_\nu\) when \(\nu \neq 0\)).

Remark 3.1. Note that \(A_{[\alpha : \beta]}\) contains a simple submodule \(\bigoplus_{\nu \neq 0} k\nu\) with a 1-dimensional trivial quotient. On the other hand, \(B_{[\alpha : \beta]}\) has the 1-dimensional trivial submodule \(k\nu_0\), and the quotient is a simple module. This is explained by the isomorphism \(B_{[\alpha : \beta]} \cong A_{[\alpha : \beta]}\), where \(\ell\) denotes the adjoint. (If \(M = \bigoplus_{\gamma \in \Gamma} k\nu_\gamma\) is a left \(\Gamma\)-graded \(W_\Gamma\)-module, the adjoint (or restricted dual) of \(M\) is the left \(\Gamma\)-graded \(W_\Gamma\)-module \(M'\) with \(M'_\gamma = \text{Hom}_k(M_{-\gamma}, k), v'_\gamma = v^*_{-\gamma}\), and \(e_\mu v'_\gamma = -v^*_{-\gamma} e_{\mu}\).)
Remark 3.2. We use a slightly different presentation of the families $A_{[\alpha;\beta]}$, $B_{[\alpha;\beta]}$ than in [Su94]. In loc.cit the last two families are replaced by $\hat{A}(a')$ defined by

$$e_\mu v'_\nu = (\nu + \mu)v'_{\mu + \nu}, \quad e_\mu v_0 = \mu(1 + (\mu + 1)a')v'_{\mu},$$

and by $\hat{B}(a')$ defined by

$$e_\mu v'_\nu = \nu v'_{\mu + \nu}, \quad e_\mu v'_{-\mu} = -\mu(1 + (\mu + 1)a')v_0,$$

for $a' \in k \cup \{\infty\}$. If $a' = \infty$ then $1 + (\mu + 1)a'$ in the above definition is regarded as $\mu + 1$. Note that $\hat{A}(a')$ (resp. $\hat{B}(a')$) is isomorphic to $A_{[1+a';a']} \times (resp. B_{[1+a';a']})$ if $a' \neq \infty$ and to $A_{[1;\infty]} \times B_{[1;\infty]}$ if $a' = \infty$, for $v_\nu = v'_\nu$ if $\nu \neq 0$, and $v_0 = v'_0$.

For the Witt algebra the choice of the basis is the same in [KSS5], however there $a' \in k$ and modules are classified up to inversion: replacing $v_\nu$ by $-v_\nu$.

Let us show how to obtain the intermediate series modules using results of Section 2.

**Proposition 3.3.** Let $\Gamma$ act on $k[a, b]$ as $t^\nu, p(a, b) = p(a + \nu, b)t^\nu$, and let $T := k[a, b] \rtimes \Gamma$. The map $\phi : W_T \to T$, $\phi(e_\mu) = (a + \mu b)t^\mu$, induces an anti-homomorphism $\Phi : U(W_T) \to T$. Consequently, $T$ is a left $U(W)$-module via $e_\mu p(a, b, t^\nu) = (a + \nu + \mu b)p(a, b, t^\nu)$.

**Proof.** Note that $k^2$ parametrises a set of intermediate series modules $N_{(\alpha, \beta)} := V_{(\alpha, \beta)}$ and $e_\mu v_0^{(\alpha, \beta)} = (a + \mu b)((\alpha, \beta))v^{(\alpha, \beta)}_\mu$. Further, $N_{(\alpha, \beta)}(\nu) \cong N_{(\alpha + \nu, \beta)}$ and hence $\sigma_\nu((\alpha, \beta)) = (\alpha + \nu, \beta)$ (using the notation of Section 2). The proposition therefore follows by Theorem 2.5 and Remark 2.7.

**Remark 3.4.** Let $\Gamma = \mathbb{Z}$ and $T = k[a, b] \rtimes \mathbb{Z}$. We may compose the map $\Phi$ of Proposition 3.3 with the canonical anti-automorphism $e_n \mapsto -e_n$ of $U(W)$ to obtain a homomorphism $\Phi' : U(W) \to T, e_n \mapsto (-a - bn)t^n$. We now discuss applications of $\Phi$ to the representation theory of $W_T$. For $p = (\alpha, \beta) \in k^2$ we denote by $I(p)$ the ideal $(a - \alpha, b - \beta)$ in $k[a, b]$. For $q$ infinitely near to $p$, corresponding to $[x : y] \in \mathbb{P}^1$, we denote by $I(q)$ the ideal $(y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2)$.

Let $B = \Phi(U(W_T))$, and note that $B$ is contained in the double idealizer $R = k[a, b] + (I(0, 0)T \cap TI(0, 1))$. From the discussion in the introduction, then, we expect three families of intermediate series $U(W_T)$-modules, one parameterised by $k^2 \setminus \{(0, 0), (0, 1)\}$ and two parameterised by $\mathbb{P}^1$. Note that because $\Phi$ is an anti-homomorphism, right $B$-modules will correspond to left $U(W_T)$-modules.

By construction of $\Phi$ we have $V(\alpha, \beta) \cong T/I(p)T$, considered as a $B$-module. Removing $V(0, 0)$ and $V(0, 1)$ we obtain the two-dimensional family we expect. We next show that we also obtain the two $\mathbb{P}^1$-families $A_{[\alpha;\beta]}$ and $B_{[\alpha;\beta]}$.

**Proposition 3.5.** Let $[x : y] \in \mathbb{P}^1$ and let $I(q) = (ya - xb, a^2, ab, b^2)$ define a point infinitely near to $(0, 0)$. Let

$$P(q) = \frac{k[a, b] + I(0, 0)T}{I(0, 0) + I(q)T}.$$  

Then $A_{[x:y]} \cong P(q)$.

**Proof.** If $w \in k[a, b] + I(0, 0)T$ let $\overline{w}$ be the image of $w$ in $P(q)$. If $x \neq 0$ we choose a basis

$$v_\nu = \begin{cases} aT \nu \neq 0, \\ 1 \nu = 0 \end{cases}$$

for $P(q)$.
Using the anti-homomorphism, we compute for \( \nu \neq 0 \)
\[
e_{\mu}v_{\nu} = \overline{at^\nu(a + b\mu)t^\mu} = \overline{a(a + b\mu + \nu)t^{\nu + \mu}} = \nu a t^{\nu + \mu} = \begin{cases} \nu v_{\nu + \mu} & \nu + \mu \neq 0, \\ 0 & \nu + \mu = 0. \end{cases}
\]
and
\[
e_{\mu}v_{0} = (a + b\mu)t^\mu = (a + \frac{y}{x}a\mu)t^\mu = \left(1 + \frac{y}{x}\mu\right)v_{\mu},
\]
so \( P(q) \cong A_{[x:y]} \) as claimed.

If \( y \neq 0 \) we pick a basis
\[
v_{\nu} = \begin{cases} \frac{b\nu}{\mu} & \nu \neq 0, \\ 1 & \nu = 0, \end{cases}
\]
and obtain \( e_{\mu}.v_{\nu} = \nu v_{\nu + \mu}, e_{\mu}.v_{0} = (\frac{x}{y} + \mu)v_{\mu}, e_{\mu}.v_{-\mu} = 0. \) Thus \( P(q) \cong A_{[x:y]} \) again. \( \square \)

In the next result, note the change of sides from the left modules \( B_{[x:y]} \) to show that \( B \) is not left or right noetherian. This in particular implies that \( U(W_{\Gamma}) \) is not left or right noetherian, which was proved earlier in [SW14, SW15].

\textbf{Proposition 3.6.} Let \([x : y] \in \mathbb{P}^1\) and let \( I(q) = (ya - x(b - 1), a^2, a(b - 1), (b - 1)^2) \) define a point infinitely near to \((0, 1)\). Let
\[
Q(q) = \frac{I(0, 1) \oplus \bigoplus_{0 \neq \nu \in \Gamma} k[a, b]t^\nu}{I(q) \oplus \bigoplus_{0 \neq \nu \in \Gamma} I(0, 1)t^\nu}.
\]
Then \( B_{[x:y]} \cong Q(q) \).

\textbf{Proof.} If \( x \neq 0 \) we choose a basis
\[
v_{\nu} = \begin{cases} \frac{\nu}{\mu} & \nu \neq 0, \\ a & \nu = 0 \end{cases}
\]
for \( Q(q) \). We compute for \( \nu + \mu \neq 0, \nu \neq 0 \)
\[
e_{\mu}v_{\nu} = (a + b\mu + \nu)t^{\nu + \mu} = (\mu + \nu)t^{\nu + \mu} = (\mu + \nu)v_{\nu + \mu}
\]
and
\[
e_{\mu}v_{0} = a(a + b\mu)t^\mu = 0, \quad e_{\mu}.v_{-\mu} = a + b\mu - \mu = \left(1 + \frac{y}{x}\mu\right)v_{0}.
\]
If \( y \neq 0 \) we pick a basis
\[
v_{\nu} = \begin{cases} \frac{\nu}{\mu} & \nu \neq 0, \\ b & \nu = 0 \end{cases}
\]
We get \( e_{\mu}.v_{\nu} = \nu v_{\mu + \nu}, e_{\mu}.v_{0} = 0, e_{\mu}.v_{-\mu} = (\frac{x}{y} + \mu)v_{0} \). \( \square \)

4. Factors of \( U(W_{\Gamma}) \)

In this section we generalise techniques from [SW15] to show that \( B = \Phi(U(W_{\Gamma})) \) is not left or right noetherian. This in particular implies that \( U(W_{\Gamma}) \) is not left or right noetherian, which was proved earlier in [SW14, SW13].

For \( 0 \neq \mu \in \Gamma \), let
\[
p_{\mu} = e_{\mu}e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}.
\]

\textbf{Lemma 4.1.} We have \( \Phi(p_{\mu}) = \mu^2b(1 - b)t^{4\mu} \).

\textbf{Proof.} Let us compute
\[
\Phi(e_{\mu}e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}) = ((a + 3\mu b)(a + \mu b + 3\mu) - (a + 2\mu b)(a + 2\mu b + 2\mu) - \mu(a + 4\mu b)) t^{4\mu} = \mu^2b(1 - b)t^{4\mu}.
\]
\( \square \)

Fix \( 0 \neq \mu \in \Gamma \) and let \( I = B\Phi(p_{\mu})B \).

\textbf{Lemma 4.2.} For all \( \nu \in \Gamma \) we have \( b(1 - b)t^\nu \in I \). In particular, \( I \) does not depend on the choice of \( \mu \). Consequently, \( I = b(1 - b)k[a, b] \times \Gamma \).
Proof. We have
\[ \Phi(e_{\nu-4\mu})b(1-b)t^\mu - b(1-b)t^\mu \Phi(e_{\nu-4\mu}) = (\Phi(e_{\nu-4\mu}) - \Phi(e_{\nu-4\mu}) - 4\mu)b(1-b)t^\mu = -4\mu b(1-b)t^\mu. \]
Thus the first claim follows by Lemma 4.1. Note that I ⊆ b(1-b)k[a, b] × Γ, and as b(1-b) ∈ I and a ∈ B, we have b(1-b)k[a] × Γ ⊆ I. Since also (a+bµ)t^\mu ∈ B, we easily obtain by induction on n that b(1-b)t^\mu k[a] × Γ ⊆ I for all n ≥ 0, and thus the last claim.

Proposition 4.3. The ideal I is not finitely generated as a left or right ideal of B.

Proof. We first compute
\begin{align}
(a + b\nu_1)t^\nu_1 \cdots (a + b\nu_l)t^\nu_l p(a, b)b(1-b)t^\lambda &= (a + b\nu_1) \cdots (a + b\nu_l + \nu_1 + \cdots + \nu_{l-1})p(a + \nu_1 + \cdots + \nu_{l-1} + \nu_l, b)b(1-b)t^{\nu_1+\cdots+\nu_l+\lambda}, \\
p(a, b)b(1-b)t^\lambda (a + b\nu_1)t^\nu_1 \cdots (a + b\nu_l)t^\nu_l &= p(a, b)b(1-b)(a + b\nu_l + \lambda) \cdots (a + b\nu_l + \lambda + \nu_1 + \cdots + \nu_{l-1})t^{\lambda+\nu_1+\cdots+\nu_l}.
\end{align}

Let us assume that I is finitely generated as a right ideal in B. Then there exist µ_1, \ldots, µ_k ∈ Γ such that I = B(I_{µ_1} + \cdots + I_{µ_k}). Let us take µ \neq µ_i, 1 ≤ i ≤ k. It follows from (4.1) that (B(I_{µ_1} + \cdots + I_{µ_k}))_µ is contained in (a, b)b(1-b)t^\nu, a contradiction to Lemma 4.2.

Let us assume now that I is finitely generated as a right ideal in B. Then there exist µ_1, \ldots, µ_k ∈ Γ such that I = (I_{µ_1} + \cdots + I_{µ_k})B. For µ \neq µ_i, 1 ≤ i ≤ k, we obtain from (4.2) that ((I_{µ_1} + \cdots + I_{µ_k})B)_µ is contained in (a + µ, b - 1)b(1-b)t^\nu, which again contradicts Lemma 4.2.

Remark 4.4. Note that the same proof works if Γ is a submonoid of k. Lemma 4.2 yields in this case b(1-b)t^{\nu_\mu} ∈ I, for n ≥ 4. The proof of Proposition 4.3 can then be adapted in an obvious way to apply to this a slightly more general situation. In particular, Φ(U(W_+)) is not noetherian, where W_+ is the subalgebra of W generated by \{e_n : n ≥ 1\}. (This last statement is proved in [SW15].)

We now show that the image B_β of the map φ_β : U(W) → B/(b - β) induced from Φ is noetherian for every β ∈ k. This is an analogue of [SW15 Proposition 2.1].

Lemma 4.5. We have B_0 ≅ k + a(k[a] × Γ), B_1 ≅ k + (k[a] × Γ)a, B_β ≅ k[a] × Γ for β ≠ 0, 1.

Proof. The lemma is obvious for β = 0, 1. Assume therefore that β ≠ 0, 1. Let us compute
\[ (a + \beta \mu)t^\mu(a + \beta \nu)t^\nu - a(a + \beta(a + \nu))t^{\mu+\nu} = (\mu a + \beta\mu(\beta\nu + \mu))t^{\mu+\nu} ∈ B_\beta. \]
Subtracting µ(a + b(µ + ν)t^{\nu+\nu}, we thus have βµν(β - 1)t^{\mu+\nu} ∈ B_β, and hence our claim.

Proposition 4.6. B_β is noetherian for every β ∈ k.

Proof. For β ≠ 0, 1 this follows by [MR01 Theorem 4.5] using Lemma 4.5. Let us note that B_0 ≅ B_1 by conjugation with a. It thus suffices to prove that B_0 is right noetherian and B_1 is left noetherian. We show that B_0 is right noetherian, and following the same argument one can show that B_1 is left noetherian.

We first note that I = a(k[a] × Γ) is a maximal right ideal in C = k[a] × Γ. To see this, let J ⊈ I be a right ideal which contains I. Take an element e = µ_\alpha t^\mu ≠ 0 in J with the minimal number of nonzero coefficients. Since ca = \sum \alpha_µ(a + µ)t^\mu ∈ J and hence \sum \alpha_µ t^\mu ∈ J, the minimality assumption implies that J = k[a] × Γ.

The proposition now follows by [Rob72 Theorem 2.2] using Lemma 4.5.

Remark 4.7. We remark that for any β the modules V(α, β) are all faithful over B_β, and it follows easily that the B_β are primitive. In general, the primitive factors of U(W_Γ) are unknown, even for Γ = Z.
References


