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THE DIVISIBILITY OF $a^n - b^n$ BY POWERS OF $n$

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Abstract

For given integers $a, b$ and $j \geq 1$ we determine the set $R_{a,b}^{(j)}$ of integers $n$ for which $a^n - b^n$ is divisible by $n^j$. For $j = 1, 2$, this set is usually infinite; we determine explicitly the exceptional cases for which $a, b$ the set $R_{a,b}^{(j)} (j = 1, 2)$ is finite. For $j = 2$, we use Zsigmondy’s Theorem for this. For $j \geq 3$ and $\gcd(a, b) = 1$, $R_{a,b}^{(j)}$ is probably always finite; this seems difficult to prove, however.

We also show that determination of the set of integers $n$ for which $a^n + b^n$ is divisible by $n^j$ can be reduced to that of $R_{a,b}^{(j)}$.

1. Introduction

Let $a, b$ and $j$ be fixed integers, with $j \geq 1$. The aim of this paper is to find the set $R_{a,b}^{(j)}$ of all positive integers $n$ such that $n^j$ divides $a^n - b^n$. For $j = 1, 2, \ldots$, these sets are clearly nested, with common intersection $\{1\}$. Our first results (Theorems 1 and 2) describe this set in the case that $\gcd(a, b) = 1$. In Section 4 we describe (Theorem 15) the set in the general situation where $\gcd(a, b)$ is unrestricted.

Theorem 1. Suppose that $\gcd(a, b) = 1$. Then the elements of the set $R_{a,b}^{(1)}$ consist of those integers $n$ whose prime factorization can be written in the form

$$n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \quad (p_1 < p_2 < \cdots < p_r, \ all \ k_i \geq 1),$$

where $p_i \mid (a^{n_i} - b^{n_i})$ ($i = 1, \ldots, r$), with $n_1 = 1$ and $n_i = p_1^{k_1} p_2^{k_2} \ldots p_{i-1}^{k_{i-1}}$ ($i = 2, \ldots, r$).

In this theorem, the $k_i$ are arbitrary positive integers. This result is a more explicit version of that proved in Győry [5], where it was shown that if $a - b > 1$ then for any positive integer $r$ the number of elements of $R_{a,b}^{(1)}$ having $r$ prime factors is infinite. The result is also essentially contained in [11], which described...
the indices \( n \) for which the generalized Fibonacci numbers \( u_n \) are divisible by \( n \). However, we present a self-contained proof in this paper.

On the other hand, for \( j \geq 2 \), the exponents \( k_i \) are more restricted.

**Theorem 2.** Suppose that \( \gcd(a, b) = 1 \), and \( j \geq 2 \). Then the elements of the set \( R_{a,b}^{(j)} \) consist of those integers \( n \) whose prime factorization can be written in the form (1), where

\[
p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \lcm(a - b, a + b) & \text{if } p_1 = 2, \end{cases}
\]

and \( p_i^{(j-1)k_i} | a^{n_i} - b^{n_i} \), with \( n_i = p_1^{k_1} p_2^{k_2} \ldots p_i^{k_i-1} (i = 2, \ldots, r) \).

Again, the result was essentially contained in [5], where it was proved that for \( a - b > 1 \) and for any given \( r \), there exists an \( n \in R_{a,b}^{(j)} \) with \( r \) distinct prime factors. Further, the number of these \( n \) is finite, and all of them can be determined. The paper [5] was stimulated by a problem from the 31st International Mathematical Olympiad, which asked for all those positive integers \( n > 1 \) for which \( 2^n + 1 \) was divisible by \( n^2 \). (For the answer, see [5], or Theorem 16.)

Thus we see that construction of \( n \in R_{a,b}^{(j)} \) depends upon finding a prime \( p_i \) not used previously with \( a^{n_i} - b^{n_i} \) being divisible by \( p_i^{j-1} \). This presents no problem for \( j = 2 \), so that \( R_{a,b}^{(2)} \), as well as \( R_{a,b}^{(1)} \), are usually infinite. See Section 5 for details, including the exceptional cases when they are finite. However, for \( j \geq 3 \) the condition \( p_i^{j-1} | a^{n_i} - b^{n_i} \) is only rarely satisfied. This suggests strongly that in this case \( R_{a,b}^{(j)} \) is always finite for \( \gcd(a, b) = 1 \). This seems very difficult to prove, even assuming the ABC Conjecture. A result of Ribenboim and Walsh [10] implies that, under ABC, the powerful part of \( a^n - b^n \) cannot often be large. But this is not strong enough for what is needed here. On the other hand, \( R_{a,b}^{(j)} (j \geq 3) \) can be made arbitrarily large by choosing \( a \) and \( b \) such that \( a - b \) is a powerful number. For instance, choosing \( a = 1 + (q_1 q_2 \ldots q_s)^{j-1} \) and \( b = 1 \), where \( q_1, q_2, \ldots, q_s \) are distinct primes, then \( R_{a,b}^{(j)} \) contains the \( 2^n \) numbers \( q_1^{\varepsilon_1} q_2^{\varepsilon_2} \ldots q_s^{\varepsilon_s} \) where the \( \varepsilon_i \) are \( 0 \) or \( 1 \). See Example 6 in Section 7.

In the next section we give preliminary results needed for the proof of the theorems. We prove them in Section 3. In Section 4 we describe (Theorem 15) \( R_{a,b}^{(j)} \) where \( \gcd(a, b) \) is unrestricted. In Section 5 we find all \( a, b \) for which \( R_{a,b}^{(2)} \) is finite (Theorem 16). In Section 6 we discuss the divisibility of \( a^n + b^n \) by powers of \( n \). In Section 7 we give some examples, and make some final remarks in Section 8.
2. Preliminary Results

We first prove a version of Fermat’s Little Theorem that gives a little bit more information in the case $x \equiv 1 \pmod{p}$.

**Lemma 3.** For $x \in \mathbb{Z}$ and $p$ an odd prime we have

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv \begin{cases} p \pmod{p^2} & \text{if } x \equiv 1 \pmod{p}; \\ 1 \pmod{p} & \text{otherwise}. \end{cases} \quad (2)$$

**Proof.** If $x \equiv 1 \pmod{p}$, say $x = 1 + kp$, then $x^j \equiv 1 + jkp \pmod{p^2}$, so that

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv p + kp \sum_{j=0}^{p-1} j \equiv p \pmod{p^2}. \quad (3)$$

Otherwise

$$x(x-1)(x^{p-2} + \cdots + x + 1) = x^p - x \equiv 0 \pmod{p}, \quad (4)$$

so that for $x \not\equiv 1 \pmod{p}$ we have $x(x^{p-2} + \cdots + x + 1) \equiv 0 \pmod{p}$, and hence

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv x(x^{p-2} + \cdots + x + 1) + 1 \equiv 1 \pmod{p}. \quad (5)$$

□

The following is a result of Birkhoff and Vandiver [2, Theorem III]. It is also special case of Lucas [9, p. 210], as corrected for $p = 2$ by Carmichael [3, Theorem X].

**Lemma 4.** Let $\gcd(a, b) = 1$ and $p$ be prime with $p \mid (a-b)$. Define $t > 0$ by $p^t \mid (a-b)$ for $p > 2$ and $2^t \mid \text{lcm}(a-b, a+b)$ if $p = 2$. Then for $\ell > 0$

$$p^{t+\ell} \parallel (a^{p^\ell} - b^{p^\ell}). \quad (6)$$

On the other hand, if $p \nmid a-b$ then for $\ell \geq 0$

$$p \nmid a^{p^\ell} - b^{p^\ell}. \quad (7)$$

**Proof.** Put $x = a/b$. First suppose that $p$ is odd and $p^t \mid (a-b)$ for some $t > 0$. Then as $\gcd(a, b) = 1$, $b$ is not divisible by $p$, and we have $x \equiv 1 \pmod{p^t}$. Then from

$$a^p - b^p = (a-b)b^{p-1}(x^{p-1} + x^{p-2} + \cdots + x + 1) \quad (8)$$
we have by Lemma 3 that  \( p^{t+1} \|(a^p - b^p) \). Applying this result \( \ell \) times, we obtain (6).

For \( p = 2 \), we have  \( p^{t+1} \|a^2 - b^2 \) and from \( a^2 \equiv b^2 \equiv 1 \pmod{8} \), we obtain \( 2^1 \|(a^4 - b^4) \), and so  \( p^{t+2} \|(a^4 - b^4) \). An easy induction then gives the required result.

Now suppose that  \( p \nmid (a - b) \). Since  \( \gcd(a, b) = 1 \), (7) clearly holds if  \( p \mid a \) or  \( p \mid b \), as must happen for  \( p = 2 \). So we can assume that  \( p \) is odd and  \( p \nmid b \). Then  \( x \equiv 1 \pmod{p} \) so that, by Lemma 3 and (8), we have \( p \nmid (a^p - b^p) \). Applying this argument \( \ell \) times, we obtain (7).

For  \( n \in R_a^{(j)} \), we now define the set  \( P_{a,b}^{(j)}(n) \) to be the set of all prime powers  \( p^k \) for which  \( np^k \in R_a^{(j)} \). Our next result describes this set precisely. (Compare with \([11, \text{Theorem ~1(a)}]\)).

**Proposition 5.** Suppose that  \( j \geq 1 \),  \( \gcd(a, b) = 1 \),  \( n \in R_a^{(j)} \) and

\[
a^n - b^n = 2^{e_2} \prod_{p > 2} p^{e_p}, \quad n = \prod_p p^{k_p} \tag{9}
\]

and define  \( e_2 \) by  \( 2^{e_2} \| \gcd(a^n - b^n, a^n + b^n) \). Then

\[
P^{(1)}(n) = \bigcup_{p|a^n - b^n} \{p^k, k \in \mathbb{N}\}, \tag{10}
\]

and for  \( j \geq 2 \)

\[
P_{a,b}^{(j)}(n) = \bigcup_{p: p^{j-1} | a^n - b^n} \left\{p^k : 1 \leq k \leq \left\lfloor \frac{e_p - jk_p}{j-1} \right\rfloor \right\}. \tag{11}
\]

Note that  \( e_2 \) is never 1. Consequently, if  \( 2m \in R_a^{(2)} \), where  \( m \) is odd, then  \( 4m \in R_a^{(2)} \). Also,  \( 2 \in R_a^{(j)} \) for  \( j \leq 3 \) when  \( a - b \) is even.

**Proof.** Taking  \( n \in R_a^{(j)} \) we have, from (9) and the definition of  \( e_2 \), that  \( jk_p \leq e_p \) for all primes  \( p \). Hence, applying Lemma 4 with  \( a, b \) replaced by  \( a^n, b^n \) we have for  \( p \) dividing  \( a^n - b^n \) that for  \( \ell > 0 \)

\[
p^{e_p + \ell} \|(a^{np^\ell} - b^{np^\ell}). \tag{12}
\]

So  \((np^\ell)^j | (a^{np^\ell} - b^{np^\ell})\) is equivalent to  \( j(k_p + \ell) \leq e_p + \ell \), or  \( (j - 1)\ell \leq e_p - jk_p \). Thus we obtain (10) for  \( j \geq 2 \), with  \( \ell \) unrestricted for  \( j = 1 \), giving (10).

On the other hand, if  \( p \nmid (a^n - b^n) \), then by Lemma 4 again,  \( p^\ell \nmid (a^{np^\ell} - b^{np^\ell}) \), so that certainly  \((np^\ell)^j \nmid (a^{np^\ell} - b^{np^\ell}). \)
We now recall some facts about the order function ord. For \( m \) an integer greater than 1 and \( x \) an integer prime to \( m \), we define \( \text{ord}_m(x) \), the order of \( x \) modulo \( m \), to be the least positive integer \( h \) such that \( x^h \equiv 1 \pmod{m} \). The next three lemmas, containing standard material on the ord function, are included for completeness.

**Lemma 6.** For \( x \in \mathbb{N} \) and prime to \( m \), we have \( m | (x^n - 1) \) if and only if \( \text{ord}_m(x) | n \).

**Proof.** Let \( \text{ord}_m(x) = h \), and assume that \( m | (x^n - 1) \). Then as \( m | (x^h - 1) \), also \( m | (x^{\text{gcd}(h,n)} - 1) \). By the minimality of \( h \), \( \text{gcd}(h,n) = h \), i.e., \( h | n \). Conversely, if \( h | n \) then \( (x^h - 1) | (x^n - 1) \), so that \( m | (x^n - 1) \).

**Corollary 7.** Let \( j \geq 1 \). We have \( n^j | (x^n - 1) \) if and only if \( \text{gcd}(x,n) = 1 \) and \( \text{ord}_{n^j}(x) | n \).

**Lemma 8.** For \( m = \prod_p p^{l_p} \) and \( x \in \mathbb{N} \) and prime to \( m \) we have

\[
\text{ord}_m(x) = \text{lcm}_p \text{ord}_{p^{l_p}}(x).
\]

**Proof.** Put \( h_p = \text{ord}_{p^{l_p}}(x) \), \( h = \text{ord}_m(x) \) and \( h' = \text{lcm}_p h_p \). Then by Lemma 6 we have \( p^{l_p} | (x^{h'} - 1) \) for all \( p \), and hence \( m | (x^{h'} - 1) \). Hence \( h | h' \). On the other hand, as \( p^{l_p} | n \) and \( m | (x^h - 1) \), we have \( p^{l_p} | (x^h - 1) \), and so \( h_p | h \), by Lemma 6. Hence \( h' = \text{lcm}_p h_p | h \).

Now put \( p_\star = \text{ord}_p(x) \), and define \( t > 0 \) by \( p^t \| (x^{p_\star} - 1) \).

**Lemma 9.** For \( \gcd(x,n) = 1 \) and \( \ell > 0 \) we have \( p_\star | (p - 1) \) and \( \text{ord}_{p^\ell}(x) = p^{\text{max}(\ell - 1,0)} p_\star \).

**Proof.** Since \( p | (x^{p-1} - 1) \), we have \( p_\star | (p - 1) \), by Lemma 6. Also, from \( p^\ell | (x^{\text{ord}_{p^\ell}(x)} - 1) \) we have \( p | (x^{\text{ord}_{p^\ell}(x)} - 1) \), and so, by Lemma 6 again, \( p_\star = \text{ord}_p(x) | \text{ord}_{p^\ell}(x) \). Further, if \( \ell \leq t \) then from \( p^\ell | (x^{p_\star} - 1) \) we have by Lemma 6 that \( \text{ord}_{p^\ell}(x) | p_\star \), so \( \text{ord}_{p^\ell}(x) = p_\star \). Further, by Lemma 4 for \( u \geq t \)

\[
P^u \| ((x^{p^u - t p_\star}) - 1),
\]

so that, taking \( u = \ell \geq t \) and using Lemma 6, \( \text{ord}_{p^\ell}(x) | p^{\ell - t} p_\star \). Also, if \( t \leq u < \ell \), then, from (14), \( x^{p^u - t p_\star} \not\equiv 1 \pmod{p^\ell} \). Hence \( \text{ord}_{p^\ell}(x) = p^{\ell - t} p_\star \) for \( \ell \geq t \).
Corollary 10. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and \( x \in \mathbb{N} \) prime to \( n \) we have \( n^j \mid x^n - 1 \) if and only if \( \gcd(x, n) = 1 \) and

\[
\text{lcm}_p p^{k'_p} \mid \prod_p p^{k_p}.
\]  

(15)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

Note that \( p_* \), \( k'_p \) and \( t_p \) in general depend on \( x \) and \( j \) as well as on \( p \).

What we actually need in our situation is the following variant of Corollary 10.

Corollary 11. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and integers \( a, b \) with \( \gcd(a, b) = 1 \) we have \( n^j \mid a^n - b^n \) if and only if \( \gcd(n, a) = \gcd(n, b) = 1 \) and

\[
\text{lcm}_p p^{k'_p} \mid \prod_p p^{k_p}.
\]  

(16)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

In this corollary, the \( x \) used to define \( p_* \) and \( t = t_p \) (see after Lemma 8) is chosen to satisfy \( bx \equiv a \pmod{n^j} \). The result is then easily deduced from Corollary 10.

By contrast with Proposition 5, our next proposition allows us to divide an element \( n \in R^{(j)}_{a,b} \) by a prime, and remain within \( R^{(j)}_{a,b} \).

Proposition 12. Let \( n \in R^{(j)}_{a,b} \) with \( n > 1 \), and suppose that \( p_{\text{max}} \) is the largest prime factor of \( n \). Then \( n/p_{\text{max}} \in R^{(j)}_{a,b} \).

Proof. Suppose \( n \in R^{(j)}_{a,b} \), so that (15) holds, with \( x = a/b \), and put \( q = p_{\text{max}} \). Then, since for every \( p \) all prime factors of \( p_* \) are less than \( p \), the only possible term on the left-hand side that divides \( q^{k_q} \) on the right-hand side is the term \( q^{k_q} \). Now reducing \( k_q \) by 1 will reduce \( k'_q \) by at least 1, unless it is already 0, when it does not change. In either case (15) will still hold with \( n \) replaced by \( n/q \), and so \( n/q \in R^{(j)}_{a,b} \). \( \square \)

Various versions and special cases of Proposition 12 for \( j = 1 \) have been known for some time, in the more general setting of Lucas sequences, due to Somer [12, Theorem 5(iv)], Jarden [7, Theorem E], Hoggatt and Bergum [6], Walsh [14], André-Jeannin [1] and others. See also Smyth [11, Theorem 3].

In order to work out for which \( a, b \) the set \( R^{(j)}_{a,b} \) is finite, we need the following classical result. Recall that \( a^n - b^n \) is said to have a primitive prime divisor \( p \) if the prime \( p \) divides \( a^n - b^n \) but does not divide \( a^k - b^k \) for any \( k \) with \( 1 \leq k < n \).
**Theorem 13** (Zsigmondy [15]). Suppose that $a$ and $b$ are nonzero coprime integers with $a > b$ and $a + b > 0$. Then, except when

- $n = 2$ and $a + b$ is a power of 2
  or
- $n = 3$, $a = 2$, $b = -1$
  or
- $n = 6$, $a = 2$, $b = 1$,

$a^n - b^n$ has a primitive prime divisor.

(Note that in this statement we have allowed $b$ to be negative, as did Zsigmondy. His theorem is nowadays often quoted with the restriction $a > b > 0$ and so has the second exceptional case omitted.)

3. **Proof of Theorems 1 and 2**

Let $n \in R_{a,b}^{(j)}$ have a factorisation (1), where $p_1 < p_2 < \cdots < p_r$ and all $k_i > 0$. First take $j \geq 1$. Then, by Proposition 12, $n/p_r^{k_r} = n_r \in R_{a,b}^{(j)}$, and hence

$$(n/p_r^{k_r})/p_r^{k_r-1} = n_{r-1}, \quad \ldots \quad p_1^{k_1} = n_2, \quad 1 = n_1$$

are all in $R_{a,b}^{(j)}$. Now separate the two cases $j = 1$ and $j \geq 2$ for Theorems 1 and 2 respectively. Now for $j = 1$ Proposition 5 gives us that $p_i \mid a^{n_i} - b^{n_i}$ ($i = 1, \ldots, r$), while for $j \geq 2$ we have, again from Proposition 5, that

$$p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \lcm(a - b, a + b) & \text{if } p_1 = 2, \end{cases}$$

and $p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i}$ ($i = 2, \ldots, r$). Here we have used the fact that $\gcd(p_i, n_i) = 1$, so that if $p_i^{k_i} \mid (a^{n_i} - b^{n_i})/n_i^2$ then $p_i^{k_i} \mid a^{n_i} - b^{n_i}$ (i.e., we are applying Proposition 5 with all the exponents $k_p$ equal to 0.)

4. **Finding $R_{a,b}^{(j)}$ When $\gcd(a, b) > 1$.**

For $a > 1$, define the set $\mathcal{F}_a$ to be the set of all $n \in \mathcal{N}$ whose prime factors all divide $a$. To find $R_{a,b}^{(j)}$ in general, we first consider the case $b = 0$. 


Proposition 14. We have $R_{a,0}^{(1)} = R_{a,0}^{(2)} = F_a$, while for $j \geq 3$ the set $R_{a,0}^{(j)} = F_a \setminus S_a^{(j)}$, where $S_a^{(j)}$ is a finite set.

Proof. From the condition $n^j \mid a^n$, all prime factors of $n$ divide $a$, so $R_{a,0}^{(j)} \subseteq F_a$, say $R_{a,0}^{(j)} = F_a \setminus S_a^{(j)}$. We need to prove that $S_a^{(j)}$ is finite. Suppose that $a = p_1^{k_1} \ldots p_r^{k_r}$, with $p_1$ the smallest prime factor of $a$. Then $n = p_1^{k_1} \ldots p_r^{k_r}$ for some $k_i \geq 0$. From $n^j \mid a^n$ we have

$$k_i \leq \frac{a_i}{j} p_1^{k_1} \ldots p_r^{k_r} \quad (i = 1, \ldots, r).$$  \hfill (17)

For these $r$ conditions to be satisfied it is sufficient that

$$\sum_{i=1}^{r} k_i \leq \frac{\min_{i=1}^{r} a_i}{j} p_1^{\sum_{i=1}^{r} k_i}.$$  \hfill (18)

Now (18) holds if $j = 1$ or 2, as in this case, from the simple inequality $k \leq 2^{k-1}$ valid for all $k \in \mathbb{N}$, we have

$$\sum_{i=1}^{r} k_i \leq \frac{1}{2^{\sum_{i=1}^{r} k_i}} \leq \frac{\min_{i=1}^{r} a_i}{j} p_1^{\sum_{i=1}^{r} k_i}.$$  \hfill (19)

Hence $S_a^{(j)}$ is empty if $j = 1$ or 2.

Now take $j \geq 3$, and let $K = K_a^{(j)}$ be the smallest integer such that $Kp_1^{-K} \leq \left( \min_{i=1}^{r} a_i \right) / j$. Then (18) holds for $\sum_{i=1}^{r} k_i \geq K$, and $S_a^{(j)}$ is contained in the finite set $S'' = \{ n \in \mathbb{N}, n = p_1^{k_1} \ldots p_r^{k_r} : \sum_{i=1}^{r} k_i < K \}$. (To compute $S_a^{(j)}$ precisely, one need just check for which $r$-tuples $(k_1, \ldots, k_r)$ with $\sum_{i=1}^{r} k_i < K$ any of the $r$ inequalities of (17) is violated.)

One (at first sight) curious consequence of the equality $R_{a,0}^{(1)} = R_{a,0}^{(2)}$ above is that $n \mid a^n$ implies $n^2 \mid a^n$.

Now let $g = \gcd(a, b)$ and $a = a_1 g, b = b_1 g$. Write $n = G n_1$, where all prime factors of $G$ divide $g$ and $\gcd(n_1, g) = 1$. Then we have the following general result.

Theorem 15. The set $R_{a,b}^{(j)}$ is given by

$$R_{a,b}^{(j)} = \{ n = G n_1 : G \in \mathcal{F}_g, n_1 \in R_{a_1, b_1}^{(j)} \setminus \mathcal{F}_{a_1, b_1}^{(j)} \text{ and } \gcd(g, n_1) = 1 \} \setminus R,$$  \hfill (20)
where $R$ is a finite set. Specifically, all $n = Gn_1 \in R$ have $1 \leq n_1 < j/2$ and
\[ G = q_1^{\ell_1} \cdots q_m^{\ell_m}, \]
where
\[ \sum_{i=1}^{m} \ell_i < R_g^{(j)}. \]
Here the $q_i$ are the primes dividing $g$, and $R_g^{(j)}$ is the constant in the proof of Proposition 14 above.

**Proof.** Supposing that $n \in R_{a,b}^{(j)}$ we have
\[ n^j \mid a^n - b^n \]
and so $n^j \mid g^n (a_1^n - b_1^n)$. Writing $n = Gn_1$, as above, we have
\[ n_1^j \mid (a_1^G)^{n_1} - (b_1^G)^{n_1} \]
and
\[ G^j \mid g^{Gn_1} ((a_1^G)^{n_1} - (b_1^G)^{n_1}). \]
Thus (23) holds with $n, a, b$ replaced by $n_1, a_1^G, b_1^G$. So we have reduced the problem of (23) to a case where $\gcd(a, b) = 1$, which we can solve for $n_1$ prime to $g$, along with the extra condition (25). Now, from the fact that $R_{g,0}^{(2)} = F_g$ from Proposition 14, we have $G^j \mid g^G$ and hence $G^j \mid g^{Gn_1}$ for all $G \in F_g$, provided that $n_1 \geq j/2$. Hence (25) can fail to hold for all $G \in F_g$ only for $1 \leq n_1 < j/2$.

Now fix $n_1$ with $1 \leq n_1 < j/2$. Then note that by Proposition 14, $G^j \mid g^{Gn_1}$ and hence (23) holds for all $G \in F_g \setminus S$, where $S$ is a finite set of $G$’s contained in the set of all $G$’s given by (21) and (22).

Note that (taking $n_1 = 1$ and using (25)) we always have $R_{g,0}^{(j)} \subset R_{a,b}^{(j)}$. See example in Section 7.

5. When Are $R_{a,b}^{(1)}$ and $R_{a,b}^{(2)}$ Finite?

First consider $R_{a,b}^{(1)}$. From Theorem 1 it is immediate that $R_{a,b}^{(1)}$ contains all powers of any primes dividing $a - b$. Thus $R_{a,b}^{(1)}$ is infinite unless $a - b = \pm 1$, in which case $R_{a,b}^{(1)} = \{1\}$. This was pointed out earlier by André-Jeannin [1, Corollary 4].

Next, take $j = 2$. Let us denote by $P_{a,b}^{(2)}$ the set of primes that divide some $n \in R_{a,b}^{(2)}$ and, as before, put $g = \gcd(a, b)$.
Theorem 16. The set $R_{a,b}^{(2)} = \{1\}$ if and only if $a$ and $b$ are consecutive integers, and $R_{a,b}^{(2)} = \{1,3\}$ if and only if $ab = -2$. Otherwise, $R_{a,b}^{(2)}$ is infinite.

If $R_{a/g,b/g}^{(2)} = \{1\}$ (respectively, $= \{1,3\}$) then $\mathcal{P}_{a,b}^{(2)}$ is the set of all prime divisors of $g$ (respectively, $3g$). Otherwise $\mathcal{P}_{a,b}^{(2)}$ is infinite.

For coprime positive integers $a, b$ with $a - b > 1$, the infiniteness of $R_{a,b}^{(2)}$ already follows from the above-mentioned results of [5].

The application of Zsigmondy’s Theorem that we require is the following.

Proposition 17. If $R_{a,b}^{(2)}$ contains some integer $n \geq 4$ then both $R_{a,b}^{(2)}$ and $\mathcal{P}_{a,b}^{(2)}$ are infinite sets.

Proof. First note that if $a = 2, b = 1$ (or more generally $a - b = \pm 1$) then by Theorem 2, $R_1^{(2)} = \{1\}$. Hence, taking $n \in R_{a,b}^{(2)}$ with $n \geq 4$ we have, by Zsigmondy’s Theorem, that $a^n - b^n$ has a primitive prime divisor, $p$ say. Now if $p \mid n$ then, by applying Proposition 12 as many times as necessary we find $p \mid n'$, where $n' \in R_{a,b}^{(2)}$ and now $p$ is the maximal prime divisor of $n'$. Hence, by Proposition 12 again, $n'' = n'/p \in R_{a,b}^{(2)}$ and so, from $n' = pn''$ and Proposition 5 we have that $p \mid a^{n''} - b^{n''}$, contradicting the primitivity of $p$.

Now using Proposition 5 again, $np \in R_{a,b}^{(2)}$. Repeating the argument with $n$ replaced by $np$ and continuing in this way we obtain an infinite sequence

$$n, \ \ np, \ \ np_1, \ \ np_1p_2, \ \ldots, \ \ np_1p_2\ldots p_\ell, \ \ldots$$

of elements of $R_{a,b}^{(2)}$, where $p < p_1 < p_2 < \cdots < p_\ell < \ldots$ are primes. \hfill \Box

Proof of Theorem 16. Assume $\gcd(a,b) = 1$, and, without loss of generality, that $a > 0$ and $a > b$. (We can ensure this by interchanging $a$ and $b$ and/or changing both their signs.) If $a - b$ is even, then $a$ and $b$ are odd, and $a^2 - b^2 \equiv 1 \pmod{2^{t+1}}$, where $t \geq 2$. Hence $4 \in R_{a,b}^{(2)}$, by Proposition 5, and so both $R_{a,b}^{(2)}$ and $\mathcal{P}_{a,b}^{(2)}$ are infinite sets, by Proposition 17.

If $a - b = 1$ then $R_{a,b}^{(2)} = \{1\}$, as we have just seen, above.

If $a - b$ is odd and at least 5, then $a - b$ must either be divisible by 9 or by a prime $p \geq 5$. Hence 9 or $p$ belong to $R_{a,b}^{(2)}$, by Proposition 5, and again both $R_{a,b}^{(2)}$ and $\mathcal{P}_{a,b}^{(2)}$ are infinite sets, by Proposition 17.

If $a - b = 3$ then $3 \in R_{a,b}^{(2)}$, and $a^3 - b^3 = 9(b^2 + 3b + 3)$. If $b = -1$ (and $a = 2$, $ab = -2$) or $-2$ (and $a = 1$, $ab = -2$) then $a^3 - b^3 = 9$ and

$$L \equiv 1 \pmod{2^{t+1}}.$$
so, by Theorem 2, so \( R^{(2)} = \{1, 3\} \). Otherwise, using \( \gcd(a, b) = 1 \) we see that \( a^3 - b^3 \geq 5 \), and so the argument for \( a - b \geq 5 \) but with \( a, b \) replaced by \( a^3, b^3 \) applies.

\[\square\]

6. The Powers of \( n \) Dividing \( a^n + b^n \)

Define \( R^{(j)}_{a, b} \) to be the set \( \{n \in \mathbb{N} : n^j \text{ divides } a^n + b^n\} \). Take \( j \geq 1 \), and assume that \( \gcd(a, b) = 1 \). (The general case \( \gcd(a, b) \geq 1 \) can be handled as in Section 4.)

We then have the following result.

**Theorem 18.** Suppose that \( j \geq 1 \), \( \gcd(a, b) = 1 \), \( a > 0 \) and \( a \geq |b| \). Then

(a) \( R^{(1)}_{a, b} \) consists of the odd elements of \( R^{(1)}_{a, -b} \), along with the numbers of the form \( 2n_1 \), where \( n_1 \) is an odd element of \( R^{(1)}_{a^2, -b^2} \);

(b) If \( j \geq 2 \) the set \( R^{(j)}_{a, b} \) consists of the odd elements of \( R^{(j)}_{a, -b} \) only.

Furthermore, for \( j = 1 \) and 2, the set \( R^{(j)}_{a, b} \) is infinite, except in the following cases:

- If \( a + b \) is 1 or a power of 2, \( (j, a, b) \neq (1, 1, 1) \), when it is \( \{1\} \);
- \( R^{(1)}_{1, 1} = \{1, 2\} \);
- \( R^{(2)}_{2, 1} = \{1, 3\} \).

**Proof.** If \( n \) is even and \( j \geq 2 \), or if \( 4 | n \) and \( j = 1 \), then \( n^j | a^n + b^n \) implies that \( 4 | a^n + b^n \), contradicting the fact that, as \( a \) and \( b \) are not both even, \( a^n + b^n \equiv 1 \) or \( 2 \) (mod 8). So either

- \( n \) is odd, in which case \( n^j | a^n + b^n \) is equivalent to finding the odd elements of the set \( R^{(j)}_{a, -b} \);

or

- \( j = 1 \) and \( n = 2n_1 \), where \( n_1 \) is odd, and belongs to \( R^{(1)}_{a^2, -b^2} \).

Now suppose that \( j = 1 \) or 2. If \( a + b \) is \( \pm 1 \) or \( \pm 2^i \) for some \( i > 0 \), then, by Theorem 2, all \( n \in R^{(j)}_{a, b} \) with \( n > 1 \) are even, so for \( j = 2 \) there are no \( n > 1 \) with \( n^j | a^n + b^n \) in this case. Otherwise, \( a + b \) will have an odd prime factor, and so at least one odd element greater than 1. By Theorem 16 and its proof, we see that \( R^{(2)}_{a, b} \) will have infinitely many odd elements unless \( a(-b) = -2 \), i.e., \( a = 2 \), \( b = 1 \) (using \( a > 0 \) and \( a \geq |b| \)).
For $j = 1$ there will be infinitely many $n$ with $n \mid a^n + b^n$, except when both $a + b$ and $a^2 + b^2$ are 1 or a power of 2. It is an easy exercise to check that, this can happen only for $a = b = 1$ or $a = 1, b = 0$. \hfill \Box$

If $g = \gcd(a, b) > 1$, then, since $R_{a, b}^{(j)}$ contains the set $R_{g, 0}^{(j)}$, it will be infinite, by Proposition 14. For $j \geq 3$ and $\gcd(a, b) = 1$, the finiteness of the set $R_{a, b}^{(j)}$ would follow from the finiteness of $R_{a, b}^{(j)}$, using Theorem 16(b).

7. Examples

The set $R_{a, b}^{(j)}$ has a natural labelled, directed-graph structure, as follows: take the vertices to be the elements of $R_{a, b}^{(j)}$, and join a vertex $n$ to a vertex $np$ as $n \rightarrow_p np$, where $p \in P^{(j)}(a, b)$. We reduce this to a spanning tree of this graph by taking only those edges $n \rightarrow_p np$ for which $p$ is the largest prime factor of $np$. For our first example we draw this tree (Figure 1).

1. Consider the set

$$R_{3, 1}^{(2)} = \{1, 2, 4, 20, 220, 1220, 2420, 5060, 13420, 14740, 23620, 55660, 145420, 147620, 162140, 237820, 259820, 290620, 308660, 339020, 447740, 847220, 899140, 1210220, \ldots \}$$

(sequence A127103 in Neil Sloane’s Integer Sequences website). Now

$$3^{20} - 1 = 2^4 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181,$$

showing that $P_{3, 1}^{(2)}(20) = \{11, 11^2, 61, 1181\}$. Also

$$3^{220} - 1 = 2^4 \cdot 5^2 \cdot 11^3 \cdot 23 \cdot 61 \cdot 67 \cdot 661 \cdot 1181 \cdot 1321 \cdot 3851 \cdot 5501 \cdot 177101 \cdot 570461 \cdot 659671 \cdot 24472341743191 \cdot 560088668384411 \cdot 927319729649066047885192700193701,$$

so that the elements of $P_{3, 1}^{(2)}(220)$ less than $10^6/220$, needed for Figure 1, are

$$11, 23, 61, 67, 661, 1181, 1321, 3851.$$
Figure 1: Part of the spanning tree for $R^{(2)}_{3,1}$, showing all elements below $10^6$.

2. Now

$$R^{(2)}_{3,-1} = \{1, 2, 3, 4, 6, 12, 21, 42, 52, 84, 156, 186, 372, \ldots \},$$

whose odd elements give

$$R^{(2)+}_{5,-1} = \{1, 3, 21, 609, 903, 2667, 9429, 26187, \ldots \}.$$ 

See Section 6.

3. We have

$$R^{(2)+}_{3,2} = R^{(2)}_{3,-2} = \{1, 5, 55, 1971145, \ldots \},$$

as all elements of $R^{(2)}_{3,-2}$ are odd. Although this set is infinite by Theorem 16, the next term is $1971145p$ where $p$ is the smallest prime factor of $3^{1971145} + 2^{1971145}$ not dividing $1971145$. This looks difficult to compute, as it could be very large.

4. We have

$$R^{(2)}_{4,-3} = R^{(2)+}_{4,3} = \{1, 7, 2653, \ldots \}.$$ 

Again, this set is infinite, but here only the three terms given are readily computable. The next term is $2653p$ where $p$ is the smallest prime factor of $4^{2653} + 3^{2653}$ not dividing $2653$.

5. This is an example of a set with more than one odd prime as a squared factor in elements of the set, in this case the primes 3 and 7. Every element greater
than 9 is of one of the forms 21m, 63m, 147m, or 441m, where m is prime to 21;

\[ R^{(2)}_{11,2} = \{1, 3, 9, 21, 63, 147, 441, 609, 1827, 4137, 4263, 7959, \\
8001, 12411, 12789, 23877, 28959, 35931, 55713, 56007, \\
86877, 107793, 119973, 167139, 212541, 216237, 230811, \\
232029, 251517, 359919, 389403, \ldots \}. \]

6. \( R^{(4)}_{27001,1} = \{1, 2, 3, 5, 6, 10, 15, 30 \} \). This is because 27001 – 1 = \( 2^3 \cdot 3^3 \cdot 5^3 \), and none of 27001\(^n \) – 1 has a factor \( p^3 \) for any prime \( p \) > 5 for any \( n = 1, 2, 3, 5, 6, 10, 15, 30 \).

7. \( R^{(3)}_{19,1} = \{1, 2, 3, 6, 42, 1806 \} \)? Is this the entire set? Yes, unless \( 19^{1806} - 1 \) is divisible by \( p^2 \) for some prime \( p \) prime to 1806, in which case 1806\( p \) would also be in the set. But determining whether or not this is the case seems to be a hard computational problem.

8. \( R^{(4)}_{56,2} \), an example with gcd\((a, b) > 1 \). It seems highly probable that

\[ R^{(4)}_{56,2} = (\mathcal{F}_2 \setminus \{2, 4, 8\}) \cup (3\mathcal{F}_2) \\
= 1, 3, 6, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 384, 512, 768, 1024, \ldots \]

However, in order to prove this, Theorem 15 tells us that we need to know that \( 28^\ell \not\equiv 1 \mod p^3 \) for every prime \( p > 3 \) and every \( \ell > 0 \). This seems very difficult! Note that \( R^{(4)}_{28,0} = \mathcal{F}_2 \setminus \{2, 4, 8\} \) and \( R^{(4)}_{28,1} = \{1, 3\} \).

8. Final Remarks

1. By finding \( R^{(j)}_{a,b} \), we are essentially solving the exponential Diophantine equation \( x^jy = a^x - b^x \), since any solutions with \( x \leq 0 \) are readily found.

2. It is known that

\[ R^{(1)}_{a,b} = \left\{ n \in \mathbb{N} : n \text{ divides } \frac{a^n - b^n}{a - b} \right\}. \]

See [11, Proposition 12] (and also André-Jeannin [1, Theorem 2] for some special cases.) This result shows that \( R^{(1)}_{a,b} = \{n \in \mathbb{N} : n \text{ divides } u_n \} \), where the \( u_n \) are the generalized Fibonacci numbers of the first kind defined by the recurrence \( u_0 = 1, u_1 = 1 \), and \( u_{n+2} = (a + b)u_{n+1} - abu_n (n \geq 0) \). This provides a link between Theorem 1 of the present paper and the results of [11].
The set \( R_{a,b}^{(1)+} \) is a special case of a set \( \{ n \in \mathbb{N} : n \text{ divides } v_n \} \), also studied in [11]. Here \( (v_n) \) is the sequence of generalized Fibonacci numbers of the second kind. For earlier work on this topic see Somer [13].

3. Earlier and related work. The study of factors of \( a^n - b^n \) dates back at least to Euler, who proved that all primitive prime factors of \( a^n - b^n \) were \( \equiv 1 \pmod{n} \). See [2, Theorem 1]. Chapter 16 of Dickson [4] is devoted to the literature on factors of \( a^n \pm b^n \).

More specifically, Kennedy and Cooper [8] studied the set \( R_{10,1}^{(1)} \). André-Jeannin [1, Corollary 4] claimed (erroneously – see Theorem 18) that the congruence \( a^n + b^n \equiv 0 \pmod{n} \) always has infinitely many solutions \( n \) for \( \gcd(a, b) = 1 \).

References


