Some notes on Lie ideals in division rings

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\begin{abstract}
A Lie ideal of a division ring \(A\) is an additive subgroup \(L\) of \(A\) such that the Lie product \([l,a] = la - al\) of any two elements \(l \in L, a \in A\) is in \(L\), or \([l,a] \in L\). The main concern of this paper is to present some dual properties of these two structures on \(A\): considering \(A\) as an associative algebra and as a Lie ring. These properties represent very similar roles of normal subgroups in \(A^*\) (the multiplicative group of unit elements) and (Lie) ideals in \(A\). In particular, when \(A\) is a division ring, we give some properties of Lie ideals of \(A\) which are analogous to similar properties of normal subgroups of \(A^*\).

A derivation on \(A\) is an additive group homomorphism \(d : A \to A\) satisfying \(d(ab) = d(a)b + ad(b)\). A derivation \(d_a : A \to A\) that is defined by \(d_a(b) = ab - ba\), for some fixed \(a \in A\) is called inner derivation; the set \(\{d_a \mid a \in A\}\) of all inner derivations of \(A\) is denoted by \(\text{InnDer}(A)\). In group theory, the normal subgroups of a group \(G\) usually are defined as the subgroups which are invariant under all inner automorphisms of \(G\) (denoted by \(\text{Inn}(G)\)). Equivalently, in the theory of Lie algebras, Lie ideals of an algebra \(A\) are defined as the submodules which are invariant under all inner derivations of \(A\). Let \(Z(A)\) denote the center of \(A\); then we have the following similar isomorphisms: \(\text{Inn}(A^*) \cong A^*/Z(A^*)\) and as a dual version \(\text{InnDer}(A) \cong \overline{A}/Z(A)\) [10, p. 73].

The following two main theorems give some more important signs in identifying a connection between the concepts of normal subgroups and Lie ideals: Let \(F\) be a field, then the Skolem-Noether theorem (in particular) states that if \(A\) is a finite-dimensional central simple \(F\)-algebra, then any \(F\)-automorphism of \(A\) is inner [3, p. 93]. A dual version of this theorem states that if \(A\) is a finite-dimensional central simple \(F\)-algebra, then any \(F\)-linear derivation of \(A\) is inner [3, p. 105].

The other one is the Cartan-Brauer-Hua theorem which states that if \(A\) is a division ring and \(B\) is a subdivision ring of \(A\) such that \(B^*\) is a normal subgroup of \(A^*\), then either \(B = A\) or \(B \subseteq Z(A)\) [6, p. 211]. A dual version of this theorem states that if \(A\) is a division ring and \(B\) is a subdivision ring of \(A\) such that \(B\) is a Lie ideal in \(A\) and \(\text{char} A \neq 2\), then either \(B = A\) or \(B \subseteq Z(A)\) [6, p. 205].

We consider some results about the structure of normal subgroups of a division ring and examine their dual versions in terms of Lie ideals of the division ring. As an example, Akbari et al. [1, 2] proved that “If

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1 Preliminary

Let \(A\) be an associative algebra over a field \(F\) with dimension \([A : F]\). If we replace the usual multiplication \(ab\) of two elements \(a\) and \(b\) of \(A\) by their Lie product \([a, b] = ab - ba\), then at the same time we have a non-associative structure of a Lie ring on \(A\), usually denoted by \(\overline{A}\). A Lie ideal in \(A\) is a regular ideal of \(\overline{A}\), and a Lie ideal in \(A\) is a regular ideal of \(\overline{A}\) with its Lie multiplication; in other words, an additive subgroup \(I\) of \(A\) is called a Lie ideal if for all \(i \in I, a \in A\) we have \([i, a] \in I\). The main concern of this paper is to present some dual properties of these two structures on \(A\): considering \(A\) as an associative algebra and as a Lie ring. These properties represent very similar roles of normal subgroups in \(A^*\) (the multiplicative group of unit elements) and (Lie) ideals in \(A\). In particular, when \(A\) is a division ring, we give some properties of Lie ideals of \(A\) which are analogous to similar properties of normal subgroups of \(A^*\).

A derivation on \(A\) is an additive group homomorphism \(d : A \to A\) satisfying \(d(ab) = d(a)b + ad(b)\). A derivation \(d_a : A \to A\) that is defined by \(d_a(b) = ab - ba\), for some fixed \(a \in A\) is called inner derivation; the set \(\{d_a \mid a \in A\}\) of all inner derivations of \(A\) is denoted by \(\text{InnDer}(A)\). In group theory, the normal subgroups of a group \(G\) usually are defined as the subgroups which are invariant under all inner automorphisms of \(G\) (denoted by \(\text{Inn}(G)\)). Equivalently, in the theory of Lie algebras, Lie ideals of an algebra \(A\) are defined as the submodules which are invariant under all inner derivations of \(A\). Let \(Z(A)\) denote the center of \(A\); then we have the following similar isomorphisms: \(\text{Inn}(A^*) \cong A^*/Z(A^*)\) and as a dual version \(\text{InnDer}(A) \cong \overline{A}/Z(A)\) [10, p. 73].

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We consider some results about the structure of normal subgroups of a division ring and examine their dual versions in terms of Lie ideals of the division ring. As an example, Akbari et al. [1, 2] proved that “If

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A is a finite-dimensional division algebra with center $F$, then any finitely generated normal subgroup of $A^*$ is central. Also, they proved that “If $A$ is an infinite division ring with center $F$ such that $[A : F] < \infty$, then $A^*$ contains no finitely generated maximal subgroups”. Here, as an analogous statement, we show that “If $A$ is a finite-dimensional division algebra with center $F$ such that $\text{char}F \neq 2$, then any finitely generated $Z$-module Lie ideal of $A$ is central”. We also show that “If $A$ is an infinite division ring with center $F$ such that $[A : F] < \infty$, then $A$ contains no finitely generated $Z$-module maximal additive subgroup”. We show that the additive commutator subgroup $[A, A]$ of $A$ is not a finitely generated $Z$-module. To sum up, the applicability of similar arguments we used to prove these dual properties reveals similar roles of these two substructures: the normal subgroups and Lie ideals in division rings [7, 8, 9].

2 Main Results

We begin by recalling the following theorem:

Theorem 1. [4, p. 5] Let $A$ be a division algebra with center $F$ and $\text{char}(A) \neq 2$. Assume that $L$ is a Lie ideal of $A$. Then either $L \subseteq F$ or $[A, A] \subseteq L$.

This result allows us to present our first main result:

Theorem 2. Let $A$ be a division ring which is finite-dimensional over its center $F$ and $\text{char}F \neq 2$. If $A$ contains a non-central Lie ideal which is a finitely generated $Z$-module, then $F$ is finitely generated over its prime subfield $P$.

Proof. Let $L$ be a non-central finitely generated $Z$-module Lie ideal of $A$. By Theorem 1, $[A, A] \subseteq L$. Let $T$ be the $F$-subdivision algebra generated by $L$. Since any non-commutative division ring is generated as a division ring by all of its additive commutators together with its center [6, p. 205], we conclude that $T = A$. Note that since $A$ is a finite-dimensional division ring, $L$ generates $A$ as an algebra, too. If $[A : F] = n$, then $A$ has a faithful matrix representation $\theta$ of degree $n$ [5, p. 82] (usually called the regular representation). Since $L$ is a finitely generated $Z$-module, there exist a finite number $k$ of matrices $M_1, \ldots, M_k$ in $GL_n(F)$ which generate $\theta(L)$ in $M_n(F)$ as a $Z$-module. Let $\Gamma \subseteq F$ be the set of elements of $F$ that appear as entries in the matrices $M_1, \ldots, M_k$. Since $L$ builds $A$ as an algebra, invoking $\theta$ one can see that this set of matrices first builds $\theta(L)$ and then builds $A \cong \theta(A)$ in $M_n(F)$. Since $\theta$ is an embedding, we may consider $A = T \subseteq M_n(P(\gamma)) \subseteq M_n(F)$, where $P(\Gamma)$ is the subfield of $F$ generated by $P \cup \Gamma$. Consequently, for all $a \in F$, its representation $aI$ is in $M_n(P(\Gamma))$, where $I$ is the identity matrix and so $a \in P(\Gamma)$ or $F = P(\Gamma)$.

We need the following lemma to present our next results.

Lemma 3. Let $D$ be a UFD with infinitely many prime ideals and let $T$ be its field of fractions. Let $A$ be a $T$-subalgebra of $M_n(T)$. Then any Lie ideal of $A$ which is finitely generated as a $Z$-module is central.

Proof. For the sake of contradiction, assume that $L$ is a non-central Lie ideal of $A$ which is finitely generated as a $Z$-module. Let $a \in A$ and $l \in L$ be such that $[a, l] \neq 0$. Since $L$ is finitely generated as a $Z$-module, there is a nonzero $d \in D$ such that $L \subseteq M_n(D_{\frac{1}{d}})$. Since $T$ is a field, for any $x \in T$ we have $x[a, l] = [xa, l] \in L$. Hence $x[a, l] = M_n(D_{\frac{1}{d}})$. Since $[a, l] \in M_n(T)$ is a nonzero matrix, one of its entries is nonzero, say $b$. Therefore, $xb \in D_{\frac{1}{d}}$ for all $x \in T$, which is a contradiction, for if $p$ is a prime element such that $p \nmid d$, then $b/p^n \notin D_{\frac{1}{d}}$ for enough large positive integer $n$.

Theorem 4. Let $A$ be a finite-dimensional division algebra with center $F$ such that $\text{char}F \neq 2$. Then any finitely generated $Z$-module Lie ideal of $A$ is central.

Proof. By Theorem 2, $F$ is finitely generated over its prime subfield $P$. Hence we may write $F$ as a finite extension of a purely transcendental extension $P(x_1, \ldots, x_d)$ of $P$, where $d$ is the transcendence degree of $F$ over $P$. We consider two cases:

Case 1. $d = 0$. If $P = F_p$, then $F$ is a finite field. Hence by the Wedderburn’s Little Theorem, $A$ is commutative [6, p. 203]. If $P = \mathbb{Q}$, then $[F : \mathbb{Q}] < \infty$ allows us to view $A = M_n(\mathbb{Q})$ via the regular representation. Now, using the above lemma, we are done.

Case 2. $d > 0$. Then $P[x_1, \ldots, x_d]$ is a UFD with infinitely many prime ideals. Let $T$ be the field of fractions of $D$. Since $[F : T] < \infty$, again we may view $A = M_n(T)$ via the regular representation. Now, applying the above Lemma completes the proof

The following is our main result:

Corollary 5. Let $A$ be a non-commutative division algebra of finite dimension over its center $F$ and $\text{char}F \neq 2$. Then the additive commutator subgroup $[A, A]$ of $A$ is not finitely generated as a $Z$-module.
The Lie ideal structure we have considered above really is a kind of additive subgroup of algebras. In what follows, we turn our attention to another kind of additive subgroups. By a maximal additive subgroup of an algebra, we mean an additive subgroup which is maximal under inclusion among proper ones. Clearly, by a maximal Lie ideal, we mean a Lie ideal which is maximal under inclusion among Lie ideals.

**Corollary 6.** Let $A$ be a division ring with center $F$ and $\text{char} F \neq 2$. Assume that $L$ is a proper maximal additive subgroup of $A$ containing $F$. If the additive group index $[L : F]$ of $L$ over $F$ is finite, then $A = F$.

**Proof.** First, consider the case $[A : F] < \infty$ and let $x_1, \ldots, x_t$ be the representations of the finite number of cosets of $F$ in $L$, so $L = (F + x_1) \cup \cdots \cup (F + x_t)$. We have $L = F + \langle \{x_1, \ldots, x_t\} \rangle$, where $\langle \{x_1, \ldots, x_t\} \rangle$ is the additive subgroup generated by $x_1, \ldots, x_t$ in $A$. Suppose that $x \in A \setminus L$. By maximality of $L$, we obtain $A = F + \{x, x_1, \ldots, x_t\}$, where $\{x, x_1, \ldots, x_t\}$ is the latter case implies that $[A : A] = [H, H] \subseteq F$ or $A = F$ as desired. Now, consider the case $[A : F] = \infty$. As in the above case, let $L = (F + x_1) \cup \cdots \cup (F + x_t)$ and take $x \in A \setminus L$. Let $V$ be the vector space generated by the set $\{1, x_1, \ldots, x_t, x\}$ over $F$. Clearly $[V : F] < \infty$ and $L \subseteq V$. Now, maximality of $L$ implies that $V = A$, a contradiction. This completes the proof.

We continue our study with the following two lemmas:

**Lemma 7.** Let $A$ be an $F$-algebra and $L$ be a maximal Lie ideal of $A$. Then

(i) $L$ contains either $F$ or $[A, A]$.

(ii) If $A$ is a division ring, then either $A = F(L)$ or $L \setminus \{0\}$ is the multiplicative group $F(L) \setminus \{0\}$, where $F(L)$ is the division ring generated by $F \cup L$.

**Proof.** (i) Assume that $L$ does not contain $F$. By maximality of $L$ and since $F + L$ is a Lie ideal containing $L$, we have $A = F + L$. Consequently, we have $[A, A] = [L, L] \subseteq L$.

(ii) Consider the division ring $F(L)$ generated by $L$ and $F$. By maximality of $L$ and since $F(L)$ is a Lie ideal containing $L$, we have either $A = F(L)$ or $L = F(L)$. In the latter case, we obtain $F(L)^n = F(L) \setminus \{0\} = L \setminus \{0\}$ is a multiplicative group.

**Lemma 8.** Let $A$ be a division ring with center $F$ and assume that $L$ is a maximal Lie ideal of $A$. Then either the multiplicative center of $L$ is equal to $F \cap L$ or $L$ is a maximal division subring of $A$.

**Proof.** By first part of the previous lemma, either $F \subseteq L$ or $[A, A] \subseteq L$. If $[A, A] \subseteq L$, then $Z(L) = C_L(L) \subseteq C_A([A, A]) = F$, where the latter inclusion is by [6, p. 205]. In other words, $Z(L) \subseteq F \cap L$ and so $Z(L) = F \cap L$. If $F \subseteq L$ and $[A, A]$ is not contained in $L$, then consider the division ring $F(L)$. Since $F(L)$ is a Lie ideal containing $L$, we have either $A = F(L)$ or $L = F(L)$ by the maximality of $L$ in $A$. In the first case, it is easily checked that $Z(L) = Z(A) = F \cap L$. Otherwise, $L = F(L)$ which means that $L$ is a maximal division subring of $A$.

Now, we can show that Theorem 4 has an analogous statement which applies to maximal additive subgroups.

**Theorem 9.** Let $A$ be a non-commutative division ring with center $F$. Then $A$ contains no finitely generated $Z$-module maximal additive subgroup.

**Proof.** Assume that $L$ is a maximal additive subgroup of $A$ that is finitely generated as a $Z$-module. For each element $x \in A \setminus L$, we have $A = L + Zx$ which means that $A$ is a finitely generated $Z$-module and this is impossible: If $\text{char} F \neq 0$, then $A$ would be finite and so commutative. If $\text{char} F = 0$, this condition makes $A$ to be finite-dimensional over the center. So $A$ is a finitely generated $Z$-module Lie ideal of a finite-dimensional division ring $A$ which by Theorem 4 is contained in $F$ and thus is commutative, contradicting our assumption that $A$ is non-commutative.

**Theorem 10.** Let $A$ be a division algebra algebraic over its center $F$ with $\text{char} F \neq 2$ and let $n$ be a natural number. Assume that $L$ is a maximal Lie ideal of $M_n(A)$. If $L$ is finite, then $A = F$.

**Proof.** Let $I \in M_n(A)$ be the identity matrix. By Lemma 7(i), either $FI \subseteq L$ or $[M_n(A), M_n(A)] \subseteq L$. The latter case implies that $[A, A]$ is finite, so $A = F$ by Corollary 6. If $FI \subseteq L$, then $F$ is finite and so $A$ a division algebra algebraic over a finite field would be commutative and thus $A = F$. \qed
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