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THE DIVISIBILITY OF $a^n - b^n$ BY POWERS OF $n$

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Abstract

For given integers $a, b$ and $j \geq 1$ we determine the set $R^{(j)}_{a,b}$ of integers $n$ for which $a^n - b^n$ is divisible by $n^j$. For $j = 1, 2$, this set is usually infinite; we determine explicitly the exceptional cases for which $a, b$ the set $R^{(j)}_{a,b} (j = 1, 2)$ is finite. For $j = 2$, we use Zsigmondy’s Theorem for this. For $j \geq 3$ and $\gcd(a, b) = 1$, $R^{(j)}_{a,b}$ is probably always finite; this seems difficult to prove, however.

We also show that determination of the set of integers $n$ for which $a^n + b^n$ is divisible by $n^j$ can be reduced to that of $R^{(j)}_{a,b}$.

1. Introduction

Let $a, b$ and $j$ be fixed integers, with $j \geq 1$. The aim of this paper is to find the set $R^{(j)}_{a,b}$ of all positive integers $n$ such that $n^j$ divides $a^n - b^n$. For $j = 1, 2, \ldots$, these sets are clearly nested, with common intersection $\{1\}$. Our first results (Theorems 1 and 2) describe this set in the case that $\gcd(a, b) = 1$. In Section 4 we describe (Theorem 15) the set in the general situation where $\gcd(a, b)$ is unrestricted.

**Theorem 1.** Suppose that $\gcd(a, b) = 1$. Then the elements of the set $R^{(1)}_{a,b}$ consist of those integers $n$ whose prime factorization can be written in the form

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \quad (p_1 < p_2 < \cdots < p_r, \text{ all } k_i \geq 1),$$

where $p_i \mid (a^{n_i} - b^{n_i}) (i = 1, \ldots, r)$, with $n_1 = 1$ and $n_i = p_1^{k_1} p_2^{k_2} \cdots p_{i-1}^{k_{i-1}}$ $(i = 2, \ldots, r)$.

In this theorem, the $k_i$ are arbitrary positive integers. This result is a more explicit version of that proved in Győry [5], where it was shown that if $a - b > 1$ then for any positive integer $r$ the number of elements of $R^{(1)}_{a,b}$ having $r$ prime factors is infinite. The result is also essentially contained in [11], which described
the indices \( n \) for which the generalized Fibonacci numbers \( u_n \) are divisible by \( n \). However, we present a self-contained proof in this paper.

On the other hand, for \( j \geq 2 \), the exponents \( k_i \) are more restricted.

**Theorem 2.** Suppose that \( \gcd(a, b) = 1 \), and \( j \geq 2 \). Then the elements of the set \( R_{a, b}^{(j)} \) consist of those integers \( n \) whose prime factorization can be written in the form (1), where

\[
p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \text{lcm}(a - b, a + b) & \text{if } p_1 = 2, \end{cases}
\]

and \( p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i} \), with \( n_i = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i-1} \) (\( i = 2, \ldots, r \)).

Again, the result was essentially contained in [5], where it was proved that for \( a - b > 1 \) and for any given \( r \), there exists an \( n \in R_{a, b}^{(j)} \) with \( r \) distinct prime factors. Further, the number of these \( n \) is finite, and all of them can be determined. The paper [5] was stimulated by a problem from the 31st International Mathematical Olympiad, which asked for all those positive integers \( n > 1 \) for which \( 2^n + 1 \) was divisible by \( n^2 \). (For the answer, see [5], or Theorem 16.)

Thus we see that construction of \( n \in R_{a, b}^{(j)} \) depends upon finding a prime \( p_i \) not used previously with \( a^{n_i} - b^{n_i} \) being divisible by \( p_i^{j-1} \). This presents no problem for \( j = 2 \), so that \( R_{a, b}^{(2)} \), as well as \( R_{a, b}^{(1)} \), are usually infinite. See Section 5 for details, including the exceptional cases when they are finite. However, for \( j \geq 3 \) the condition \( p_i^{j-1} \mid a^{n_i} - b^{n_i} \) is only rarely satisfied. This suggests strongly that in this case \( R_{a, b}^{(j)} \) is always finite for \( \gcd(a, b) = 1 \). This seems very difficult to prove, even assuming the ABC Conjecture. A result of Ribenboim and Walsh [10] implies that, under ABC, the powerful part of \( a^n - b^n \) cannot often be large. But this is not strong enough for what is needed here. On the other hand, \( R_{a, b}^{(j)} (j \geq 3) \) can be made arbitrarily large by choosing \( a, b \) such that \( a - b \) is a powerful number. For instance, choosing \( a = 1 + (q_1 q_2 \ldots q_s)^{j-1} \) and \( b = 1 \), where \( q_1, q_2, \ldots, q_s \) are distinct primes, then \( R_{a, b}^{(j)} \) contains the 2\(^s\) numbers \( q_1^{\varepsilon_1} q_2^{\varepsilon_2} \cdots q_s^{\varepsilon_s} \) where the \( \varepsilon_i \) are 0 or 1. See Example 6 in Section 7.

In the next section we give preliminary results needed for the proof of the theorems. We prove them in Section 3. In Section 4 we describe (Theorem 15) \( R_{a, b}^{(j)} \), where \( \gcd(a, b) \) is unrestricted. In Section 5 we find all \( a, b \) for which \( R_{a, b}^{(2)} \) is finite (Theorem 16). In Section 6 we discuss the divisibility of \( a^n + b^n \) by powers of \( n \). In Section 7 we give some examples, and make some final remarks in Section 8.
2. Preliminary Results

We first prove a version of Fermat’s Little Theorem that gives a little bit more information in the case \( x \equiv 1 \pmod{p} \).

**Lemma 3.** For \( x \in \mathbb{Z} \) and \( p \) an odd prime we have

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv \begin{cases} p \pmod{p^2} & \text{if } x \equiv 1 \pmod{p}; \\ 1 \pmod{p} & \text{otherwise}. \end{cases}
\]  

(2)

**Proof.** If \( x \equiv 1 \pmod{p} \), say \( x = 1 + kp \), then \( x^j \equiv 1 + jkp \pmod{p^2} \), so that

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv p + kp \sum_{j=0}^{p-1} j \equiv p \pmod{p^2}.
\]  

(3)

Otherwise

\[
x(x-1)(x^{p-2} + \cdots + x + 1) = x^p - x \equiv 0 \pmod{p},
\]  

(4)

so that for \( x \not\equiv 1 \pmod{p} \) we have \( x(x^{p-2} + \cdots + x + 1) \equiv 0 \pmod{p} \), and hence

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv x(x^{p-2} + \cdots + x + 1) + 1 \equiv 1 \pmod{p}.
\]  

(5)

\[\square\]

The following is a result of Birkhoff and Vandiver [2, Theorem III]. It is also special case of Lucas [9, p. 210], as corrected for \( p = 2 \) by Carmichael [3, Theorem X].

**Lemma 4.** Let \( \gcd(a, b) = 1 \) and \( p \) be prime with \( p \nmid (a - b) \). Define \( t > 0 \) by \( p^t \mid (a - b) \) for \( p > 2 \) and \( 2^t \mid \text{lcm}(a - b, a + b) \) if \( p = 2 \). Then for \( \ell > 0 \)

\[
p^t \mid (a^p - b^p).
\]  

(6)

On the other hand, if \( p \nmid a - b \) then for \( \ell \geq 0 \)

\[
p \nmid a^p - b^p.
\]  

(7)

**Proof.** Put \( x = a/b \). First suppose that \( p \) is odd and \( p^t \| (a - b) \) for some \( t > 0 \). Then as \( \gcd(a, b) = 1 \), \( b \) is not divisible by \( p \), and we have \( x \equiv 1 \pmod{p^t} \). Then from

\[
a^p - b^p = (a - b)b^{p-1}(x^{p-1} + x^{p-2} + \cdots + x + 1)
\]  

(8)
we have by Lemma 3 that $p^{t+1}||(a^p - b^p)$. Applying this result $\ell$ times, we obtain (6).

For $p = 2$, we have $p^{t+1}||a^2 - b^2$ and from $a^2 \equiv b^2 \equiv 1 \pmod{8}$, we obtain $2^1||(a^4 - b^4)$, and so $p^{t+2}||(a^4 - b^4)$. An easy induction then gives the required result.

Now suppose that $p \nmid (a - b)$. Since $\gcd(a, b) = 1$, (7) clearly holds if $p | a$ or $p | b$, as must happen for $p = 2$. So we can assume that $p$ is odd and $p \nmid b$. Then $x \not\equiv 1 \pmod{p}$ so that, by Lemma 3 and (8), we have $p \nmid (a^p - b^p)$. Applying this argument $\ell$ times, we obtain (7). 

For $n \in R^{(j)}_{a,b}$, we now define the set $P^{(j)}_{a,b}(n)$ to be the set of all prime powers $p^k$ for which $np^k \in R^{(j)}_{a,b}$. Our next result describes this set precisely. (Compare with [11, Theorem 1(a)]).

**Proposition 5.** Suppose that $j \geq 1$, $\gcd(a, b) = 1$, $n \in R^{(j)}_{a,b}$ and

$$a^n - b^n = 2^{\nu_2} \prod_{p > 2} p^{\nu_p}, \quad n = \prod_{p} p^{k_p}$$

(9) and define $e_2$ by $2^{\nu_2}||\text{lcm}(a^n - b^n, a^n + b^n)$. Then

$$P^{(1)}(n) = \bigcup_{p | a^n - b^n} \{p^k, k \in \mathbb{N}\},$$

(10) and for $j \geq 2$

$$P^{(j)}_{a,b}(n) = \bigcup_{p: p^{j-1} | a^n - b^n} \left\{p^k : 1 \leq k \leq \left\lfloor \frac{e_p - jk_p}{j - 1} \right\rfloor \right\},$$

(11)

Note that $e_2$ is never 1. Consequently, if $2m \in R^{(2)}_{a,b}$, where $m$ is odd, then $4m \in R^{(2)}_{a,b}$. Also, $2 \in R^{(j)}_{a,b}$ for $j \leq 3$ when $a - b$ is even.

**Proof.** Taking $n \in R^{(j)}_{a,b}$ we have, from (9) and the definition of $e_2$, that $jk_p \leq e_p$ for all primes $p$. Hence, applying Lemma 4 with $a, b$ replaced by $a^n, b^n$ we have for $p$ dividing $a^n - b^n$ that for $\ell > 0$

$$p^{\nu_p + \ell}||(a^{np^\ell} - b^{np^\ell}).$$

(12)

So $(np^\ell)^j | (a^{np^\ell} - b^{np^\ell})$ is equivalent to $j(k_p + \ell) \leq e_p + \ell$, or $(j - 1)\ell \leq e_p - jk_p$. Thus we obtain (10) for $j \geq 2$, with $\ell$ unrestricted for $j = 1$, giving (10).

On the other hand, if $p \nmid (a^n - b^n)$, then by Lemma 4 again, $p^\ell \nmid (a^{np^\ell} - b^{np^\ell})$, so that certainly $(np^\ell)^j \nmid (a^{np^\ell} - b^{np^\ell})$. \qed
We now recall some facts about the order function \( \text{ord} \). For \( m \) an integer greater than 1 and \( x \) an integer prime to \( m \), we define \( \text{ord}_m(x) \), the order of \( x \) modulo \( m \), to be the least positive integer \( h \) such that \( x^h \equiv 1 \pmod{m} \). The next three lemmas, containing standard material on the order function, are included for completeness.

**Lemma 6.** For \( x \in \mathbb{N} \) and prime to \( m \), we have \( m \mid (x^n - 1) \) if and only if \( \text{ord}_m(x) \mid n \).

**Proof.** Let \( \text{ord}_m(x) = h \), and assume that \( m \mid (x^n - 1) \). Then as \( m \mid (x^h - 1) \), also \( m \mid (x^{\gcd(h,n)} - 1) \). By the minimality of \( h \), \( \gcd(h,n) = h \), i.e., \( h \mid n \). Conversely, if \( h \mid n \) then \( (x^h - 1) \mid (x^n - 1) \), so that \( m \mid (x^n - 1) \).

**Corollary 7.** Let \( j \geq 1 \). We have \( n^j \mid (x^n - 1) \) if and only if \( \gcd(x,n) = 1 \) and \( \text{ord}_{n^j}(x) \mid n \).

**Lemma 8.** For \( m = \prod_p p^{f_p} \) and \( x \in \mathbb{N} \) and prime to \( m \) we have

\[
\text{ord}_m(x) = \text{lcm}_p \text{ord}_{p^{f_p}}(x).
\]

**Proof.** Put \( h_p = \text{ord}_{p^{f_p}}(x) \), \( h = \text{ord}_m(x) \) and \( h' = \text{lcm}_p h_p \). Then by Lemma 6 we have \( p^{f_p} \mid (x^{h'} - 1) \) for all \( p \), and hence \( m \mid (x^{h'} - 1) \). Hence \( h \mid h' \). On the other hand, as \( p^{f_p} \mid n \) and \( m \mid (x^h - 1) \), we have \( p^{f_p} \mid (x^h - 1) \), and so \( h_p \mid h \), by Lemma 6. Hence \( h' = \text{lcm}_p h_p \mid h \).

Now put \( p_* = \text{ord}_p(x) \), and define \( t > 0 \) by \( p^t \mid (x^{p_*} - 1) \).

**Lemma 9.** For \( \gcd(x,n) = 1 \) and \( \ell > 0 \) we have \( p_* \mid (p-1) \) and \( \text{ord}_{p^\ell}(x) = p^{\max(\ell-1,0)} p_* \).

**Proof.** Since \( p \mid (x^{p-1} - 1) \), we have \( p_* \mid (p-1) \), by Lemma 6. Also, from \( p^\ell \mid (x^{\text{ord}_{p^\ell}(x)} - 1) \) we have \( p \mid (x^{\text{ord}_{p^\ell}(x)} - 1) \), and so, by Lemma 6 again, \( p_* = \text{ord}_p(x) \mid \text{ord}_{p^\ell}(x) \). Further, if \( \ell \leq t \) then from \( p^\ell \mid (x^{p_*} - 1) \) we have by Lemma 6 that \( \text{ord}_{p^\ell}(x) \mid p_* \), so \( \text{ord}_{p^\ell}(x) = p_* \). Further, by Lemma 4 for \( u \geq t \)

\[
 p^u \mid (x^{p^{u-t} p_*} - 1),
\]

so that, taking \( u = \ell \geq t \) and using Lemma 6, \( \text{ord}_{p^\ell}(x) \mid p^{\ell-t} p_* \). Also, if \( t \leq u < \ell \), then, from (14), \( x^{p^{u-t} p_*} \not\equiv 1 \pmod{p^\ell} \). Hence \( \text{ord}_{p^\ell}(x) = p^{\ell-t} p_* \) for \( \ell \geq t \).  

\[ \square \]
Corollary 10. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and \( x \in \mathbb{N} \) prime to \( n \) we have \( n^j \mid x^n - 1 \) if and only if \( \gcd(x, n) = 1 \) and

\[
\text{lcm}_p p^{k'_p}p_s \mid \prod_p p^{k_p}.
\]

(15)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

Note that \( p_s, k'_p \) and \( t_p \) in general depend on \( x \) and \( j \) as well as on \( p \).

What we actually need in our situation is the following variant of Corollary 10.

Corollary 11. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and integers \( a, b \) with \( \gcd(a, b) = 1 \) we have \( n^j \mid a^n - b^n \) if and only if \( \gcd(n, a) = \gcd(n, b) = 1 \) and

\[
\text{lcm}_p p^{k'_p}p_s \mid \prod_p p^{k_p}.
\]

(16)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

In this corollary, the \( x \) used to define \( p_s \) and \( t = t_p \) (see after Lemma 8) is chosen to satisfy \( bx \equiv a \pmod{n^j} \). The result is then easily deduced from Corollary 10.

By contrast with Proposition 5, our next proposition allows us to divide an element \( n \in R^{(j)}_{a,b} \) by a prime, and remain within \( R^{(j)}_{a,b} \).

Proposition 12. Let \( n \in R^{(j)}_{a,b} \) with \( n > 1 \), and suppose \( p_{\text{max}} \) is the largest prime factor of \( n \). Then \( n/p_{\text{max}} \in R^{(j)}_{a,b} \).

Proof. Suppose \( n \in R^{(j)}_{a,b} \), so that (15) holds, with \( x = a/b \), and put \( q = p_{\text{max}} \). Then, since for every \( p \) all prime factors of \( p_s \) are less than \( p \), the only possible term on the left-hand side that divides \( q^{k_q} \) on the right-hand side is the term \( q^{k_q} \).

Now reducing \( k_q \) by 1 will reduce \( k'_p \) by at least 1, unless it is already 0, when it does not change. In either case (15) will still hold with \( n \) replaced by \( n/q \), and so \( n/q \in R^{(j)}_{a,b} \). \( \square \)

Various versions and special cases of Proposition 12 for \( j = 1 \) have been known for some time, in the more general setting of Lucas sequences, due to Somer [12, Theorem 5(iv)], Jarden [7, Theorem E], Hoggatt and Bergum [6], Walsh [14], André-Jeannin [1] and others. See also Smyth [11, Theorem 3].

In order to work out for which \( a, b \) the set \( R^{(j)}_{a,b} \) is finite, we need the following classical result. Recall that \( a^n - b^n \) is said to have a primitive prime divisor \( p \) if the prime \( p \) divides \( a^n - b^n \) but does not divide \( a^k - b^k \) for any \( k \) with \( 1 \leq k < n \).
Theorem 13 (Zsigmondy [15]). Suppose that \( a \) and \( b \) are nonzero coprime integers with \( a > b \) and \( a + b > 0 \). Then, except when

1. \( n = 2 \) and \( a + b \) is a power of 2
2. \( n = 3, \ a = 2, \ b = -1 \)
3. \( n = 6, \ a = 2, \ b = 1, \)

\( a^n - b^n \) has a primitive prime divisor.

(Note that in this statement we have allowed \( b \) to be negative, as did Zsigmondy. His theorem is nowadays often quoted with the restriction \( a > b > 0 \) and so has the second exceptional case omitted.)

3. Proof of Theorems 1 and 2

Let \( n \in R^{(j)}_{a,b} \) have a factorisation (1), where \( p_1 < p_2 < \cdots < p_r \) and all \( k_i > 0 \). First take \( j \geq 1 \). Then, by Proposition 12, \( n/p_i^{k_i} = n_r \in R^{(j)}_{a,b} \), and hence

\[
\frac{(n/p_i^{k_i})}{p_i^{k_i-1}} = n_{r-1}, \quad \ldots, \quad p_1^{k_1} = n_2, \quad 1 = n_1
\]

are all in \( R^{(j)}_{a,b} \). Now separate the two cases \( j = 1 \) and \( j \geq 2 \) for Theorems 1 and 2 respectively. Now for \( j = 1 \) Proposition 5 gives us that \( p_i | a^{n_i} - b^{n_i} (i = 1, \ldots, r) \), while for \( j \geq 2 \) we have, again from Proposition 5, that

\[
p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b \text{ if } p_1 > 2; \\ \text{lcm}(a - b, a + b) \text{ if } p_1 = 2,
\end{cases}
\]

and \( p_i^{(j-1)k_i} | a^{n_i} - b^{n_i} (i = 2, \ldots, r) \). Here we have used the fact that \( \gcd(p_i, n_i) = 1 \), so that if \( p_i^{k_i} | (a^{n_i} - b^{n_i})/n_i^2 \) then \( p_i^{k_i} | a^{n_i} - b^{n_i} \) (i.e., we are applying Proposition 5 with all the exponents \( k_p \) equal to 0.)

4. Finding \( R^{(j)}_{a,b} \) When \( \gcd(a, b) > 1 \).

For \( a > 1 \), define the set \( F_a \) to be the set of all \( n \in \mathcal{N} \) whose prime factors all divide \( a \). To find \( R^{(j)}_{a,b} \) in general, we first consider the case \( b = 0 \).
Proposition 14. We have \( R_{a,0}^{(1)} = R_{a,0}^{(2)} = \mathcal{F}_a \), while for \( j \geq 3 \) the set \( R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_a^{(j)} \), where \( S_a^{(j)} \) is a finite set.

Proof. From the condition \( n^j \mid a^n \), all prime factors of \( n \) divide \( a \), so \( R_{a,0}^{(j)} \subseteq \mathcal{F}_a \), say \( R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_a^{(j)} \). We need to prove that \( S_a^{(j)} \) is finite. Suppose that \( a = p_1^{k_1} \cdots p_r^{k_r} \), with \( p_1 \) the smallest prime factor of \( a \). Then \( n = p_1^{k_1} \cdots p_r^{k_r} \) for some \( k_i \geq 0 \). From \( n^j \mid a^n \) we have

\[
k_i \leq \frac{a_i}{j} p_1^{k_1} \cdots p_r^{k_r} \quad (i = 1, \ldots, r).
\]

(17)

For these \( r \) conditions to be satisfied it is sufficient that

\[
\sum_{i=1}^r k_i \leq \frac{\min_{i=1}^r a_i}{j} p_1^{\sum_{i=1}^r k_i}.
\]

(18)

Now (18) holds if \( j = 1 \) or 2, as in this case, from the simple inequality \( k \leq 2^{k-1} \) valid for all \( k \in \mathbb{N} \), we have

\[
\sum_{i=1}^r k_i \leq \frac{1}{2} 2^{\sum_{i=1}^r k_i} \leq \frac{\min_{i=1}^r a_i}{j} p_1^{\sum_{i=1}^r k_i}.
\]

(19)

Hence \( S_a^{(j)} \) is empty if \( j = 1 \) or 2.

Now take \( j \geq 3 \), and let \( K = K_a^{(j)} \) be the smallest integer such that \( K p_1^{-K} \leq (\min_{i=1}^r a_i)/j \). Then (18) holds for \( \sum_{i=1}^r k_i \geq K \), and \( S_a^{(j)} \) is contained in the finite set \( S'' = \{ n \in \mathbb{N}, n = p_1^{k_1} \cdots p_r^{k_r} : \sum_{i=1}^r k_i < K \} \). (To compute \( S_a^{(j)} \) precisely, one need just check for which \( r \)-tuples \((k_1, \ldots, k_r)\) with \( \sum_{i=1}^r k_i < K \) any of the \( r \) inequalities of (17) is violated.)

One (at first sight) curious consequence of the equality \( R_{a,0}^{(1)} = R_{a,0}^{(2)} \) above is that \( n \mid a^n \) implies \( n^2 \mid a^n \).

Now let \( g = \gcd(a, b) \) and \( a = a_1 g, \ b = b_1 g \). Write \( n = G n_1 \), where all prime factors of \( G \) divide \( g \) and \( \gcd(n_1, g) = 1 \). Then we have the following general result.

Theorem 15. The set \( R_{a,b}^{(j)} \) is given by

\[
R_{a,b}^{(j)} = \{ n = G n_1 : G \in \mathcal{F}_g, n_1 \in R_{a_1, b_1}^{(j)} \cap \mathcal{F}_{a_1} \text{ and } \gcd(g, n_1) = 1 \} \setminus R,
\]

(20)
where $R$ is a finite set. Specifically, all $n = Gn_1 \in R$ have $1 \leq n_1 < j/2$ and
\[
G = q_1^{\ell_1} \cdots q_m^{\ell_m},
\]
where
\[
\sum_{i=1}^{m} \ell_i < R_g^{(j)},
\]
Here the $q_i$ are the primes dividing $g$, and $R_g^{(j)}$ is the constant in the proof of Proposition 14 above.

Proof. Supposing that $n \in R_{a,b}^{(j)}$ we have
\[
n^j \mid a^n - b^n
\]
and so $n^j \mid g^n(a_1^n - b_1^n)$. Writing $n = Gn_1$, as above, we have
\[
n_1^j \mid (a_1^G)^{n_1} - (b_1^G)^{n_1}
\]
and
\[
G^j \mid g^{Gn_1}((a_1^G)^{n_1} - (b_1^G)^{n_1}).
\]
Thus (23) holds with $n, a, b$ replaced by $n_1, a_1^G, b_1^G$. So we have reduced the problem of (23) to a case where gcd$(a, b) = 1$, which we can solve for $n_1$ prime to $g$, along with the extra condition (25). Now, from the fact that $R_{g,0}^{(2)} = \mathcal{F}_g$ from Proposition 14, we have $G^2 \mid g^G$ and hence $G^j \mid g^{Gn_1}$ for all $G \in \mathcal{F}_g$, provided that $n_1 \geq j/2$. Hence (25) can fail to hold for all $G \in \mathcal{F}_g$ only for $1 \leq n_1 < j/2$.

Now fix $n_1$ with $1 \leq n_1 < j/2$. Then note that by Proposition 14, $G^j \mid g^{Gn_1}$ and hence (23) holds for all $G \in \mathcal{F}_{g^{n_1}} \setminus S$, where $S$ is a finite set of $G$’s contained in the set of all $G$’s given by (21) and (22).

Note that (taking $n_1 = 1$ and using (25)) we always have $R_{g,0}^{(j)} \subset R_{a,b}^{(j)}$. See example in Section 7.

5. When Are $R_{a,b}^{(1)}$ and $R_{a,b}^{(2)}$ Finite?

First consider $R_{a,b}^{(1)}$. From Theorem 1 it is immediate that $R_{a,b}^{(1)}$ contains all powers of any primes dividing $a - b$. Thus $R_{a,b}^{(1)}$ is infinite unless $a - b = \pm 1$, in which case $R_{a,b}^{(1)} = \{1\}$. This was pointed out earlier by André-Jeannin [1, Corollary 4].

Next, take $j = 2$. Let us denote by $\mathcal{P}_{a,b}^{(2)}$ the set of primes that divide some $n \in R_{a,b}^{(2)}$ and, as before, put $g = \gcd(a, b)$.
Theorem 16. The set \( R_{a,b}^{(2)} = \{1\} \) if and only if \( a \) and \( b \) are consecutive integers, and \( R_{a,b}^{(2)} = \{1, 3\} \) if and only if \( ab = -2 \). Otherwise, \( R_{a,b}^{(2)} \) is infinite.

If \( R_{a,b}^{(2)} = \{1\} \) (respectively, \( \{1, 3\} \)) then \( \mathcal{P}_{a,b}^{(2)} \) is the set of all prime divisors of \( g \) (respectively, \( 3g \)). Otherwise \( \mathcal{P}_{a,b}^{(2)} \) is infinite.

For coprime positive integers \( a, b \) with \( a - b > 1 \), the infiniteness of \( R_{a,b}^{(2)} \) already follows from the above-mentioned results of [5].

The application of Zsigmondy’s Theorem that we require is the following.

Proposition 17. If \( R_{a,b}^{(2)} \) contains some integer \( n \geq 4 \) then both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets.

Proof. First note that if \( a = 2, \ b = 1 \) (or more generally \( a - b = \pm 1 \)) then by Theorem 2, \( R^{(2)} = \{1\} \). Hence, taking \( n \in R_{a,b}^{(2)} \) with \( n \geq 4 \) we have, by Zsigmondy’s Theorem, that \( a^n - b^n \) has a primitive prime divisor, \( p \) say. Now if \( p \mid n \) then, by applying Proposition 12 as many times as necessary we find \( p \mid n' \), where \( n' \in R_{a,b}^{(2)} \) and now \( p \) is the maximal prime divisor of \( n' \). Hence, by Proposition 12 again, \( n'' = n'/p \in R_{a,b}^{(2)} \) and so, from \( n' = pn'' \) and Proposition 5 we have that \( p \mid a^{n''} - b^{n''} \), contradicting the primitivity of \( p \).

Now using Proposition 5 again, \( np \in R_{a,b}^{(2)} \). Repeating the argument with \( n \) replaced by \( np \) and continuing in this way we obtain an infinite sequence

\[
n, \ np, \ np_1, \ np_1p_2, \ldots, \ np_1p_2\ldots p_t, \ldots
\]

of elements of \( R_{a,b}^{(2)} \), where \( p < p_1 < p_2 < \cdots < p_t < \ldots \) are primes. \( \square \)

Proof of Theorem 16. Assume \( \gcd(a, b) = 1 \), and, without loss of generality, that \( a > 0 \) and \( a > b \). (We can ensure this by interchanging \( a \) and \( b \) and/or changing both their signs.) If \( a - b \) is even, then \( a \) and \( b \) are odd, and \( a^2 - b^2 \equiv 1 \pmod{2^{t+1}} \), where \( t \geq 2 \). Hence \( 4 \in R_{a,b}^{(2)} \), by Proposition 5, and so both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets, by Proposition 17.

If \( a - b = 1 \) then \( R^{(2)} = \{1\} \), as we have just seen, above.

If \( a - b \) is odd and at least 5, then \( a - b \) must either be divisible by 9 or by a prime \( p \geq 5 \). Hence 9 or \( p \) belong to \( R_{a,b}^{(2)} \), by Proposition 5, and again both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets, by Proposition 17.

If \( a - b = 3 \) then \( 3 \in R_{a,b}^{(2)} \), and \( a^3 - b^3 = 9(b^2 + 3b + 3) \). If \( b = -1 \) (and \( a = 2, ab = -2 \)) or \( -2 \) (and \( a = 1, ab = -2 \)) then \( a^3 - b^3 = 9 \) and
so, by Theorem 2, so \( R^{(2)} = \{1, 3\} \). Otherwise, using \( \gcd(a, b) = 1 \) we see that \( a^3 - b^3 \geq 5 \), and so the argument for \( a - b \geq 5 \) but with \( a, b \) replaced by \( a^3, b^3 \) applies.

6. The Powers of \( n \) Dividing \( a^n + b^n \)

Define \( R_{a,b}^{(j)} \) to be the set \( \{n \in \mathbb{N} : n^j \) divides \( a^n + b^n \} \). Take \( j \geq 1 \), and assume that \( \gcd(a, b) = 1 \). (The general case \( \gcd(a, b) \geq 1 \) can be handled as in Section 4.) We then have the following result.

**Theorem 18.** Suppose that \( j \geq 1 \), \( \gcd(a, b) = 1 \), \( a > 0 \) and \( a \geq |b| \). Then

(a) \( R_{a,b}^{(1)+} \) consists of the odd elements of \( R_{a,b}^{(1)} \), along with the numbers of the form \( 2n_1 \), where \( n_1 \) is an odd element of \( R_{a^2,b^2}^{(1)} \);

(b) If \( j \geq 2 \) the set \( R_{a,b}^{(j)+} \) consists of the odd elements of \( R_{a,b}^{(j)} \) only.

Furthermore, for \( j = 1 \) and \( 2 \), the set \( R_{a,b}^{(j)+} \) is infinite, except in the following cases:

- If \( a + b \) is 1 or a power of 2, \( (j, a, b) \neq (1, 1, 1) \), when it is \( \{1\} \);
- \( R_{1,1}^{(1)+} = \{1, 2\} \);
- \( R_{2,1}^{(2)+} = \{1, 3\} \).

**Proof.** If \( n \) is even and \( j \geq 2 \), or if \( 4 \mid n \) and \( j = 1 \), then \( n^j \mid a^n + b^n \) implies that \( 4 \mid a^n + b^n \), contradicting the fact that, as \( a \) and \( b \) are not both even, \( a^n + b^n \equiv 1 \) or \( 2 \) (mod 8). So either

- \( n \) is odd, in which case \( n^j \mid a^n + b^n \) is equivalent to finding the odd elements of the set \( R_{a,-b}^{(j)} \);

or

- \( j = 1 \) and \( n = 2n_1 \), where \( n_1 \) is odd, and belongs to \( R_{a^2,-b^2}^{(1)} \).

Now suppose that \( j = 1 \) or \( 2 \). If \( a + b \) is \( \pm 1 \) or \( \pm 2^i \) for some \( i > 0 \), then, by Theorem 2, all \( n \in R_{a,-b}^{(j)} \) with \( n > 1 \) are even, so for \( j = 2 \) there are no \( n > 1 \) with \( n^j \mid a^n + b^n \) in this case. Otherwise, \( a + b \) will have an odd prime factor, and so at least one odd element greater than 1. By Theorem 16 and its proof, we see that \( R_{a,-b}^{(2)} \) will have infinitely many odd elements unless \( a(-b) = -2 \), i.e., \( a = 2 \), \( b = 1 \) (using \( a > 0 \) and \( a \geq |b| \)).
For $j = 1$ there will be infinitely many $n$ with $n | a^n + b^n$, except when both $a + b$ and $a^2 + b^2$ are 1 or a power of 2. It is an easy exercise to check that, this can happen only for $a = b = 1$ or $a = 1$, $b = 0$.

If $g = \gcd(a, b) > 1$, then, since $R_{a,b}^{(j)+}$ contains the set $R_{a,b}^{(j)}$, it will be infinite, by Proposition 14. For $j \geq 3$ and $\gcd(a, b) = 1$, the finiteness of the set $R_{a,b}^{(j)+}$ would follow from the finiteness of $R_{a,b}^{(j)}$, using Theorem 16(b).

7. Examples

The set $R_{a,b}^{(j)}$ has a natural labelled, directed-graph structure, as follows: take the vertices to be the elements of $R_{a,b}^{(j)}$, and join a vertex $n$ to a vertex $np$ as $n \rightarrow_p np$, where $p \in \mathcal{P}_{a,b}^{(j)}(n)$. We reduce this to a spanning tree of this graph by taking only those edges $n \rightarrow_p np$ for which $p$ is the largest prime factor of $np$. For our first example we draw this tree (Figure 1).

1. Consider the set

$$R_{3,1}^{(2)} = \{1, 2, 4, 20, 220, 1220, 2420, 5060, 13420, 14740, 23620, 55660,$$

$$145420, 147620, 162140, 237820, 299820, 290620, 308660,$$

$$339020, 447740, 847220, 899140, 1210220, \ldots \}$$

(sequence A127103 in Neil Sloane’s Integer Sequences website). Now

$$3^{20} - 1 = 2^4 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181,$$

showing that $\mathcal{P}_{3,1}^{(2)}(20) = \{11, 11^2, 61, 1181\}$. Also

$$3^{220} - 1 = 2^4 \cdot 5^2 \cdot 11^3 \cdot 23 \cdot 61 \cdot 67 \cdot 661 \cdot 1181 \cdot 1321 \cdot 3851 \cdot 5501 \cdot 177101 \cdot 570461 \cdot 659671 \cdot 24472341743191 \cdot 560088668384411 \cdot 927319729649066047885192700193701,$$

so that the elements of $\mathcal{P}_{3,1}^{(2)}(220)$ less than $10^6/220$, needed for Figure 1, are

$$11, 23, 61, 67, 661, 1181, 1321, 3851.$$


2. Now

\[ R_{5,-1}^{(2)} = \{1, 2, 3, 4, 6, 12, 21, 42, 52, 84, 156, 186, 372, \ldots \} , \]

whose odd elements give

\[ R_{5,-1}^{(2)+} = \{1, 3, 21, 609, 903, 2667, 9429, 26187, \ldots \} . \]

See Section 6.

3. We have

\[ R_{3,-2}^{(2)+} = R_{3,2}^{(2)} = \{1, 5, 55, 1971145, \ldots \} , \]

as all elements of \( R_{3,-2}^{(2)} \) are odd. Although this set is infinite by Theorem 16, the next term is 1971145\(p\) where \(p\) is the smallest prime factor of \(3^{1971145} + 2^{1971145}\) not dividing 1971145. This looks difficult to compute, as it could be very large.

4. We have

\[ R_{4,-3}^{(2)} = R_{4,3}^{(2)+} = \{1, 7, 2653, \ldots \} . \]

Again, this set is infinite, but here only the three terms given are readily computable. The next term is 2653\(p\) where \(p\) is the smallest prime factor of \(4^{2653} + 3^{2653}\) not dividing 2653.

5. This is an example of a set with more than one odd prime as a squared factor in elements of the set, in this case the primes 3 and 7. Every element greater
than 9 is of one of the forms 21m, 63m, 147m, or 441m, where m is prime to 21;

\[ R_{11,2}^{(2)} = \{1, 3, 9, 21, 63, 147, 441, 609, 1827, 4137, 4263, 7959, 8001, 12411, 12789, 23877, 28959, 35931, 55713, 56007, 86877, 107793, 119973, 167139, 212541, 216237, 230811, 232029, 251517, 359919, 389403, \ldots \}. \]

6. \( R_{27001,1}^{(4)} = \{1, 2, 3, 5, 6, 10, 15, 30\} \). This is because 27001 − 1 = 2^3 \cdot 3^3 \cdot 5^3, and none of 27001^n − 1 has a factor \( p^3 \) for any prime \( p > 5 \) for any \( n = 1, 2, 3, 5, 6, 10, 15, 30 \).

7. \( R_{91,1}^{(3)} = \{1, 2, 3, 6, 42, 1806\} \)? Is this the entire set? Yes, unless \( 19^{1806} - 1 \) is divisible by \( p^2 \) for some prime \( p \) prime to 1806, in which case 1806p would also be in the set. But determining whether or not this is the case seems to be a hard computational problem.

8. \( R_{56,2}^{(4)} \), an example with \( \gcd(a, b) > 1 \). It seems highly probable that

\[ R_{56,2}^{(4)} = (\mathcal{F}_2 \setminus \{2, 4, 8\}) \cup (3\mathcal{F}_2) = 1, 3, 6, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 384, 512, 768, 1024, \ldots \]

However, in order to prove this, Theorem 15 tells us that we need to know that \( 28^{2^\ell} \not\equiv 1 \pmod{p^3} \) for every prime \( p > 3 \) and every \( \ell > 0 \). This seems very difficult! Note that \( R_{2,0}^{(4)} = \mathcal{F}_2 \setminus \{2, 4, 8\} \) and \( R_{28,1}^{(4)} = \{1, 3\} \).

8. Final Remarks

1. By finding \( R_{a,b}^{(j)} \), we are essentially solving the exponential Diophantine equation \( x^j y = a^x - b^x \), since any solutions with \( x \leq 0 \) are readily found.

2. It is known that

\[ R_{a,b}^{(3)} = \left\{ n \in \mathbb{N} : n \text{ divides } \frac{a^n - b^n}{a - b} \right\}. \]

See [11, Proposition 12] (and also André-Jeannin [1, Theorem 2] for some special cases.) This result shows that \( R_{a,b}^{(1)} = \{ n \in \mathbb{N} : n \text{ divides } u_n \} \), where the \( u_n \) are the generalized Fibonacci numbers of the first kind defined by the recurrence \( u_0 = 1, u_1 = 1, \) and \( u_{n+2} = (a + b)u_{n+1} - abu_n \ (n \geq 0) \). This provides a link between Theorem 1 of the present paper and the results of [11].
The set \( R_{a,b}^{(1)+} \) is a special case of a set \( \{ n \in \mathbb{N} : n \text{ divides } v_n \} \), also studied in [11]. Here \( (v_n) \) is the sequence of generalized Fibonacci numbers of the second kind. For earlier work on this topic see Somer [13].

3. Earlier and related work. The study of factors of \( a^n - b^n \) dates back at least to Euler, who proved that all primitive prime factors of \( a^n - b^n \) were \( \equiv 1 \) (mod \( n \)). See [2, Theorem 1]. Chapter 16 of Dickson [4] is devoted to the literature on factors of \( a^n \pm b^n \).

More specifically, Kennedy and Cooper [8] studied the set \( R_{10,1}^{(1)} \). André-Jeannin [1, Corollary 4] claimed (erroneously – see Theorem 18) that the congruence \( a^n + b^n \equiv 0 \) (mod \( n \)) always has infinitely many solutions \( n \) for \( \gcd(a, b) = 1 \).

References


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