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A NOTE ON OMITTING THE REPLACEMENT SCHEMA

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In [1] Heath considers a formalisation of primitive recursive arithmetic similar to that given in Goodstein [2], in which the replacement schema (Goodstein's $\text{Sb}_2$) is deduced from special cases of itself, using a double recursive uniqueness rule. The deduction of $\text{Sb}_2$ given in [1] is, however, incomplete. This is rectified in the present note. The special cases of $\text{Sb}_2$ taken by Heath are:

(i) $A = B \vdash SA = SB$
(ii) $A = B \vdash x + A = x + B$
(iii) $A = B \vdash A + x = B + x$
(iv) $A = B \vdash x - A = x - B$
(v) $A = B \vdash A - x = B - x$

Remark In fact either (ii) or (iii) can be omitted since $x + y = y + x$ can be proved without using (ii) or (iii) and then one can be derived from the other.

In order to derive the full $\text{Sb}_2$, i.e., $A = B \vdash f(A) = f(B)$, for any primitive recursive function $f$, it is necessary to show that the substitution theorem, $x = y \rightarrow f(x) = f(y)$, persists under definition by a primitive recursive schema. Heath shows that it persists under the recursion without parameter, which I shall call $R$,

\[
\begin{align*}
 f(0) &= (0), \\
 f(Sx) &= g(x, f(x)),
\end{align*}
\]

i.e., that from $x = y \& w = z \rightarrow g(x, w) = g(y, z)$ we can deduce $x = y \rightarrow f(x) = f(y)$. He then quotes a theorem of R. M. Robinson that all primitive recursive functions are generated from $0, x, Sx, x + y$ and $x \cdot y$ by substitution and the recursion $R$. To complete the proof it would be sufficient to show that Robinson's reduction of primitive recursion can be carried out in the restricted primitive recursive arithmetic (i.e., without full $\text{Sb}_2$). This would involve defining the pairing functions $J(x, y)$, $K(x)$ and $L(x)$ given by Robinson, deriving their main properties, e.g. $L(Sx) \neq 0 \rightarrow K(Sx) = K(x)$ & $L(Sx) = S(Lx)$, and checking that the substitution theorem is satisfied by them. This part was omitted by Heath, and it is not clear that this programme could be carried out.

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However it is fairly easy to check that the substitution theorem persists under full recursion, by a simple adaptation of Heath's proof for the recursion scheme $R$, as the following theorem shows.

**Theorem** Suppose $f$ is defined by primitive recursion from $h$ and $g$, i.e.,

\[ f(u_0, \ldots, u_n, 0) = h(u_0, \ldots, u_n) \]  
\[ f(u_0, \ldots, u_n, Sx) = g(u_0, \ldots, u_n, x, f(u_0, \ldots, u_n, x)) \]

and the substitution theorem has already been proved for $h$ and $g$, i.e.,

\[ u_0 = v_0 & \ldots & u_n = v_n \rightarrow h(u_0, \ldots, u_n) = h(v_0, \ldots, v_n) \]  
\[ u_0 = v_0 & \ldots & u_{n+2} = v_{n+2} \rightarrow h(u_0, \ldots, u_{n+2}) = h(v_0, \ldots, v_{n+2}) \]

Then the substitution theorem holds for $f$, i.e.,

\[ u_0 = v_0 & \ldots & u_n = v_n \rightarrow f(u_0, \ldots, u_n, 0) = f(v_0, \ldots, v_n, 0) \]

**Proof**

**Lemma I** \( u_0 = v_0 & \ldots & u_n = v_n \rightarrow f(u_0, \ldots, u_n, x) = f(v_0, \ldots, v_n, x) \)

By induction on $x$, prove the basis

\[ u_0 = v_0 & \ldots & u_n = v_n \rightarrow f(u_0, \ldots, u_n, 0) = f(v_0, \ldots, v_n, 0) \]

by hypotheses (a) and (c) and the step

\[ u_0 = v_0 & \ldots & u_n = v_n & \rightarrow f(u_0, \ldots, u_n, x) = f(v_0, \ldots, v_n, x) \]

\[ f(u_0, \ldots, u_n, x) = f(v_0, \ldots, v_n, x) \rightarrow f(u_0, \ldots, u_n, Sx) = f(v_0, \ldots, v_n, Sx) \]

by hypotheses (b) and (d).

**Lemma II** \( x = y \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y) \)

By double induction on $x$ and $y$, prove

\[ x = 0 \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, 0) \]

and

\[ 0 = y \rightarrow f(u_0, \ldots, u_n, 0) = f(u_0, \ldots, u_n, y) \]

by schema $F$ on $x$ and $y$ respectively. Then use the deduction theorem to prove

\( (x = y \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y)) \rightarrow (SX = SY \rightarrow f(u_0, \ldots, u_n, SX) = f(u_0, \ldots, u_n, SY)) \)

Assume \( x = y \rightarrow f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y) \) and \( SX = SY \) and without using $Sb_1$ on any of the variables $u_0, \ldots, u_n, x, y$, deduce, in turn,

\[ x = y \]
\[ f(u_0, \ldots, u_n, x) = f(u_0, \ldots, u_n, y) \]
\[ g(u_0, \ldots, u_n, x, f(u_0, \ldots, u_n, x)) = g(u_0, \ldots, u_n, y, f(u_0, \ldots, u_n, y)) \]

by modus ponens

by hypothesis (d).
Therefore
\[ f(u_0, \ldots, u_n, Sx) = f(u_0, \ldots, u_n, Sy) \]
by hypothesis (b).

The theorem follows from Lemmas I and II.

REFERENCES


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