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THE DIVISIBILITY OF $a^n - b^n$ BY POWERS OF $n$

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Abstract

For given integers $a, b$ and $j \geq 1$ we determine the set $R_{a,b}^{(j)}$ of integers $n$ for which $a^n - b^n$ is divisible by $n^j$. For $j = 1, 2$, this set is usually infinite; we determine explicitly the exceptional cases for which $a, b$ the set $R_{a,b}^{(j)} (j = 1, 2)$ is finite. For $j = 2$, we use Zsigmondy’s Theorem for this. For $j \geq 3$ and gcd($a, b$) = 1, $R_{a,b}^{(j)}$ is probably always finite; this seems difficult to prove, however.

We also show that determination of the set of integers $n$ for which $a^n + b^n$ is divisible by $n^j$ can be reduced to that of $R_{a,b}^{(j)}$.

1. Introduction

Let $a, b$ and $j$ be fixed integers, with $j \geq 1$. The aim of this paper is to find the set $R_{a,b}^{(j)}$ of all positive integers $n$ such that $n^j$ divides $a^n - b^n$. For $j = 1, 2, \ldots$, these sets are clearly nested, with common intersection $\{1\}$. Our first results (Theorems 1 and 2) describe this set in the case that gcd($a, b$) = 1. In Section 4 we describe (Theorem 15) the set in the general situation where gcd($a, b$) is unrestricted.

**Theorem 1.** Suppose that gcd($a, b$) = 1. Then the elements of the set $R_{a,b}^{(1)}$ consist of those integers $n$ whose prime factorization can be written in the form

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \ (p_1 < p_2 < \cdots < p_r, \ all \ k_i \geq 1), \quad (1)$$

where $p_i \mid (a^{n_i} - b^{n_i})$ ($i = 1, \ldots, r$), with $n_1 = 1$ and $n_i = p_1^{k_1} p_2^{k_2} \cdots p_{i-1}^{k_{i-1}}$ ($i = 2, \ldots, r$).

In this theorem, the $k_i$ are arbitrary positive integers. This result is a more explicit version of that proved in Győry [5], where it was shown that if $a - b > 1$ then for any positive integer $r$ the number of elements of $R_{a,b}^{(1)}$ having $r$ prime factors is infinite. The result is also essentially contained in [11], which described
the indices \( n \) for which the generalized Fibonacci numbers \( u_n \) are divisible by \( n \). However, we present a self-contained proof in this paper.

On the other hand, for \( j \geq 2 \), the exponents \( k_i \) are more restricted.

**Theorem 2.** Suppose that \( \gcd(a, b) = 1 \), and \( j \geq 2 \). Then the elements of the set \( R_{a,b}^{(j)} \) consist of those integers \( n \) whose prime factorization can be written in the form

\[
(p^{(j-1)k_1} \divides a - b \quad \text{if } p_1 > 2; \quad \text{lcm}(a - b, a + b) \quad \text{if } p_1 = 2),
\]

and \( p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i} \), with \( n_i = p_i^{k_1}p_2^{k_2} \cdots p_i^{k_i-1} \) \( (i = 2, \ldots, r) \).

Again, the result was essentially contained in [5], where it was proved that for \( a - b > 1 \) and for any given \( r \), there exists an \( n \in R_{a,b}^{(j)} \) with \( r \) distinct prime factors. Further, the number of these \( n \) is finite, and all of them can be determined. The paper [5] was stimulated by a problem from the 31st International Mathematical Olympiad, which asked for all those positive integers \( n > 1 \) for which \( 2^n + 1 \) was divisible by \( n^2 \). (For the answer, see [5], or Theorem 16.)

Thus we see that construction of \( n \in R_{a,b}^{(j)} \) depends upon finding a prime \( p_i \) not used previously with \( a^{n_i} - b^{n_i} \) being divisible by \( p_i^{j-1} \). This presents no problem for \( j = 2 \), so that \( R_{a,b}^{(2)} \), as well as \( R_{a,b}^{(1)} \), are usually infinite. See Section 5 for details, including the exceptional cases when they are finite. However, for \( j \geq 3 \) the condition \( p_i^{j-1} \mid a^{n_i} - b^{n_i} \) is only rarely satisfied. This suggests strongly that in this case \( R_{a,b}^{(j)} \) is always finite for \( \gcd(a, b) = 1 \). This seems very difficult to prove, even assuming the ABC Conjecture. A result of Ribenboim and Walsh [10] implies that, under ABC, the powerful part of \( a^n - b^n \) cannot often be large. But this is not strong enough for what is needed here. On the other hand, \( R_{a,b}^{(j)} (j \geq 3) \) can be made arbitrarily large by choosing \( a \) and \( b \) such that \( a - b \) is a powerful number. For instance, choosing \( a = 1 + (q_1q_2 \cdots q_s)^{j-1} \) and \( b = 1 \), where \( q_1, q_2, \ldots, q_s \) are distinct primes, then \( R_{a,b}^{(j)} \) contains the \( 2^n \) numbers \( q_1^{\varepsilon_1}q_2^{\varepsilon_2} \cdots q_s^{\varepsilon_s} \), where the \( \varepsilon_i \) are 0 or 1. See Example 6 in Section 7.

In the next section we give preliminary results needed for the proof of the theorems. We prove them in Section 3. In Section 4 we describe (Theorem 15) \( R_{a,b}^{(j)} \) where \( \gcd(a, b) \) is unrestricted. In Section 5 we find all \( a, b \) for which \( R_{a,b}^{(2)} \) is finite (Theorem 16). In Section 6 we discuss the divisibility of \( a^n + b^n \) by powers of \( n \). In Section 7 we give some examples, and make some final remarks in Section 8.
2. Preliminary Results

We first prove a version of Fermat’s Little Theorem that gives a little bit more information in the case \( x \equiv 1 \pmod{p} \).

Lemma 3. For \( x \in \mathbb{Z} \) and \( p \) an odd prime we have

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv \begin{cases} 
p \pmod{p^2} & \text{if } x \equiv 1 \pmod{p}; \\
1 \pmod{p} & \text{otherwise}.
\end{cases}
\] (2)

Proof. If \( x \equiv 1 \pmod{p} \), say \( x = 1 + kp \), then \( x^j \equiv 1 + jkp \pmod{p^2} \), so that

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv p + kp \sum_{j=0}^{p-1} j \equiv p \pmod{p^2}.
\] (3)

Otherwise

\[
x(x - 1)(x^{p-2} + \cdots + x + 1) = x^p - x \equiv 0 \pmod{p},
\] (4)

so that for \( x \not\equiv 1 \pmod{p} \) we have \( x(x^{p-2} + \cdots + x + 1) \equiv 0 \pmod{p} \), and hence

\[
x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv x(x^{p-2} + \cdots + x + 1) + 1 \equiv 1 \pmod{p}.
\] (5)

\[\square\]

The following is a result of Birkhoff and Vandiver [2, Theorem III]. It is also special case of Lucas [9, p. 210], as corrected for \( p = 2 \) by Carmichael [3, Theorem X].

Lemma 4. Let \( \gcd(a, b) = 1 \) and \( p \) be prime with \( p \nmid (a - b) \). Define \( t > 0 \) by \( p^t || (a - b) \) for \( p > 2 \) and \( 2^t || \lcm(a - b, a + b) \) if \( p = 2 \). Then for \( \ell > 0 \)

\[
p^{t+\ell} || (a^{p^\ell} - b^{p^\ell}).
\] (6)

On the other hand, if \( p \nmid a - b \), then for \( \ell \geq 0 \)

\[
p \nmid a^{p^\ell} - b^{p^\ell}.
\] (7)

Proof. Put \( x = a/b \). First suppose that \( p \) is odd and \( p^t || (a - b) \) for some \( t > 0 \). Then as \( \gcd(a, b) = 1 \), \( b \) is not divisible by \( p \), and we have \( x \equiv 1 \pmod{p^t} \). Then from

\[
a^p - b^p = (a - b)b^{p-1}(x^{p-1} + x^{p-2} + \cdots + x + 1)
\] (8)
we have by Lemma 3 that \( p^{\ell+1} \| (a^p - b^p) \). Applying this result \( \ell \) times, we obtain (6).

For \( p = 2 \), we have \( p^{\ell+1} \| a^2 - b^2 \) and from \( a^2 \equiv b^2 \equiv 1 \) (mod 8), we obtain \( 2^1 \| (a^4 + b^4) \), and so \( p^{\ell+2} \| (a^4 - b^4) \). An easy induction then gives the required result.

Now suppose that \( p \nmid (a - b) \). Since \( \gcd(a, b) = 1 \), (7) clearly holds if \( p = 2 \) or \( p \nmid b \), as must happen for \( p = 2 \). So we can assume that \( p \) is odd and \( p \nmid b \). Then \( x \not\equiv 1 \) (mod \( p \)) so that, by Lemma 3 and (8), we have \( p \nmid (a^p - b^p) \). Applying this argument \( \ell \) times, we obtain (7).

For \( n \in R_{a,b}^{(j)} \), we now define the set \( \mathcal{P}_{a,b}^{(j)}(n) \) to be the set of all prime powers \( p^k \) for which \( np^k \in R_{a,b}^{(j)} \). Our next result describes this set precisely. (Compare with [11, Theorem 1(a)]).

**Proposition 5.** Suppose that \( j \geq 1 \), \( \gcd(a, b) = 1 \), \( n \in R_{a,b}^{(j)} \) and

\[
a^n - b^n = 2^{e_2} \prod_{p>2} p^{e_p}, \quad n = \prod_p p^{k_p}
\]

and define \( e_2 \) by \( 2^{e_2} \| \text{lcm}(a^n - b^n, a^n + b^n) \). Then

\[
\mathcal{P}^{(1)}(n) = \bigcup_{p|a^n-b^n} \{ p^k, k \in \mathbb{N} \},
\]

and for \( j \geq 2 \)

\[
\mathcal{P}_{a,b}^{(j)}(n) = \bigcup_{p: p^{j-1}|a^n-b^n} \left\{ p^k : 1 \leq k \leq \left\lfloor \frac{e_p - jk_p}{j-1} \right\rfloor \right\}.
\]

Note that \( e_2 \) is never 1. Consequently, if \( 2m \in R_{a,b}^{(2)} \), where \( m \) is odd, then \( 4m \in R_{a,b}^{(2)} \). Also, \( 2 \in R_{a,b}^{(j)} \) for \( j \leq 3 \) when \( a - b \) is even.

**Proof.** Taking \( n \in R_{a,b}^{(j)} \) we have, from (9) and the definition of \( e_2 \), that \( jk_p \leq e_p \) for all primes \( p \). Hence, applying Lemma 4 with \( a, b \) replaced by \( a^n, b^n \) we have for \( p \) dividing \( a^n - b^n \) that for \( \ell > 0 \)

\[
p^{e_p+\ell} \| (a^{np^\ell} - b^{np^\ell}).
\]

So \( (np^\ell)^j \mid (a^{np^\ell} - b^{np^\ell}) \) is equivalent to \( j(k_p + \ell) \leq e_p + \ell \), or \( (j-1)\ell \leq e_p - jk_p \). Thus we obtain (10) for \( j \geq 2 \), with \( \ell \) unrestricted for \( j = 1 \), giving (10).

On the other hand, if \( p \nmid (a^n - b^n) \), then by Lemma 4 again, \( p^\ell \nmid (a^{np^\ell} - b^{np^\ell}) \), so that certainly \((np^\ell)^j \nmid (a^{np^\ell} - b^{np^\ell}) \). \( \square \)
We now recall some facts about the order function $\text{ord}$. For $m$ an integer greater than 1 and $x$ an integer prime to $m$, we define $\text{ord}_m(x)$, the order of $x$ modulo $m$, to be the least positive integer $h$ such that $x^h \equiv 1 \pmod{m}$. The next three lemmas, containing standard material on the ord function, are included for completeness.

**Lemma 6.** For $x \in \mathbb{N}$ and prime to $m$, we have $m \mid (x^n - 1)$ if and only if $\text{ord}_m(x) \mid n$.

**Proof.** Let $\text{ord}_m(x) = h$, and assume that $m \mid (x^n - 1)$. Then as $m \mid (x^h - 1)$, also $m \mid (x^{\text{gcd}(h,n)} - 1)$. By the minimality of $h$, $\text{gcd}(h,n) = h$, i.e., $h \mid n$. Conversely, if $h \mid n$ then $(x^h - 1) \mid (x^n - 1)$, so that $m \mid (x^n - 1)$. □

**Corollary 7.** Let $j \geq 1$. We have $n^j \mid (x^n - 1)$ if and only if $\text{gcd}(x,n) = 1$ and $\text{ord}_{n^j}(x) \mid n$.

**Lemma 8.** For $m = \prod_p p^t_p$ and $x \in \mathbb{N}$ and prime to $m$ we have

\[ \text{ord}_m(x) = \text{lcm}_p \text{ord}_{p^t_p}(x). \]  

(13)

**Proof.** Put $h = \text{ord}_{p^t_p}(x)$, $h = \text{ord}_m(x)$ and $h' = \text{lcm}_p h_p$. Then by Lemma 6 we have $p^t_p \mid (x^{h'} - 1)$ for all $p$, and hence $m \mid (x^{h'} - 1)$. Hence $h \mid h'$. On the other hand, as $p^t_p \mid n$ and $m \mid (x^h - 1)$, we have $p^t_p \mid (x^h - 1)$, and so $h_p \mid h$, by Lemma 6. Hence $h' = \text{lcm}_p h_p \mid h$. □

Now put $p_\star = \text{ord}_p(x)$, and define $t > 0$ by $p^t_p \|(x^{p_\star} - 1)$.

**Lemma 9.** For $\text{gcd}(x,n) = 1$ and $\ell > 0$ we have $p_\star \mid (p - 1)$ and $\text{ord}_{p_\star}(x) = p^{|\text{max}(\ell-t,0)|}_p$.

**Proof.** Since $p \mid (x^{p-1} - 1)$, we have $p_\star \mid (p - 1)$, by Lemma 6. Also, from $p^\ell \mid (x^{\text{ord}_p(x)} - 1)$ we have $p \mid (x^{\text{ord}_p(x)} - 1)$, and so, by Lemma 6 again, $p_\star = \text{ord}_p(x) \mid \text{ord}_p(x)$. Further, if $\ell \leq t$ then from $p^\ell \mid (x^{p_\star} - 1)$ we have by Lemma 6 that $\text{ord}_{p_\star}(x) \mid p_\star$, so $\text{ord}_{p_\star}(x) = p_\star$. Further, by Lemma 4 for $u \geq t$

\[ p^u\|(x^{p^{u-t}p_\star} - 1), \]

(14)

so that, taking $u = \ell \geq t$ and using Lemma 6, $\text{ord}_{p_\star}(x) \mid p^{\ell-t}p_\star$. Also, if $t \leq u < \ell$, then, from (14), $x^{p^{u-t}p_\star} \not\equiv 1 \pmod{p^\ell}$. Hence $\text{ord}_{p_\star}(x) = p^{\ell-t}p_\star$ for $\ell \geq t$. □
Corollary 10. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and \( x \in \mathbb{N} \) prime to \( n \) we have \( n^j | x^n - 1 \) if and only if \( \gcd(x, n) = 1 \) and

\[
\text{lcm}_p p^{k'_p} | \prod_p p^{k_p}.
\]  

(15)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

Note that \( p_s, k'_p \) and \( t_p \) in general depend on \( x \) and \( j \) as well as on \( p \).

What we actually need in our situation is the following variant of Corollary 10.

Corollary 11. Let \( j \geq 1 \). For \( n = \prod_p p^{k_p} \) and integers \( a, b \) with \( \gcd(a, b) = 1 \) we have \( n^j | a^n - b^n \) if and only if \( \gcd(n, a) = \gcd(n, b) = 1 \) and

\[
\text{lcm}_p p^{k'_p} | \prod_p p^{k_p}.
\]  

(16)

Here the \( k'_p = \max(jk_p - t_p, 0) \) are integers with \( t_p > 0 \).

In this corollary, the \( x \) used to define \( p_s \) and \( t = t_p \) (see after Lemma 8) is chosen to satisfy \( bx \equiv a \pmod{n^j} \). The result is then easily deduced from Corollary 10.

By contrast with Proposition 5, our next proposition allows us to divide an element \( n \in R^{(j)}_{a,b} \) by a prime, and remain within \( R^{(j)}_{a,b} \).

Proposition 12. Let \( n \in R^{(j)}_{a,b} \) with \( n > 1 \), and suppose \( p_{\text{max}} \) is the largest prime factor of \( n \). Then \( n/p_{\text{max}} \in R^{(j)}_{a,b} \).

Proof. Suppose \( n \in R^{(j)}_{a,b} \), so that (15) holds, with \( x = a/b \), and put \( q = p_{\text{max}} \). Then, since for every \( p \) all prime factors of \( p_s \) are less than \( p \), the only possible term on the left-hand side that divides \( q^k \) on the right-hand side is the term \( q^k \).

Now reducing \( k_q \) by 1 will reduce \( k'_q \) by at least 1, unless it is already 0, when it does not change. In either case (15) will still hold with \( n \) replaced by \( n/q \), and so \( n/q \in R^{(j)}_{a,b} \).

Various versions and special cases of Proposition 12 for \( j = 1 \) have been known for some time, in the more general setting of Lucas sequences, due to Somer [12, Theorem 5(iv)], Jarden [7, Theorem E], Hoggatt and Bergum [6], Walsh [14], André-Jeannin [1] and others. See also Smyth [11, Theorem 3].

In order to work out for which \( a, b \) the set \( R^{(j)}_{a,b} \) is finite, we need the following classical result. Recall that \( a^n - b^n \) is said to have a primitive prime divisor \( p \) if the prime \( p \) divides \( a^n - b^n \) but does not divide \( a^k - b^k \) for any \( k \) with \( 1 \leq k < n \).
Theorem 13 (Zsigmondy [15]). Suppose that \(a\) and \(b\) are nonzero coprime integers with \(a > b\) and \(a + b > 0\). Then, except when

- \(n = 2\) and \(a + b\) is a power of 2

or

- \(n = 3\), \(a = 2\), \(b = -1\)

or

- \(n = 6\), \(a = 2\), \(b = 1\),

\(a^n - b^n\) has a primitive prime divisor.

(Note that in this statement we have allowed \(b\) to be negative, as did Zsigmondy. His theorem is nowadays often quoted with the restriction \(a > b > 0\) and so has the second exceptional case omitted.)

3. Proof of Theorems 1 and 2

Let \(n \in R_{a,b}^{(j)}\) have a factorisation (1), where \(p_1 < p_2 < \cdots < p_r\) and all \(k_i > 0\). First take \(j \geq 1\). Then, by Proposition 12, \(n/p_r^{k_r} = n_r \in R_{a,b}^{(j)}\), and hence

\[
\frac{n}{p_r^{k_r}}/p_{r-1}^{k_{r-1}} = n_{r-1}, \ldots, \quad p_1^{k_1} = n_2, \quad 1 = n_1
\]

are all in \(R_{a,b}^{(j)}\). Now separate the two cases \(j = 1\) and \(j \geq 2\) for Theorems 1 and 2 respectively. Now for \(j = 1\) Proposition 5 gives us that \(p_i \mid a^{n_i} - b^{n_i} \quad (i = 1, \ldots, r)\), while for \(j \geq 2\) we have, again from Proposition 5, that

\[
p_1^{(j-1)k_1} \text{ divides } \begin{cases} a - b & \text{if } p_1 > 2; \\ \text{lcm}(a - b, a + b) & \text{if } p_1 = 2, \end{cases}
\]

and \(p_i^{(j-1)k_i} \mid a^{n_i} - b^{n_i} \quad (i = 2, \ldots, r)\). Here we have used the fact that \(\gcd(p_i, n_i) = 1\), so that if \(p_i^{k_i} \mid (a^{n_i} - b^{n_i})/n_i^2\) then \(p_i^{k_i} \mid a^{n_i} - b^{n_i}\) (i.e., we are applying Proposition 5 with all the exponents \(k_p\) equal to 0.)

4. Finding \(R_{a,b}^{(j)}\) When \(\gcd(a, b) > 1\).

For \(a > 1\), define the set \(\mathcal{F}_a\) to be the set of all \(n \in \mathcal{N}\) whose prime factors all divide \(a\). To find \(R_{a,b}^{(j)}\) in general, we first consider the case \(b = 0\).
Proposition 14. We have $R_{a,0}^{(1)} = R_{a,0}^{(2)} = \mathcal{F}_a$, while for $j \geq 3$ the set $R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_{a}^{(j)}$, where $S_{a}^{(j)}$ is a finite set.

Proof. From the condition $n^j \mid a^n$, all prime factors of $n$ divide $a$, so $R_{a,0}^{(j)} \subset \mathcal{F}_a$, say $R_{a,0}^{(j)} = \mathcal{F}_a \setminus S_{a}^{(j)}$. We need to prove that $S_{a}^{(j)}$ is finite. Suppose that $a = p_1^{k_1} \cdots p_r^{k_r}$, with $p_1$ the smallest prime factor of $a$. Then $n = p_1^{k_1} \cdots p_r^{k_r}$ for some $k_i \geq 0$. From $n^j \mid a^n$ we have

$$k_i \leq \frac{a_i}{j} p_1^{i k_i} \cdots p_r^{i k_r} \quad (i = 1, \ldots, r). \tag{17}$$

For these $r$ conditions to be satisfied it is sufficient that

$$\sum_{i=1}^{r} k_i \leq \frac{\min_{i=1}^{r} a_i}{j} \sum_{i=1}^{r} k_i. \tag{18}$$

Now (18) holds if $j = 1$ or 2, as in this case, from the simple inequality $k \leq 2^{k-1}$ valid for all $k \in \mathbb{N}$, we have

$$\sum_{i=1}^{r} k_i \leq \frac{1}{2} 2^{\sum_{i=1}^{r} k_i} \leq \frac{\min_{i=1}^{r} a_i}{j} \sum_{i=1}^{r} k_i. \tag{19}$$

Hence $S_{a}^{(j)}$ is empty if $j = 1$ or 2.

Now take $j \geq 3$, and let $K = K_{a}^{(j)}$ be the smallest integer such that $K p_1^{-K} \leq (\min_{i=1}^{r} a_i)/j$. Then (18) holds for $\sum_{i=1}^{r} k_i \geq K$, and $S_{a}^{(j)}$ is contained in the finite set $S'' = \{ n \in \mathbb{N}, n = p_1^{k_1} \cdots p_r^{k_r} : \sum_{i=1}^{r} k_i < K \}$. (To compute $S_{a}^{(j)}$ precisely, one need just check for which $r$-tuples $(k_1, \ldots, k_r)$ with $\sum_{i=1}^{r} k_i < K$ any of the $r$ inequalities of (17) is violated.)

One (at first sight) curious consequence of the equality $R_{a,0}^{(1)} = R_{a,0}^{(2)}$ above is that $n \mid a^n$ implies $n^2 \mid a^n$.

Now let $g = \gcd(a, b)$ and $a = a_1 g$, $b = b_1 g$. Write $n = G n_1$, where all prime factors of $G$ divide $g$ and $\gcd(n_1, g) = 1$. Then we have the following general result.

Theorem 15. The set $R_{a,b}^{(j)}$ is given by

$$R_{a,b}^{(j)} = \{ n = G n_1 : G \in \mathcal{F}_g, n_1 \in R_{a_1, b_1}^{(j)} \text{ and } \gcd(g, n_1) = 1 \} \setminus R, \quad \tag{20}$$
where $R$ is a finite set. Specifically, all $n = G n_1 \in R$ have $1 \leq n_1 < j/2$ and

$$G = q_1^{\ell_1} \cdots q_m^{\ell_m},$$  \hspace{1cm} (21)

where

$$\sum_{i=1}^{m} \ell_i < R_g^{(j)}.$$  \hspace{1cm} (22)

Here the $q_i$ are the primes dividing $g$, and $R_g^{(j)}$ is the constant in the proof of Proposition 14 above.

\textbf{Proof.} Supposing that $n \in R_{a,b}^{(j)}$ we have

$$n^j \mid a^n - b^n$$  \hspace{1cm} (23)

and so $n^j \mid g^n(a_1^n - b_1^n)$. Writing $n = G n_1$, as above, we have

$$n_1^j \mid (a_1^{G})^{n_1} - (b_1^{G})^{n_1}$$  \hspace{1cm} (24)

and

$$G^j \mid g^{G n_1} \left((a_1^{G})^{n_1} - (b_1^{G})^{n_1}\right).$$  \hspace{1cm} (25)

Thus (23) holds with $n, a, b$ replaced by $n_1, a_1^{G}, b_1^{G}$. So we have reduced the problem of (23) to a case where $\gcd(a, b) = 1$, which we can solve for $n_1$ prime to $g$, along with the extra condition (25). Now, from the fact that $R_{g,0}^{(2)} = \mathcal{F}_g$ from Proposition 14, we have $G^2 \mid g^G$ and hence $G^j \mid g^{G n_1}$ for all $G \in \mathcal{F}_g$, provided that $n_1 \geq j/2$. Hence (25) can fail to hold for all $G \in \mathcal{F}_g$ only for $1 \leq n_1 < j/2$.

Now fix $n_1$ with $1 \leq n_1 < j/2$. Then note that by Proposition 14, $G^j \mid g^{G n_1}$ and hence (23) holds for all $G \in \mathcal{F}_g \setminus S$, where $S$ is a finite set of $G$’s contained in the set of all $G$’s given by (21) and (22).

Note that (taking $n_1 = 1$ and using (25)) we always have $R_{g,0}^{(j)} \subset R_{a,b}^{(j)}$. See example in Section 7.

\textbf{5. When Are $R_{a,b}^{(1)}$ and $R_{a,b}^{(2)}$ Finite?}

First consider $R_{a,b}^{(1)}$. From Theorem 1 it is immediate that $R_{a,b}^{(1)}$ contains all powers of any primes dividing $a - b$. Thus $R_{a,b}^{(1)}$ is infinite unless $a - b = \pm 1$, in which case $R_{a,b}^{(1)} = \{1\}$. This was pointed out earlier by André-Jeannin [1, Corollary 4].

Next, take $j = 2$. Let us denote by $\mathcal{P}_{a,b}^{(2)}$ the set of primes that divide some $n \in R_{a,b}^{(2)}$ and, as before, put $g = \gcd(a, b)$.
Theorem 16. The set \( R_{a,b}^{(2)} = \{1\} \) if and only if \( a \) and \( b \) are consecutive integers, and \( R_{a,b}^{(2)} = \{1,3\} \) if and only if \( ab = -2 \). Otherwise, \( R_{a,b}^{(2)} \) is infinite.

If \( R_{a,b}^{(2)} \) and \( b/g \) = \( \{1\} \) (respectively, \( \{1,3\} \)) then \( \mathcal{P}_{a,b}^{(2)} \) is the set of all prime divisors of \( g \) (respectively, 3g). Otherwise \( \mathcal{P}_{a,b}^{(2)} \) is infinite.

For coprime positive integers \( a, b \) with \( a - b > 1 \), the infiniteness of \( R_{a,b}^{(2)} \) already follows from the above-mentioned results of [5].

The application of Zsigmondy’s Theorem that we require is the following.

Proposition 17. If \( R_{a,b}^{(2)} \) contains some integer \( n \geq 4 \) then both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets.

Proof. First note that if \( a = 2, b = 1 \) (or more generally \( a - b = \pm 1 \)) then by Theorem 2, \( R_{a,b}^{(2)} = \{1\} \). Hence, taking \( n \in R_{a,b}^{(2)} \) with \( n \geq 4 \) we have, by Zsigmondy’s Theorem, that \( a^n - b^n \) has a primitive prime divisor, \( p \) say. Now if \( p | n \) then, by applying Proposition 12 as many times as necessary we find \( p | n' \), where \( n' \in R_{a,b}^{(2)} \) and now \( p \) is the maximal prime divisor of \( n' \). Hence, by Proposition 12 again, \( n'' = n'/p \in R_{a,b}^{(2)} \) and so, from \( n' = pn'' \) and Proposition 5 we have that \( p | a^{n''} - b^{n''} \), contradicting the primitivity of \( p \).

Now using Proposition 5 again, \( np \in R_{a,b}^{(2)} \). Repeating the argument with \( n \) replaced by \( np \) and continuing in this way we obtain an infinite sequence

\[
n, \quad np, \quad np_1, \quad np_1p_2, \quad \ldots, \quad np_1p_2\ldots p_t, \quad \ldots
\]

of elements of \( R_{a,b}^{(2)} \), where \( p < p_1 < p_2 < \cdots < p_t < \ldots \) are primes.$\square$

Proof of Theorem 16. Assume \( \gcd(a,b) = 1 \), and, without loss of generality, that \( a > 0 \) and \( a > b \). (We can ensure this by interchanging \( a \) and \( b \) and/or changing both their signs.) If \( a - b \) is even, then \( a \) and \( b \) are odd, and \( a^2 - b^2 \equiv 1 \) (mod \( 2^{t + 1} \)), where \( t \geq 2 \). Hence \( 4 \in R_{a,b}^{(2)} \), by Proposition 5, and so both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets, by Proposition 17.

If \( a - b = 1 \) then \( R_{a,b}^{(2)} = \{1\} \), as we have just seen, above.

If \( a - b \) is odd and at least 5, then \( a - b \) must either be divisible by 9 or by a prime \( p \geq 5 \). Hence 9 or \( p \) belong to \( R_{a,b}^{(2)} \), by Proposition 5, and again both \( R_{a,b}^{(2)} \) and \( \mathcal{P}_{a,b}^{(2)} \) are infinite sets, by Proposition 17.

If \( a - b = 3 \) then \( 3 \in R_{a,b}^{(2)} \), and \( a^3 - b^3 = 9(b^2 + 3b + 3) \). If \( b = -1 \) (and \( a = 2, ab = -2 \)) or \( -2 \) (and \( a = 1, ab = -2 \)) then \( a^3 - b^3 = 9 \) and
so, by Theorem 2, so \( R^{(2)} = \{1, 3\} \). Otherwise, using \( \gcd(a, b) = 1 \) we see that 
\[ a^3 - b^3 \geq 5, \]
and so the argument for \( a - b \geq 5 \) but with \( a, b \) replaced by \( a^3, b^3 \) applies.

\[ \square \]

6. The Powers of \( n \) Dividing \( a^n + b^n \)

Define \( R_{a,b}^{(j)+} \) to be the set \( \{n \in \mathbb{N} : n^j \text{ divides } a^n + b^n \} \). Take \( j \geq 1 \), and assume that \( \gcd(a, b) = 1 \). (The general case \( \gcd(a, b) \geq 1 \) can be handled as in Section 4.) We then have the following result.

**Theorem 18.** Suppose that \( j \geq 1, \gcd(a, b) = 1, a > 0 \) and \( a \geq |b| \). Then

(a) \( R_{a,b}^{(1)+} \) consists of the odd elements of \( R_{a,-b}^{(1)+} \), along with the numbers of the form \( 2n_1 \), where \( n_1 \) is an odd element of \( R_{a^2,-b^2}^{(1)+} \);

(b) If \( j \geq 2 \) the set \( R_{a,b}^{(j)+} \) consists of the odd elements of \( R_{a,-b}^{(j)} \) only.

Furthermore, for \( j = 1 \) and \( 2 \), the set \( R_{a,b}^{(j)+} \) is infinite, except in the following cases:

- If \( a + b = 1 \) or a power of 2, \((j, a, b) \neq (1, 1, 1)\), when it is \( \{1\}\);
- \( R_{1,1}^{(1)+} = \{1, 2\} \);
- \( R_{2,1}^{(2)+} = \{1, 3\} \).

**Proof.** If \( n \) is even and \( j \geq 2 \), or if \( 4 \mid n \) and \( j = 1 \), then \( n^j \mid a^n + b^n \) implies that \( 4 \mid a^n + b^n \), contradicting the fact that, as \( a \) and \( b \) are not both even, \( a^n + b^n \equiv 1 \) or \( 2 \) (mod 8). So either

- \( n \) is odd, in which case \( n^j \mid a^n + b^n \) is equivalent to finding the odd elements of \( R_{a,-b}^{(j)} \);
- or

\[ j = 1 \text{ and } n = 2n_1, \text{ where } n_1 \text{ is odd, and belongs to } R_{a^2,-b^2}^{(1)}. \]

Now suppose that \( j = 1 \) or \( 2 \). If \( a + b \) is \( \pm 1 \) or \( \pm 2^i \) for some \( i > 0 \), then, by Theorem 2, all \( n \in R_{a,-b}^{(j)} \) with \( n > 1 \) are even, so for \( j = 2 \) there are no \( n > 1 \) with \( n^j \mid a^n + b^n \) in this case. Otherwise, \( a + b \) will have an odd prime factor, and so at least one odd element greater than 1. By Theorem 16 and its proof, we see that \( R_{a,-b}^{(2)} \) will have infinitely many odd elements unless \( a(-b) = -2 \), i.e., \( a = 2, b = 1 \) (using \( a > 0 \) and \( a \geq |b| \)).
For \( j = 1 \) there will be infinitely many \( n \) with \( n \mid a^n + b^n \), except when both \( a + b \) and \( a^2 + b^2 \) are 1 or a power of 2. It is an easy exercise to check that, this can happen only for \( a = b = 1 \) or \( a = 1, b = 0 \). \( \square \)

If \( g = \gcd(a, b) > 1 \), then, since \( R_{a,b}^{(j)^+} \) contains the set \( R_{a,b}^{(j)} \), it will be infinite, by Proposition 14. For \( j \geq 3 \) and \( \gcd(a, b) = 1 \), the finiteness of the set \( R_{a,b}^{(j)^+} \) would follow from the finiteness of \( R_{a,b}^{(j)} \), using Theorem 16(b).

7. Examples

The set \( R_{a,b}^{(j)} \) has a natural labelled, directed-graph structure, as follows: take the vertices to be the elements of \( R_{a,b}^{(j)} \), and join a vertex \( n \) to a vertex \( np \) as \( n \rightarrow_p np \), where \( p \in \mathcal{P}_{a,b}^{(j)}(n) \). We reduce this to a spanning tree of this graph by taking only those edges \( n \rightarrow_p np \) for which \( p \) is the largest prime factor of \( np \). For our first example we draw this tree (Figure 1).

1. Consider the set

\[
R_{3,1}^{(2)} = \{1, 2, 4, 20, 220, 1220, 2420, 5060, 13420, 14740, 23620, 55660, 145420, 147620, 162140, 237820, 259820, 290620, 308660, 339020, 447740, 847220, 899140, 1210220, \ldots \}
\]

(sequence A127103 in Neil Sloane’s Integer Sequences website). Now

\[
3^{20} - 1 = 2^4 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181,
\]

dislaying that \( \mathcal{P}_{3,1}^{(2)}(20) = \{11, 11^2, 61, 1181\} \). Also

\[
3^{220} - 1 = 2^4 \cdot 5^2 \cdot 11^3 \cdot 23 \cdot 61 \cdot 67 \cdot 661 \cdot 1181 \cdot 1321 \cdot 3851 \cdot 5501
\]

\[
\cdot 177101 \cdot 570461 \cdot 659671 \cdot 24472341743191 \cdot 560088668384411
\]

\[
\cdot 927319729649066047885192700193701,
\]

so that the elements of \( \mathcal{P}_{3,1}^{(2)}(220) \) less than \( 10^6/220 \), needed for Figure 1, are

\[
11, 23, 61, 67, 661, 1181, 1321, 3851.
\]
2. Now
\[ R_{5, -1}^{(2)} = \{1, 2, 3, 4, 6, 12, 21, 42, 52, 84, 156, 186, 372, \ldots \}, \]
whose odd elements give
\[ R_{5, -1}^{(2)} = \{1, 3, 21, 609, 903, 2667, 9429, 26187, \ldots \}. \]
See Section 6.

3. We have
\[ R_{3, 2}^{(2)} = R_{3, -2}^{(2)} = \{1, 5, 55, 1971145, \ldots \}, \]
as all elements of \( R_{3, -2}^{(2)} \) are odd. Although this set is infinite by Theorem 16, the next term is 1971145p where p is the smallest prime factor of \( 3^{1971145} + 2^{1971145} \) not dividing 1971145. This looks difficult to compute, as it could be very large.

4. We have
\[ R_{4, -3}^{(2)} = R_{4, 3}^{(2)} = \{1, 7, 2653, \ldots \}. \]
Again, this set is infinite, but here only the three terms given are readily computable. The next term is 2653p where p is the smallest prime factor of \( 4^{2653} + 3^{2653} \) not dividing 2653.

5. This is an example of a set with more than one odd prime as a squared factor in elements of the set, in this case the primes 3 and 7. Every element greater
than 9 is of one of the forms 21m, 63m, 147m, or 441m, where m is prime to 21:

\[ R_{11,2}^{(2)} = \{1, 3, 9, 21, 63, 147, 441, 609, 1827, 4137, 4263, 7959, \\
8001, 12411, 12789, 23877, 28959, 35931, 55713, 56007, \\
86877, 107793, 119973, 167139, 212541, 216237, 230811, \\
232029, 251517, 359919, 389403, \ldots \}. \]

6. \( R_{27001,1}^{(4)} = \{1, 2, 3, 5, 6, 10, 15, 30\} \). This is because 27001 − 1 = \( 2^3 \cdot 3^3 \cdot 5^3 \), and none of 27001\(n \) − 1 has a factor \( p^3 \) for any prime \( p > 5 \) for any \( n = 1, 2, 3, 5, 6, 10, 15, 30. \)

7. \( R_{19,1}^{(3)} = \{1, 2, 3, 6, 42, 1806\}? \) Is this the entire set? Yes, unless \( 19^{1806} − 1 \) is divisible by \( p^2 \) for some prime \( p \) prime to 1806, in which case 1806\(p \) would also be in the set. But determining whether or not this is the case seems to be a hard computational problem.

8. \( R_{56,2}^{(4)} \), an example with \( \gcd(a, b) > 1 \). It seems highly probable that

\[
R_{56,2}^{(4)} = (\mathcal{F}_2 \setminus \{2, 4, 8\}) \cup (3\mathcal{F}_2) \\
= 1, 3, 6, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 384, 512, 768, 1024, \ldots .
\]

However, in order to prove this, Theorem 15 tells us that we need to know that \( 28^{2\ell} \not\equiv 1 \pmod{p^3} \) for every prime \( p > 3 \) and every \( \ell > 0. \) This seems very difficult! Note that \( R_{2,0}^{(4)} = \mathcal{F}_2 \setminus \{2, 4, 8\} \) and \( R_{28,1}^{(4)} = \{1, 3\}. \)

8. Final Remarks

1. By finding \( R_{a,b}^{(j)} \), we are essentially solving the exponential Diophantine equation \( x^j y = a^x - b^x \), since any solutions with \( x \leq 0 \) are readily found.

2. It is known that

\[
R_{a,b}^{(3)} = \left\{ n \in \mathbb{N} : n \text{ divides } \frac{a^n - b^n}{a - b} \right\} .
\]

See [11, Proposition 12] (and also André-Jeannin [1, Theorem 2] for some special cases.) This result shows that \( R_{a,b}^{(1)} = \{n \in \mathbb{N} : n \text{ divides } u_n\}, \) where the \( u_n \) are the generalized Fibonacci numbers of the first kind defined by the recurrence \( u_0 = 1, u_1 = 1, \) and \( u_{n+2} = (a + b)u_{n+1} - abu_n \) \( (n \geq 0) \). This provides a link between Theorem 1 of the present paper and the results of [11].
The set $R_{a,b}^{(1)_+}$ is a special case of a set \( \{ n \in \mathbb{N} : n \text{ divides } v_n \} \), also studied in [11]. Here \( (v_n) \) is the sequence of generalized Fibonacci numbers of the second kind. For earlier work on this topic see Somer [13].

3. Earlier and related work. The study of factors of $a^n - b^n$ dates back at least to Euler, who proved that all primitive prime factors of $a^n - b^n$ were $\equiv 1 \pmod{n}$. See [2, Theorem 1]. Chapter 16 of Dickson [4] is devoted to the literature on factors of $a^n \pm b^n$.

More specifically, Kennedy and Cooper [8] studied the set $R_{10,1}^{(1)}$. André-Jeannin [1, Corollary 4] claimed (erroneously – see Theorem 18) that the congruence $a^n + b^n \equiv 0 \pmod{n}$ always has infinitely many solutions $n$ for $\gcd(a,b) = 1$.

References


[14] WALSH, G. On integers \( n \) with the property \( n \mid f_n \). 5pp., unpublished, 1986.