Matching Schur Complement Approximations for Certain Saddle-Point Systems

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Fast interior point solvers for $H^1$-regularized PDE-constrained optimization problems

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We consider Newton systems arising from the interior point solution of PDE-constrained optimization problems. In particular, we examine problems where the control variable is regularized by an $H^1$-norm within the cost functional. We present preconditioned iterative methods for the resulting matrix systems, and justify the potency of our approach through numerical experiments.

1 Problem statement

We consider linear, time-independent PDE-constrained optimization problems with additional bound constraints, of the form:

$$\min_{y,u} \frac{1}{2} \|y - \hat{y}\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\Omega)}^2$$

s.t. $
Dy = u,$ in $\Omega,$
$y = f,$ on $\partial\Omega,$
$y_a \leq y \leq y_b,$ a.e. in $\Omega,$
$u_a \leq u \leq u_b,$ a.e. in $\Omega,$

where $y$ and $u$ denote state and control variables which we wish to determine, $\hat{y}$ is a given desired state, $\beta > 0$ a regularization parameter, $D$ some differential operator, and $y_a,$ $y_b,$ $u_a,$ $u_b$ prescribed bound constraints on the state and control variables. The problem is solved in a domain $\Omega \subset \mathbb{R}^d,$ $d \in \{2,3\},$ with boundary $\partial\Omega.$ We highlight that the regularization term corresponding to the control is $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$ — problems with an $L_2$-norm regularization term for the control are considered in [2], as are time-dependent variants of such problems.

Applying an interior point method and a discretize-then-optimize approach, as in [2], leads to a discrete Lagrangian of the following form:

$$L(y, \bar{u}, \bar{p}) = \frac{1}{2} \bar{y}^T M \bar{y} - \bar{y}_d^T \hat{y} + \frac{\beta}{2} \bar{u}^T (M + K) \bar{u} + \bar{p}^T (A \bar{y} - M \bar{u} - \bar{f})$$

$$- \mu \sum_j \log (y_j - y_{a,j}) - \mu \sum_j \log (y_{b,j} - y_j) - \mu \sum_j \log (u_j - u_{a,j}) - \mu \sum_j \log (u_{b,j} - u_j),$$

of which we wish to find the stationary point. Here $\bar{p}$ and $\bar{y}$ correspond to the discretized adjoint variable and desired state, $y_{a,j}, y_{b,j}, u_{a,j}$, and $u_{b,j}$ represent the values of $y,$ $y_a,$ $y_b,$ $u,$ $u_a,$ and $u_b$ at the $j$-th finite element node, $K$ and $M$ are the finite element stiffness and mass matrices, $A$ is the finite element matrix related to the PDE operator $D$, and $\mu$ is the chosen barrier parameter within the interior point method.

Applying Newton iteration (with Newton steps $\bar{s}_y, \bar{s}_u, \bar{s}_p,$ and previous iterates $\bar{y}^*, \bar{u}^*, \bar{p}^*$) to the resulting first-order optimality conditions leads to matrix systems of the form

$$\begin{bmatrix} M + D_y & 0 & A^T \\ 0 & \beta (M + K) + D_u & -M \\ A & -M & 0 \end{bmatrix} \begin{bmatrix} \bar{s}_y \\ \bar{s}_u \\ \bar{s}_p \end{bmatrix} = \begin{bmatrix} \mu (Y - Y_a) - \bar{f} - \mu (Y_b - Y)^{-1} \bar{e} + \bar{y}_d - M \bar{y}^* + A^T \bar{p}^* \\ \mu (U - U_a) - \bar{f} - \mu (U_b - U)^{-1} \bar{e} - (M + K) \bar{u}^* - \bar{y}^* \\ \bar{f} - \bar{y}^* + M \bar{u}^* \end{bmatrix},$$

where

$$D_y = (Y - Y_a)^{-1} Z_{y,a} + (Y_b - Y)^{-1} Z_{y,b}, \quad D_u = (U - U_a)^{-1} Z_{u,a} + (U_b - U)^{-1} Z_{u,b}.$$

Here, $Y,$ $U,$ $Y_a,$ $Y_b,$ $U_a,$ $U_b$ are diagonal matrices containing the entries of $y,$ $u$ (from the previous Newton iteration), $\bar{y}_d,$ $\bar{y}_a,$ $\bar{y}_b,$ $\bar{u}_a,$ $\bar{u}_b$ at each finite element node, and $Z_{y,a},$ $Z_{y,b},$ $Z_{u,a},$ $Z_{u,b}$ contain entries of the form $\frac{\mu}{y - y_a},$ $\frac{\mu}{y_b - y},$ $\frac{\mu}{u - u_a},$ $\frac{\mu}{u_b - u}$, respectively.

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2 Preconditioning

We now note that the Newton system is of saddle point form (see [1] for a survey of such systems). Using the justification provided in [3,4], we consider the block triangular preconditioners

$$
\mathcal{P}_1 = \begin{bmatrix}
M + D_u & 0 & 0 \\
0 & \beta(M + K) + D_u & 0 \\
A & -M & -S_1
\end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix}
-S_2 & 0 & A^T \\
0 & \beta(M + K) + D_u & -M \\
0 & -M & 0
\end{bmatrix},
$$

where $S_1$ and $S_2$ are derived using ‘matching strategies’, as follows:

$$S_1 = (A + \hat{M})(M + D_u)^{-1} (A + \hat{M})^\top,$$

$$S_2 = -(A + \hat{M})^\top \hat{M}^{-1}(\beta(M + K) + D_u)\hat{M}^{-1}(A + \hat{M}),$$

with $\hat{M} = M [\text{diag}(\beta(M + K) + D_u)]^{-1/2} [\text{diag}(M + D_u)]^{1/2}$. Eigenvalue analysis concerning similar preconditioners for interior point methods can be found in [2].

In practice it is sensible to use multigrid methods to apply the inverse of the Schur complement inverses $S_1^{-1}$ and $S_2^{-1}$. We apply both preconditioners $\mathcal{P}_1$ and $\mathcal{P}_2$ within the GMRES algorithm.

3 Numerical results

We now implement an interior point method, coupled with our GMRES solver (with preconditioners $\mathcal{P}_1$ and $\mathcal{P}_2$), for a particular test problem. In more detail, we set $D = -\nabla^2, \hat{\beta} = e^{-64(x_1-0.5)^2+(x_2-0.5)^2}$, where $\mathbf{x} = [x_1, x_2]^\top \in \Omega = [0, 1]^2$, with state and control constraints prescribed based on the physical properties of the problem. The iterative solvers are run to a tolerance of $10^{-8}$, with the outer (interior point) solver set to a tolerance of $10^{-6}$. We solve for a range of step-sizes $h$ and values of $\beta$, using MATLAB R2015a, on a quad-core 3.2 GHz processor. We observe that the number of interior point iterations, as well as the GMRES iteration count, behave robustly for a wide range of parameters. We therefore conclude that our preconditioning strategies are highly effective for parameter consideration.

**Table 1:** Number of interior point (Newton) iterations required to achieve convergence (blue, left), and average number of GMRES steps per interior point iteration before a relative preconditioned residual norm of $10^{-8}$ is achieved (black, right), for the test problem considered.

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References


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