A NOTE ON OMITTING THE REPLACEMENT SCHEMA

A. BUNDY

In [1] Heath considers a formalisation of primitive recursive arithmetic similar to that given in Goodstein [2], in which the replacement schema (Goodstein's $S_{b_2}$) is deduced from special cases of itself, using a double recursive uniqueness rule. The deduction of $S_{b_2}$ given in [1] is, however, incomplete. This is rectified in the present note. The special cases of $S_{b_2}$ taken by Heath are:

(i) $A = B \vdash SA = SB$
(ii) $A = B \vdash x + A = x + B$
(iii) $A = B \vdash A + x = B + x$
(iv) $A = B \vdash x \cdot A = x \cdot B$
(v) $A = B \vdash A \cdot x = B \cdot x$

Remark  In fact either (ii) or (iii) can be omitted since $x + y = y + x$ can be proved without using (ii) or (iii) and then one can be derived from the other.

In order to derive the full $S_{b_2}$, i.e., $A = B \vdash f(A) = f(B)$, for any primitive recursive function $f$, it is necessary to show that the substitution theorem, $x = y \rightarrow f(x) = f(y)$, persists under definition by a primitive recursive schema. Heath shows that it persists under the recursion without parameter, which I shall call $R$,

\[ f(0) = (0), \]
\[ f(Sx) = g(x, f(x)), \]

i.e., that from $x = y \& w = z \rightarrow g(x, w) = g(y, z)$ we can deduce $x = y \rightarrow f(x) = f(y)$. He then quotes a theorem of R. M. Robinson that all primitive recursive functions are generated from $0, x, Sx, x + y$ and $x \cdot y$ by substitution and the recursion $R$. To complete the proof it would be sufficient to show that Robinson's reduction of primitive recursion can be carried out in the restricted primitive recursive arithmetic (i.e., without full $S_{b_2}$). This would involve defining the pairing functions $J(x, y), K(x)$ and $L(x)$ given by Robinson, deriving their main properties, e.g. $L(Sx) \neq 0 \rightarrow K(Sx) = K(x) \& L(Sx) = S(Lx)$, and checking that the substitution theorem is satisfied by them. This part was omitted by Heath, and it is not clear that this programme could be carried out.

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However it is fairly easy to check that the substitution theorem persists under full recursion, by a simple adaptation of Heath’s proof for the recursion scheme R, as the following theorem shows.

**Theorem**

Suppose $f$ is defined by primitive recursion from $h$ and $g$, i.e.,

\[
\begin{align*}
  f(u_0, \ldots, u_n, 0) &= h(u_0, \ldots, u_n) \\
  f(u_0, \ldots, u_n, Sx) &= g(u_0, \ldots, u_n, x, f(u_0, \ldots, u_n, x))
\end{align*}
\]

and the substitution theorem has already been proved for $h$ and $g$, i.e.,

\[
\begin{align*}
  u_0 = v_0 & \land \ldots \land u_n = v_n \rightarrow h(u_0, \ldots, u_n) = h(v_0, \ldots, v_n) \\
  u_0 = v_0 & \land \ldots \land u_n = v_n \rightarrow g(u_0, \ldots, u_n, f(u_0, \ldots, u_n, x)) = g(v_0, \ldots, v_n, f(v_0, \ldots, v_n, x))
\end{align*}
\]

Then the substitution theorem holds for $f$, i.e.,

\[
\begin{align*}
  u_0 = v_0 & \land \ldots \land u_n = v_n \rightarrow f(u_0, \ldots, u_n, 0) = f(v_0, \ldots, v_n, 0)
\end{align*}
\]

**Proof**

**Lemma I**

$u_0 = v_0 \land \ldots \land u_n = v_n \rightarrow f(u_0, \ldots, u_n, x) = f(v_0, \ldots, v_n, x)$

By induction on $x$, prove the basis

\[
\begin{align*}
  u_0 = v_0 & \land \ldots \land u_n = v_n \rightarrow f(u_0, \ldots, u_n, 0) = f(v_0, \ldots, v_n, 0)
\end{align*}
\]

by hypotheses (a) and (c)

and the step

\[
\begin{align*}
  u_0 = v_0 & \land \ldots \land u_n = v_n \rightarrow f(u_0, \ldots, u_n, x) = f(v_0, \ldots, v_n, x)
\end{align*}
\]

by hypotheses (b) and (d).

**Lemma II**

$x = y \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y)$

By double induction on $x$ and $y$, prove

\[
\begin{align*}
  x = 0 & \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, 0)
\end{align*}
\]

and

\[
\begin{align*}
  0 = y & \rightarrow f(u_0, \ldots, u_n, 0) = f(u_0, \ldots, u_n, y)
\end{align*}
\]

by schema $\mathbf{F}$ on $x$ and $y$ respectively. Then use the deduction theorem to prove

\[
\begin{align*}
  (x = y & \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y)) \rightarrow \\
  (Sx = Sy & \rightarrow f(u_0, \ldots, u_n, Sx) = f(u_0, \ldots, u_n, Sy))
\end{align*}
\]

Assume $x = y \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y)$ and $Sx = Sy$ and without using $\mathbf{Sb}$ on any of the variables $u_0, \ldots, u_n, x, y$, deduce, in turn,

\[
\begin{align*}
  x &= y \\
  f(u_0, \ldots, u_n, x) &= f(u_0, \ldots, u_n, y)
\end{align*}
\]

by modus ponens

\[
\begin{align*}
  g(u_0, \ldots, u_n, x, f(u_0, \ldots, u_n, x)) &= g(u_0, \ldots, u_n, y, f(u_0, \ldots, u_n, y))
\end{align*}
\]

by hypothesis (d).
Therefore
\[ f(u_0, \ldots, u_n, Sx) = f(u_0, \ldots, u_n, Sy) \]
by hypothesis (b).

The theorem follows from Lemmas I and II.

REFERENCES


*University of Leicester,
Leicester, England*