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Solving algebraic equations in roots of unity

Iskander Aliev and Chris Smyth

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Summary

This paper is devoted to finding solutions of polynomial equations in roots of unity. It was conjectured by S. Lang and proved by M. Laurent that all such solutions can be described in terms of a finite number of parametric families called maximal torsion cosets. We obtain new explicit upper bounds for the number of maximal torsion cosets on an algebraic subvariety of the complex algebraic $n$-torus $\mathbb{G}_m^n$. In contrast to earlier works that give the bounds of polynomial growth in the maximum total degree of defining polynomials, the proofs of our results are constructive. This allows us to obtain a new algorithm for determining maximal torsion cosets on an algebraic subvariety of $\mathbb{G}_m^n$.


1 Introduction

Let $f_1, \ldots, f_t$ be the polynomials in $n$ variables defined over $\mathbb{C}$. In this paper we deal with solutions of the system

$$\begin{align*}
&f_1(X_1, \ldots, X_n) = 0 \\
&\vdots \\
&f_t(X_1, \ldots, X_n) = 0
\end{align*}$$

in roots of unity. It will be convenient to think of such solutions as torsion points on the subvariety $\mathcal{V}(f_1, \ldots, f_t)$ of the complex algebraic torus $\mathbb{G}_m^n$ defined by the system (1). As an affine variety, we identify $\mathbb{G}_m^n$ with the Zariski open subset $x_1x_2\cdots x_n \neq 0$ of affine space $\mathbb{A}^n$, with the usual multiplication

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = (x_1y_1, x_2y_2, \ldots, x_ny_n).$$

By algebraic subvariety of $\mathbb{G}_m^n$ we understand a Zariski closed subset. An algebraic subgroup of $\mathbb{G}_m^n$ is a Zariski closed subgroup. A subtorus of $\mathbb{G}_m^n$ is a geometrically irreducible algebraic subgroup. A torsion coset is a coset $\omega H$, where $H$ is a
subtorus of $\mathbb{G}_m^n$ and $\omega = (\omega_1, \ldots, \omega_n)$ is a torsion point. Given an algebraic subvariety $V$ of $\mathbb{G}_m^n$, a torsion coset $C$ is called maximal in $V$ if $C \subset V$ and it is not properly contained in any other torsion coset in $V$. A maximal 0-dimensional torsion coset will be also called isolated torsion point.

Let $N_{\text{tor}}(V)$ denote the number of maximal torsion cosets contained in $V$. A famous conjecture by Lang ([17], p. 221) proved by McQuillan [22] implies as a special case that $N_{\text{tor}}(V)$ is finite. This special case had been settled by Ihara, Serre and Tate (see Lang [17], p. 201) when $\dim(V) = 1$, and by Laurent [18] if $\dim(V) > 1$. A different proof of this result was also given by Sarnak and Adams [26]. It follows that all solutions of the system (1) in roots of unity can be described in terms of a finite number of maximal torsion cosets on the subvariety $V(f_1, \ldots, f_t)$. It is then of interest to obtain an upper bound for this number.

Zhang [29] and Bombieri and Zannier [6] showed that if $V$ is defined over a number field $K$ then $N_{\text{tor}}(V)$ is effectively bounded in terms of $d$, $n$, $[K : \mathbb{Q}]$ and $M$, when the defining polynomials were of total degrees at most $d$ and heights at most $M$. Schmidt [28] found an explicit upper bound for the number of maximal torsion cosets on an algebraic subvariety of $\mathbb{G}_m^n$ that depends only on the dimension $n$ and the maximum total degree $d$ of the defining polynomials. Indeed, let

$$N_{\text{tor}}(n, d) = \max_V N_{\text{tor}}(V),$$

where the maximum is taken over all subvarieties $V \subset \mathbb{G}_m^n$ defined by polynomial equations of total degree at most $d$. The proof of Schmidt’s bound is based on a result of Schlickewei [27] about the number of nondegenerate solutions of a linear equation in roots of unity. This latter result was significantly improved by Evertse [13], and the resulting Evertse–Schmidt bound can then be stated as

$$N_{\text{tor}}(n, d) \leq (11d)^{n^2} \binom{n + d}{d} 3^{(n+d)^2}.$$  \hspace{1cm} (2)

Applying techniques from arithmetic algebraic geometry, David and Philippon [10] went even further and obtained a polynomial in $d$ upper bound for the number of isolated torsion points, with the exponent being essentially $7^k$, where $k$ is the dimension of the subvariety. This result have been since slightly improved by Amoroso and David [2]. A polynomial bound for the number of all maximal torsion cosets also appears in the main result of Rémond [24], with the exponent $(k + 1)^3(k+1)^2$.

It should be mentioned here that the last two bounds are special cases of more general results. David and Philippon [10] in fact study the number of algebraic points with small height and Rémond [24] deals with subgroups of finite rank and even with thickness of such subgroups in the sense of the height. The high generality of the results requires applying sophisticated tools from arithmetic algebraic geometry. This approach involves work with heights in the fields of algebraic numbers and a delicate specialization argument (see e.g. Proposition
6.9 in David and Philippon [11]) that allows to transfer the results to algebraically closed fields of characteristics 0.

In this paper we present a constructive and more elementary approach to this problem which is based on well–known arithmetic properties of the roots of unity. Roughly speaking, we use the Minkowski geometry of numbers to reduce the problem to a very special case and then apply an intersection/elimination argument. This allows us to obtain a polynomial bound with the exponent $5^n$ for the number of maximal torsion cosets lying on a subvariety of $\mathbb{G}_m^n$ defined over $\mathbb{C}$ and implies an algorithm for finding all such cosets. The algorithm is presented in Section 6.

One should point out here that other algorithms for finding all the maximal torsion cosets on a subvariety of $\mathbb{G}_m^n$ were proposed by Sarnak and Adams in [26] and by Ruppert [25]. In view of its high complexity, the algorithm of Ruppert is described in [25] only for a special choice of defining polynomials. Note also that different algorithms implicitly follow from the papers by Mann [20], Conway and Jones [9] and Dvornicich and Zannier [12].

1.1 The main results

We shall start with the case of hypersurfaces.

**Theorem 1.1.** Let $f \in \mathbb{C}[X_1, \ldots, X_n]$, $n \geq 2$, be a polynomial of total degree $d$ and let $\mathcal{H} = \mathcal{H}(f)$ be the hypersurface in $\mathbb{G}_m^n$ defined by $f$. Then

$$N_{\text{tor}}(\mathcal{H}) \leq c_1(n) \cdot d^{c_2(n)};$$

(3)

with

$$c_1(n) = n^{\frac{1}{2}(2+n)5^n} \text{ and } c_2(n) = \frac{1}{16}(49 \cdot 5^{n-2} - 4n - 9).$$

Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial of degree $d_i$ in $X_i$. Ruppert [25] conjectured that the number of isolated torsion points on $\mathcal{H}(f)$ is bounded by $c(n) \cdot d_1 \cdots d_n$. Theorem 1.1 is a step towards proving this conjecture. Furthermore, the results of Beukers and Smyth [3] for plane curves (see Lemma 2.2 below) indicate that the following stronger conjecture might be true.

**Conjecture.** The number of isolated torsion points on the hypersurface $\mathcal{H}(f)$ is bounded by $c(n) \cdot \text{vol}_n(f)$, where $\text{vol}_n(f)$ is the $n$-volume of the Newton polytope of the polynomial $f$.

Concerning general varieties, we obtained the following result.
Theorem 1.2. For \( n \geq 2 \) we have
\[
N_{\text{tor}}(n, d) \leq c_3(n) d^{c_4(n)},
\]
where
\[
c_3(n) = n^{(2+n)2^n - 2} \prod_{i=2}^{n} c_2(i) \quad \text{and} \quad c_4(n) = \sum_{i=2}^{n} c_2(i) 2^{n-i} + 2^{n-1}.
\]

It should be pointed out that the constants \( c_i(n) \) in Theorems 1.1 and 1.2 could be certainly improved. To simplify the presentation, we tried to avoid painstaking estimates.

The proof of Theorem 1.1 is based on Theorem 1.3, formulated in the next section. Theorem 1.2, in its turn, is a consequence of Theorem 1.1.

1.2 An intersection argument

For \( i \in \mathbb{Z}^n \), we abbreviate \( X^i = X_1^{i_1} \cdots X_n^{i_n} \). Let
\[
f(X) = \sum_{i \in \mathbb{Z}^n} a_i X^i
\]
be a Laurent polynomial. By the support of \( f \) we mean the set
\[
S_f = \{ i \in \mathbb{Z}^n : a_i \neq 0 \}
\]
and by the exponent lattice of \( f \) we mean the lattice \( L(f) \) generated by the difference set \( D(S_f) = S_f - S_f \), so that
\[
L(f) = \text{span}_\mathbb{Z} \{ D(S_f) \}.
\]

Our next result and its proof is a generalization of that for \( n = 2 \) in Beukers and Smyth [3].

Theorem 1.3. Let \( f \in \mathbb{C}[X_1, \ldots, X_n], \ n \geq 2, \) be an irreducible polynomial with \( L(f) = \mathbb{Z}^n \). Then for some \( m \) with \( 1 \leq m \leq 2^{n+1} - 1 \) there exist \( m \) polynomials \( f_1, f_2, \ldots, f_m \) with the following properties:

(i) \( \deg(f_i) \leq 2 \deg(f) \) for \( i = 1, \ldots, m; \)

(ii) For \( 1 \leq i \leq m \) the polynomials \( f \) and \( f_i \) have no common factor;

(iii) For any torsion coset \( C \) lying on the hypersurface \( \mathcal{H}(f) \) there exists some \( f_i, 1 \leq i \leq m \), such that the coset \( C \) also lies on the hypersurface \( \mathcal{H}(f_i) \).
2 Lemmas required for the proofs

In this section, we give the definitions and basic lemmas we need in the rest of paper.

2.1 Finding the cyclotomic part of a polynomial in one variable

Let us consider the following one-variable version of the problem: given a polynomial \( f \in \mathbb{C}[X] \), find all roots of unity \( \omega \) that are zeroes of \( f \). This is equivalent to finding the factor of \( f \) consisting of the product of all distinct irreducible cyclotomic polynomial factors of \( f \), which we shall call the cyclotomic part of \( f \).

Algorithms for finding the cyclotomic part of \( f \) follow from several papers, for instance the papers by Mann [20], Conway and Jones [9] and Dvornicich and Zannier [12]. In this paper we use the approach of Bradford and Davenport [7] and Beukers and Smyth [3] who proposed the algorithms based on the following properties of roots of unity.

Lemma 2.1 (Beukers and Smyth [3], Lemma 1).

(i) If \( g \in \mathbb{C}[X] \), \( g(0) \neq 0 \), is a polynomial with the property that for every zero \( \alpha \) of \( g \), at least one of \( \pm \alpha^2 \) is also a zero, then all zeroes of \( g \) are roots of unity.

(ii) If \( \omega \) is a root of unity, then it is conjugate to \( \omega^p \) where

\[
\begin{align*}
    p &= 2k + 1, \quad \omega^p = -\omega \quad \text{for \( \omega \) a primitive \( 4k \)-th root of unity}; \\
    p &= k + 2, \quad \omega^p = -\omega^2 \quad \text{for \( \omega \) a primitive \( 2k \)-th root of unity, \( k \) odd}; \\
    p &= 2, \quad \omega^p = \omega^2 \quad \text{for \( \omega \) a \( k \)-th root of unity, \( k \) odd}.
\end{align*}
\]

In the special case \( f \in \mathbb{Z}[X] \), Filaseta and Schinzel [14] constructed a deterministic algorithm for finding the cyclotomic part of \( f \) that works especially well when the number of nonzero terms is small compared to the degree of \( f \).

2.2 Torsion points on plane curves

Let \( f \in \mathbb{C}[X^\pm, Y^\pm] \) be a Laurent polynomial. The problem of finding torsion points on the curve \( C \) defined by the polynomial equation \( f(X,Y) = 0 \) was implicitly solved already in work of Lang [16] and Liardet [19], as well as in the papers by Mann [20], Conway and Jones [9] and Dvornicich and Zannier [12], already referred to. More recently, it has been also addressed in Beukers and Smyth [3] and Ruppert [25].

The polynomial \( f \) can be written in the form

\[
f(X,Y) = g(X,Y) \prod_i (X^{a_i}Y^{b_i} - \omega_i),
\]
where the \( \omega_j \) are roots of unity and \( g \) is a polynomial (possibly reducible) that has no factor of the form \( X^a Y^b - \omega \), for \( \omega \) a root of unity.

**Lemma 2.2** (Beukers and Smyth [3], Main Theorem). The curve \( \mathcal{C} \) has at most 22 \( \text{vol}_2(g) \) isolated torsion points.

Hence, for \( f \in \mathbb{C}[X,Y], \) the number of isolated torsion points on the curve \( \mathcal{C} = \mathcal{H}(f) \) is at most 11 \( (\deg(f))^2 \). Furthermore, by Lemma 2.7 below, each factor \( X^a Y^b - \omega_i \) of the polynomial \( f \) gives precisely one torsion coset. Summarizing the above observations, we get the inequality

\[
N_{\text{tor}}(\mathcal{C}) \leq 11(\deg(f))^2 + \deg(f) .
\]  

\((5)\)

### 2.3 Geometry of numbers

The bijection \( i \leftrightarrow X^i \) allows us to study polynomials by the use of the geometry of numbers. The following technical tools will be needed.

We first recall some basic definitions. A **lattice** is a discrete subgroup of \( \mathbb{R}^n \). Given a lattice \( L \) of rank \( k \), any set of vectors \( \{b_1, \ldots, b_k\} \) with \( L = \text{span}_\mathbb{Z}\{b_1, \ldots, b_k\} \) or the matrix \( B = (b_1, \ldots, b_k) \) with rows \( b_i \) will be called a basis of \( L \). The **determinant** of a lattice \( L \) with a basis \( B \) is defined to be

\[
\det(L) = \sqrt{\det(BB^T)} .
\]

Let \( B^n_p \) with \( p = 1, 2, \infty \) denote the unit \( n \)-ball with respect to the \( l_p \)-norm, and let \( \gamma_n \) be the Hermite constant for dimension \( n \) – see Section 38.1 of Gruber and Lekkerkerker [15]. For a convex body \( K \) and a lattice \( L \), we also denote by \( \lambda_i(K,L) \) the \( i \)th successive minimum of \( K \) with respect to \( L \) – see Section 9.1 ibid.

**Lemma 2.3.** Let \( S \) be a subspace of \( \mathbb{R}^n \) with \( \dim(S) = \text{rank}(S \cap \mathbb{Z}^n) = r < n \). Then there exists a basis \( \{b_1, b_2, \ldots, b_n\} \) of the lattice \( \mathbb{Z}^n \) such that

\[
\begin{align*}
(i) & \quad S \subset \text{span}_\mathbb{R}\{b_1, \ldots, b_{n-1}\} ; \\
(ii) & \quad |b_i| < 1 + \frac{1}{2}(n - 1) \frac{\gamma_{n-1}^{\frac{n-1}{n}}}{\gamma_{n-r}^{\frac{2n}{n}}} \det(S \cap \mathbb{Z}^n)^{\frac{1}{n-r}}, \quad i = 1, \ldots, n .
\end{align*}
\]

**Proof.** Suppose first that \( r < n - 1 \). By Proposition 1 (ii) of Aliev, Schinzel and Schmidt [1], there exists a subspace \( T \subset \mathbb{R}^n \) with \( \dim(T) = n - 1 \) such that \( S \subset T \) and

\[
\det(T \cap \mathbb{Z}^n) \leq \gamma_{n-r}^{\frac{1}{n-r}} \det(S \cap \mathbb{Z}^n)^{\frac{1}{n-r}} .
\]  

\((6)\)

In the case \( r = n - 1 \) we will put \( T = S \).
The subspace $T$ can be considered as a standard $(n-1)$-dimensional euclidean space. Then by the Minkowski’s second theorem for balls (see Theorem I, Ch. VIII of Cassels [8]) we have

\[ \prod_{i=1}^{n-1} \lambda_i(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \gamma_{n-1}^{\frac{n-1}{2}} \operatorname{det}(T \cap \mathbb{Z}^n). \]

Noting that $1 \leq \lambda_1(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \ldots \leq \lambda_{n-1}(T \cap B_2^n, T \cap \mathbb{Z}^n)$, we get

\[ \lambda_{n-1}(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \gamma_{n-2}^{\frac{n-1}{2}} \operatorname{det}(T \cap \mathbb{Z}^n). \]

(7)

Next, by Corollary of Theorem VII, Ch. VIII of Cassels [8], there exists a basis $B = (b_1, \ldots, b_{n-1})$ of the lattice $T \cap \mathbb{Z}^n$ with $|b_j| \leq \max\{1, j/2\} \lambda_j(T \cap B_2^n, T \cap \mathbb{Z}^n)$, $j = 1, \ldots, n - 1$. Consequently,

\[ |b_i| \leq \frac{n-1}{2} \lambda_{n-1}(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \frac{n-1}{2} \gamma_{n-1}^{\frac{n-1}{2}} \operatorname{det}(T \cap \mathbb{Z}^n) \]

\[ \leq \frac{n-1}{2} \gamma_{n-1}^{\frac{n-1}{2}} \gamma_{n-r}^{\frac{n-2}{2}} \operatorname{det}(S \cap \mathbb{Z}^n)^{\frac{1}{n-r}}, \quad i = 1, \ldots, n - 1. \]

Further, we need to extend $B$ to a basis of the lattice $\mathbb{Z}^n$. Let $\mathbf{a}$ be a primitive integer vector from $\operatorname{span}_\mathbb{R}(T \cap \mathbb{Z}^n)$. Clearly, all possible vectors $\mathbf{b}$ such that $(b_1, \ldots, b_{n-1}, \mathbf{b})$ is a basis of $\mathbb{Z}^n$ form the set $\{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{a} \rangle = \pm 1 \} \cap \mathbb{Z}^n$, and this set contains a point $\mathbf{b}_n$ with

\[ |b_n| \leq \frac{1}{|a|} + \mu(T \cap B_2^n, T \cap \mathbb{Z}^n), \tag{8} \]

where $\mu(\cdot, \cdot)$ is the inhomogeneous minimum – see Section 13.1 of Gruber–Lekkerkerker [15]. By Jarnik’s inequality (see Theorem 1 on p. 99 ibid.)

\[ \mu(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i(T \cap B_2^n, T \cap \mathbb{Z}^n) \leq \frac{n-1}{2} \lambda_{n-1}(T \cap B_2^n, T \cap \mathbb{Z}^n). \]

Consequently, by (8), (7) and (6), we have

\[ |b_n| < 1 + \frac{n-1}{2} \gamma_{n-1}^{\frac{n-1}{2}} \gamma_{n-r}^{\frac{n-2}{2}} \operatorname{det}(S \cap \mathbb{Z}^n)^{\frac{1}{n-r}}. \]

When $L$ is a lattice on rank $n$, its polar lattice $L^*$ is defined as

\[ L^* = \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{Z} \text{ for all } \mathbf{y} \in L \}. \]

Given a basis $B = (b_1, \ldots, b_n)$ of $L$, the basis of $L^*$ polar to $B$ is the basis $B^* = (b_1^*, \ldots, b_n^*)$ with

\[ \langle b_i, b_j^* \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n, \]

where $\delta_{ij}$ is the Kronecker delta.
Corollary 2.1. Let $S$ be a subspace of $\mathbb{R}^n$ with $\dim(S) = \text{rank}(S \cap \mathbb{Z}^n) = r < n$. Then there exists a basis $A = (a_1, a_2, \ldots, a_n)$ of the lattice $\mathbb{Z}^n$ such that $a_1 \in S^\perp$ and the vectors of the polar basis $A^* = (a_1^*, a_2^*, \ldots, a_n^*)$ satisfy the inequalities

$$|a_i^*| < 1 + \frac{n-1}{2} \gamma^{\frac{r-1}{2}} \gamma^{\frac{r}{2} - n} \det(S \cap \mathbb{Z}^n)^{\frac{1}{n-r}}, \quad i = 1, \ldots, n.$$  \hfill (9)

Proof. Applying Lemma 2.3 to the subspace $S$ we get a basis $\{b_1, b_2, \ldots, b_n\}$ of $\mathbb{Z}^n$ satisfying conditions (i)–(ii). Observe that its polar basis $\{b_1^*, b_2^*, \ldots, b_n^*\}$ has its last vector $b_n^*$ in $S^\perp$. Therefore, we can put $a_1 = b_n^*$, $a_2 = b_2^*$, $\ldots$, $a_{n-1} = b_{n-1}^*$, $a_n = b_1^*$.

2.4 Lattices and torsion cosets

In the subsection we describe the standard bijection between lattices and algebraic subgroups of $G_n^m$. By an integer lattice we understand a lattice $A \subset \mathbb{Z}^n$. An integer lattice is called primitive if $A = \text{span}_\mathbb{R}(A) \cap \mathbb{Z}^n$. For an integer lattice $A$, we define the subgroup $H_A$ of $G_n^m$ by

$$H_A = \{ x \in G_n^m : x a = 1 \text{ for all } a \in A \}. $$

Then, for instance, $H_{\mathbb{Z}^n}$ is the trivial subgroup.

Lemma 2.4 (See Schmidt [28], Lemmas 1 and 2). The map $A \mapsto H_A$ sets up a bijection between integer lattices and algebraic subgroups of $G_n^m$. A subgroup $H = H_A$ is irreducible if and only if the lattice $A$ is primitive.

Let $\omega = (\omega_1, \ldots, \omega_n)$ be a torsion point and let $C = \omega H_A$ be an $r$-dimensional torsion coset with $r \geq 1$. We will need the following parametric representation of $C$. Let $\text{span}_\mathbb{R}(A)$ denote the orthogonal complement of $\text{span}_\mathbb{R}(A)$ in $\mathbb{R}^n$ and let $G = (g_{ij})$ be an $r \times n$ integer matrix of rank $r$ whose rows $g_1, \ldots, g_r$ form a basis of the lattice $\text{span}_\mathbb{R}(A) \cap \mathbb{Z}^n$. Then the coset $C$ can be represented in the form

$$C = \left( \omega_1 \prod_{j=1}^r t_j^{g_{1j}}, \ldots, \omega_n \prod_{j=1}^r t_j^{g_{nj}} \right)$$

with parameters $t_1, \ldots, t_r \in \mathbb{C}^\ast$. We will say that $G$ is an exponent matrix for the coset $C$. If $f \in \mathbb{C}[X_1^{\pm1}, \ldots, X_n^{\pm1}]$ is a Laurent polynomial and for $j \in \mathbb{Z}^r$

$$f_j(X) = \sum_{i \in S_j : i G^T = j} a_i X^i,$$

then $f(X) = \sum_{j \in \mathbb{Z}^r} f_j(X)$ and

the coset $C$ lies on $\mathcal{H}(f)$ if and only if $f_j(\omega) = 0$ for all $j \in \mathbb{Z}^r$.

(10)
Let \( U = (u_1, u_2, \ldots, u_n) \) be a basis of the lattice \( \mathbb{Z}^n \). We will associate with \( U \) the new coordinates \((Y_1, \ldots, Y_n)\) in \( \mathbb{G}_m^n \) defined by
\[
Y_1 = X^{u_1}, \quad Y_2 = X^{u_2}, \ldots, \quad Y_n = X^{u_n}.
\] (11)

Suppose that the matrix \( U^{-1} \) has rows \( v_1, v_2, \ldots, v_n \). By the image of a Laurent polynomial \( f \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) in coordinates \((Y_1, \ldots, Y_n)\) we mean the Laurent polynomial
\[
f^U(Y) = f(Y^{v_1}, \ldots, Y^{v_n}).
\]

By the image of a torsion coset \( C = \omega H_A \) in coordinates \((Y_1, \ldots, Y_n)\) we mean the torsion coset
\[
C^U = (\omega^{u_1}, \ldots, \omega^{u_n})H_B,
\]
where \( B = \{aU^{-1} : a \in A\} \).

Lemma 2.5. The map \( C \mapsto C^U \) sets up a bijection between maximal torsion cosets on the subvarieties \( \mathcal{V}(f_1, \ldots, f_t) \) and \( \mathcal{V}(f_1^U, \ldots, f_t^U) \).

Proof. It is enough to observe that the map \( \phi : \mathbb{G}_m^n \to \mathbb{G}_m^n \) defined by
\[
\phi(x) = (x^{u_1}, \ldots, x^{u_n})
\] (12)

is an automorphism of \( \mathbb{G}_m^n \) (see Ch. 3 in Bombieri and Gubler [4] and Section 2 in Schmidt [28]). \( \Box \)

Remark. The automorphism (12) is called a monoidal transformation. We introduced the coordinates (11) to make the inductive argument used in the proofs of Theorems 1.1–1.2 more transparent.

For \( f \in \mathbb{C}[X_1, \ldots, X_n] \) and \( k \geq n \), we will denote by \( T_i^k(f) \) the number of \( i \)-dimensional maximal torsion cosets on \( \mathcal{H}(f) \), regarded as a hypersurface in \( \mathbb{G}_m^n \). Let \( A \subset \mathbb{Z}^n \) be an integer lattice of rank \( n \) with \( \det(A) > 1 \) and let \( A = (a_1, \ldots, a_n) \) be a basis of \( A \).

Lemma 2.6. Suppose that the Laurent polynomials \( f, f^* \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) satisfy
\[
f = f^*(X^{a_1}, \ldots, X^{a_n}).
\] (13)

Then the inequalities
\[
T_i^n(f^*) \leq T_i^n(f) \leq \det(A) T_i^n(f^*), \quad i = 0, \ldots, n - 1
\] (14)

hold.
Proof. First, for any torsion point $\zeta = (\zeta_1, \ldots, \zeta_n)$ on $H(f^*)$, we will find all torsion points $\omega$ on $H(f)$ with $\zeta = (\omega^{a_1}, \ldots, \omega^{a_n})$. Putting the matrix $A$ into Smith Normal Form (see Newman [23], p. 26) yields two matrices $V$ and $W$ in $\text{GL}_n(\mathbb{Z})$ with $WAV = \text{D}$, where $\text{D} = \text{diag}(d_1, \ldots, d_n)$. Therefore, by Lemma 2.5, we may assume without loss of generality that $A = \text{diag}(d_1, \ldots, d_n)$. Let $\theta_1, \ldots, \theta_n$ be primitive $d_1$st, $d_2$nd, $\ldots$, $d_n$th roots of $\zeta_1, \ldots, \zeta_n$, respectively. Then as we let $\theta_1, \ldots, \theta_n$ vary over all possible such choices of these primitive roots

the torsion point $\zeta \in H(f^*)$ gives precisely $\det(A)$ torsion points $\omega = (\theta_1, \ldots, \theta_n)$ on $H(f)$ with $\zeta = (\omega^{a_1}, \ldots, \omega^{a_n})$. (15)

Let now $M_f$ and $M_{f^*}$ denote the sets of all maximal torsion cosets of positive dimension on $H(f)$ and $H(f^*)$ respectively. We will define a map $\tau : M_f \rightarrow M_{f^*}$ as follows. Let $C \in M_f$ be an $r$-dimensional maximal torsion coset. Given any torsion point $\omega = (\omega_1, \ldots, \omega_n) \in C$, we can write the coset as $C = \omega H_B$ for some primitive integer lattice $B$. Recall that $C$ can be also represented in the form

$$C = \left( \omega_1 \prod_{j=1}^r t_{j_1}^{g_{j_1}}, \ldots, \omega_n \prod_{j=1}^r t_{j_n}^{g_{j_n}} \right),$$

where $t_1, \ldots, t_r \in \mathbb{C}^*$ are parameters and the vectors $g_j = (g_{j_1}, \ldots, g_{j_n})$, $j = 1, \ldots, r$, form a basis of the lattice $\text{span}_{\mathbb{R}}(B) \cap \mathbb{Z}^n$. Let $M = \text{span}_{\mathbb{Z}}\{g_1 A^T, \ldots, g_r A^T\}$ and $L = \text{span}_{\mathbb{R}}(M) \cap \mathbb{Z}^n$. Then we define

$$\tau(C) = \left( \omega^{a_1} \prod_{k=1}^r t_{k_1}^{s_{k_1}}, \ldots, \omega^{a_n} \prod_{k=1}^r t_{k_n}^{s_{k_n}} \right),$$

where $t_1, \ldots, t_r \in \mathbb{C}^*$ are parameters and the vectors $s_k = (s_{k_1}, \ldots, s_{k_n})$, $k = 1, \ldots, r$, form a basis of the lattice $L$. Let us show that $\tau$ is well-defined. First, the observation (10) implies that $\tau(C)$ is a maximal $r$-dimensional torsion coset on $H(f^*)$. Now we have to show that $\tau(C)$ does not depend on the choice of $\omega \in C$. Observe that any torsion point $\eta \in C$ has the form

$$\eta = \left( \omega_1 \prod_{j=1}^r \nu_{j_1}^{g_{j_1}}, \ldots, \omega_n \prod_{j=1}^r \nu_{j_n}^{g_{j_n}} \right),$$

where $\nu_1, \ldots, \nu_r$ are some roots of unity. Put $h_j = g_j A^T$, $j = 1, \ldots, r$. It is enough to show that for any roots of unity $\nu_1, \ldots, \nu_r$ there exist roots of unity $\mu_1, \ldots, \mu_r$ such that

$$\prod_{j=1}^r \nu_{j_i}^{h_{j_i}} = \prod_{k=1}^r \mu_{k_i}^{g_{k_i}}, \quad i = 1, \ldots, n.$$

Since $M \subset L$, we have $h_j \in L$, so that

$$h_j = l_{j_1} s_1 + \cdots + l_{j_r} s_r, \quad l_{j_1}, \ldots, l_{j_r} \in \mathbb{Z}.$$
Now we can put
\[ \mu_k = \nu_1^{s_{1k}} \nu_2^{s_{2k}} \cdots \nu_r^{s_{rk}}, \quad k = 1, \ldots, r. \]
Thus, the map \( \tau \) is well-defined. It can be also easily shown that the map \( \tau \) is surjective. This observation immediately implies the left hand side inequality in (14) for positive \( i \). Moreover, by (15), we clearly have
\[ T_0^n(f) = \det(A) T_0^n(f^*), \]
so that the lemma is proved for the isolated torsion points.

Let now \( D = \zeta H' \in \mathcal{M}^r \) be an \( r \)-dimensional maximal torsion coset. Suppose that \( D = \tau(C) \) for some \( C \in M_f \). We will show that \( C = \omega H \), where \( \omega \) can be chosen among \( \det(A) \) torsion points listed in (15). This will immediately imply the right hand side inequality in (14) for positive \( i \). We may assume without loss of generality that \( H = H_B \) and \( H' = H_{\text{span}_R(L) \cap \mathbb{Z}^n} \), with the lattices \( B \) and \( L \) defined as above. Let \( \mu_1, \ldots, \mu_r \) be any roots of unity. Then the coset \( D \) can be represented as
\[ D = \left( \zeta_1 \prod_{k=1}^r \mu_k^{s_{1k}} \prod_{k=1}^r t_k^{s_{2k}}, \ldots, \zeta_n \prod_{k=1}^r \mu_k^{s_{nk}} \prod_{k=1}^r t_k^{s_{nk}} \right) \]
for \( \zeta = (\zeta_1, \ldots, \zeta_n) \). Thus, it is enough to prove the existence of roots of unity \( \nu_1, \ldots, \nu_r \) with
\[ \prod_{k=1}^r \mu_k^{s_{ki}} = \prod_{j=1}^r \nu_j^{h_{ji}}, \quad i = 1, \ldots, n. \]

The lattice \( M \) is a sublattice of \( L \) and \( \text{rank}(M) = \text{rank}(L) \). Therefore there exist positive integers \( n_1, \ldots, n_r \) such that \( n_i s_i \in M, \ i = 1, \ldots, r \), and, consequently, we have
\[ n_i s_i = m_{i1} h_1 + \cdots + m_{ir} h_r, \quad m_{i1}, \ldots, m_{ir} \in \mathbb{Z}. \]
Now, if the roots of unity \( \rho_1, \ldots, \rho_r \) satisfy \( \rho_i^{n_i} = \mu_i, \ i = 1, \ldots, r \), we can put
\[ \nu_j = \rho_1^{m_{1j}} \rho_2^{m_{2j}} \cdots \rho_r^{m_{rj}}, \quad j = 1, \ldots, r. \]

2.5 Torsion cosets of codimension 1 in \( \mathbb{G}^\mu_\mathbb{m} \)

The next lemma is an immediate consequence of the structure of torsion cosets, explained for example in Bombieri and Gubler [4]. We give a proof here for the sake of completeness.
Lemma 2.7. Suppose that the hypersurface $H$ is defined by the polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ with $f = \prod_i h_i$, where $h_i$ are irreducible polynomials. Then the $(n-1)$-dimensional torsion cosets on $H$ are precisely the hypersurfaces $H(h_j)$ defined by the factors $h_j$ of the form $X^{m_j} - \omega_j X^{n_j}$, where $\omega_j$ are roots of unity.

Proof. Let $\omega$ be a root of unity and let $h = X^m - \omega X^n$ be a factor of $f$. Multiplying $h$ by a monomial we may assume that $h$ is a Laurent polynomial of the form $X^a - \omega$, where $a = (a_1, \ldots, a_n)$ is a primitive integer vector, so that $\gcd(a_1, \ldots, a_n) = 1$. Let $A$ be the integer lattice generated by the vector $a$, $b = (b_1, \ldots, b_n)$ be an integer vector with $\langle b, a \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the usual inner product, and put $\omega = (\omega^{b_1}, \ldots, \omega^{b_n})$.

Now, all points of the torsion coset $C = \omega H_A$ clearly satisfy the equation $X^a = \omega$. To show that any solution $x = (x_1, \ldots, x_n)$ of this equation belongs to $C$ we observe that the point $(x_1 \omega^{-b_1}, \ldots, x_n \omega^{-b_n})$ belongs to the subtorus $H_A$.

Conversely, let $C = \omega H$ be an $(n-1)$-dimensional coset on $H$. Since the exponent matrix of the coset $C$ has rank $n-1$, there exists a primitive integer vector $a$ such that and for all $j \in \mathbb{Z}^{n-1}$ we have $\text{span}_\mathbb{Z}(L(f_j)) \cap \mathbb{Z}^n = \text{span}_\mathbb{Z}\{a\}$. Since $f_j(\omega) = 0$, the Laurent polynomial $h_C = X^a - \omega^a$ will divide all $f_j$ and, consequently, $f$. Multiplying by a monomial, we may assume that $h_C$ is a factor of the desired form. Finally, noting that $H = H_{\text{span}_\mathbb{Z}\{a\}}$ and applying the result of the previous paragraph, we see that $C = H(h_C)$.

\[\square\]

3 Proof of Theorem 1.3

The proof of Theorem 1.3 is divided into several cases, in the similar way to Section 3 of Beukers and Smyth [3].

3.1 $f$ with rational coefficients

Suppose that $f \in \mathbb{Q}[X_1, \ldots, X_n]$, $n \geq 2$, is irreducible and has $L(f) = \mathbb{Z}^n$. We will show that $2^{n+1}-1$ polynomials

\begin{align*}
    f(\epsilon_1X_1, \ldots, \epsilon_nX_n), & \quad \epsilon_i = \pm 1, \text{ not all } \epsilon_i = 1 \\
    f(\epsilon_1X_1^2, \ldots, \epsilon_nX_n^2), & \quad \epsilon_i = \pm 1.
\end{align*}

(18) (19)

satisfy all conditions of the theorem.

The condition (i) clearly holds for all polynomials (18)–(19). Suppose now that $f$ divides one of the polynomials (18). Let us consider the lattice

\[L_2 = \left\{ (x_1, \ldots, x_n) \in \mathbb{Z}^n : \frac{1 - \epsilon_1}{2} x_1 + \ldots + \frac{1 - \epsilon_n}{2} x_n \equiv 0 \mod 2 \right\},\]
with the same choice of $\epsilon_i$. Note that $\det(L_2) = 2$ and thus $L_2 \not\subseteq \mathbb{Z}^n$. Then, for some $z \in \mathbb{Z}^n$, we have $z + S_f \subset L_2$. Therefore the lattice $L(f)$ cannot coincide with $\mathbb{Z}^n$, a contradiction. This argument also implies that the polynomials (18) are pairwise coprime. Next, if $f$ divides a polynomial $f'$ from (19) then, since $f' \in \mathbb{Q}[X_1^2, \ldots, X_n^2]$, we have that each of the polynomials (18) also divides $f'$. Hence $2^n \deg f \leq \deg f' = 2 \deg f$, so that $n = 1$, a contradiction. Consequently, the set of polynomials $f_1, \ldots, f_m$ consists of all the polynomials (18)–(19). Then condition (ii) is satisfied.

It remains only to check that the condition (iii) holds. Let $C = \omega H$ be a torsion $r$-dimensional coset on the hypersurface $H = H(f)$. There is a root of unity $\omega$ such that $\omega = (\omega_1^{i_1}, \ldots, \omega_n^{i_n})$, where we may assume that $\gcd(i_1, \ldots, i_n) = 1$ so that, in particular, not all of the $i_1, \ldots, i_n$ are even. Next, we have

$$f(\omega^{i_1}, \ldots, \omega^{i_n}) = 0$$

and by part (ii) of Lemma 2.1, also at least one of the $2^{n+1} - 1$ equalities

$$f(\epsilon_1 \omega^{i_1}, \ldots, \epsilon_n \omega^{i_n}) = 0, \quad \epsilon_i = \pm 1, \text{ not all } \epsilon_i = 1$$

holds. Therefore, the torsion point $\omega$ lies on a hypersurface $H' = H(f')$, where $f'$ is one of the polynomials $f_1, \ldots, f_m$. This settles the case $r = 0$.

Suppose now that $r \geq 1$. We claim that the torsion coset $C$ lies on $H'$. To see this we observe that for all $j \in \mathbb{Z}^r$ we have

$$f_j'(\omega) = f_j(\omega^{p_{i_1}}, \ldots, \omega^{p_{i_n}}) = 0,$$

where $p$ is the exponent from the part (ii) of Lemma 2.1. Hence by (10), $C$ lies on $H'$.

### 3.2 $f$ with coefficients in $\mathbb{Q}^{ab}$

We now define the polynomials $f_1, \ldots, f_m$ in the case of $f$ having coefficients lying in a cyclotomic field. Let us choose $N$ to be the smallest integer such that, for some roots of unity $\zeta_1, \ldots, \zeta_n$, the polynomial $f(\zeta_1 x_1, \ldots, \zeta_n x_n)$ has all its coefficients in $K = \mathbb{Q}(\omega_N)$, for $\omega_N$ a primitive $N$th root of unity. Since for $N$ odd $-\omega_N$ is a primitive $(2N)$th root of unity, we may assume either that $N$ is odd or a multiple of 4.

We then replace $f$ by this polynomial. When we have found the polynomials $f_1, \ldots, f_m$ for this new $f$, it is easy to go back and find those for the original $f$. 

3.2.1 \( N \) odd

Take \( \sigma \) to be an automorphism of \( K \) taking \( \omega_N \) to \( \omega_N^2 \). We keep the polynomials \( f_i \) that come from (18) and replace the polynomials that come from (19) by

\[
  f^\sigma(\epsilon_1 X_1^2, \ldots, \epsilon_n X_n^2), \quad \epsilon_i = \pm 1, \quad \text{not divisible by } f.
\]

We then claim that any torsion coset of \( H(f) \) either lies on one of the \( 2^n - 1 \) hypersurfaces defined by (18) or on one of the \( 2^n \) hypersurfaces defined by one of the polynomials (20). Take a torsion coset \( C = (\omega_I^{i_1}, \ldots, \omega_I^{i_n})H \) of \( H(f) \), with \( \gcd(i_1, \ldots, i_n) = 1 \). If \( 4 \nmid l \) then we can extend \( \sigma \) to an automorphism of \( K(\omega_l) \) which takes \( \omega_l \) to one of \( \pm \omega_l^2 \). Therefore, the coset \( C \) also lies on a hypersurface defined by one of the polynomials (20). On the other hand, if \( 4 \mid l \), we put \( 4k = \text{lcm}(l, N) \). Then the automorphism, \( \tau \) say, of \( K(\omega_l) = \mathbb{Q}(\omega_{4k}) \) mapping \( \omega_{4k} \mapsto \omega_{4k}^{2k+1} \) takes \( \omega_l \mapsto \omega_l^{2k+1} = -\omega_l \) and \( \omega_N \mapsto \omega_N^{2k+1} = \omega_N \). Thus, \( C \) lies on a hypersurface defined by one of the polynomials (18).

3.2.2 \( 4 \mid N \)

We take the same coset \( C \) as in the previous case, again put \( 4k = \text{lcm}(l, N) \), and use the same automorphism \( \tau \). Then \( \tau \) takes \( \omega_l \mapsto \omega_{4k}^{2k} \omega_l = \pm \omega_l \) and \( \omega_N \mapsto \omega_N^{2k} \omega_N = \pm \omega_N \). We now consider separately the four possibilities for these signs.

Firstly, from the definition of \( k \) they cannot both be + signs.

If

\[
  \tau(\omega_l) = \omega_l, \quad \tau(\omega_N) = -\omega_N
\]

then \( C \) also lies on \( H(f^{\tau}) \). Note that \( f^{\tau} \neq f \), by the minimality of \( N \), so that they have a proper intersection.

If

\[
  \tau(\omega_l) = -\omega_l, \quad \tau(\omega_N) = \omega_N
\]

then \( C \) also lies on a hypersurface defined by one of the polynomials (18). As \( L(f) = \mathbb{Z}^n \), each has proper intersection with \( f \), as we saw in Section 3.1.

Finally, if

\[
  \tau(\omega_l) = -\omega_l, \quad \tau(\omega_N) = -\omega_N
\]

then \( C \) also lies on one of the hypersurfaces \( H(f^{\tau}) \), for \( f_i \) in (18). Suppose that for instance \( f \) and \( f^{\tau}(-X_1, X_2, \ldots, X_n) \) have a common component, so that \( f^{\tau}(-X_1, X_2, \ldots, X_n) = f(X_1, X_2, \ldots, X_n) \). Then we have

\[
  f(\omega_N X_1, X_2, \ldots, X_n)^\tau = f^{\tau}(-\omega_N X_1, X_2, \ldots, X_n) = f(\omega_N X_1, X_2, \ldots, X_n).
\]

For any coefficient \( c \) of \( f(\omega_N X_1, X_2, \ldots, X_n) \), write \( c = a + \omega_N b \), where \( a, b \in \mathbb{Q}(\omega_N^2) \). Then \( c^\tau = a - \omega_N b = c \), so that \( b = 0 \), \( c \in \mathbb{Q}(\omega_N^2) \). Consequently,
\[ f(\omega N X_1, X_2, \ldots, X_n) \in \mathbb{Q}(\omega^2_n)[X_1, \ldots, X_n], \] contradicting the minimality of \( N \).
The same argument applies for other polynomials (18). Thus, \( C \) lies on one of \( 2^{n+1} - 1 \) subvarieties defined by the polynomials (18) and the polynomials
\[ f^\sigma(\epsilon_1 X_1, \ldots, \epsilon_n X_n), \quad \epsilon_i = \pm 1. \]

### 3.3 \( f \) with coefficients in \( \mathbb{C} \)

Let \( L \) be the coefficient field of \( f \). Suppose that \( L \) is not a subfield of \( \mathbb{Q}^{ab} \). Without loss of generality, assume that at least one coefficient of \( f \) is equal to 1 and choose an automorphism \( \sigma \in \text{Gal}(L/\mathbb{Q}^{ab}) \) which does not fix \( f \). Then since all roots of unity belong to \( \mathbb{Q}^{ab} \), \( f \) and \( f^\sigma \) have the same torsion cosets. Further, \( f \) and \( f^\sigma \) have no common component. Thus in this case we can take the set of \( f_i \) to be the single polynomial \( f^\sigma \).

### 4 Proof of Theorem 1.1

The lemmas of the next two subsections will allow us to assume that \( L(f) = \mathbb{Z}^n \).

#### 4.1 \( L(f) \) of rank less than \( n \)

**Lemma 4.1.** Let \( f \in \mathbb{C}[X_1, \ldots, X_n], n \geq 2 \), be a polynomial of (total) degree \( d \). Suppose that \( L(f) \) has rank \( r \) less than \( n \). Then there exists a polynomial \( f^* \in \mathbb{C}[X_1, \ldots, X_r] \) of degree at most \( d \) such that \( L(f^*) \) also has rank \( r \) and
\[ T^n_i(f) \leq T^n_{i-n+r}(f^*), \quad i = n - r, \ldots, n - 1. \] (21)

**Proof.** Multiplying \( f \) by a monomial, we will assume without loss of generality that \( S_f \subset L(f) \). Then there exists an integer vector \( s = (s_1, \ldots, s_n) \in \text{span}_\mathbb{Z}(S_f) \) and we may assume that \( s_n \neq 0 \). Consider the integer lattice \( A \subset \mathbb{Z}^n \) with the basis
\[ A = \begin{pmatrix} 1 & 0 & \ldots & 0 & s_1 \\ 0 & 1 & \ldots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & s_{n-1} \\ 0 & 0 & \ldots & 0 & s_n \end{pmatrix} \]

Observe that
\[ f(X_1, \ldots, X_{n-1}, 1) = f(X^{a_1}, \ldots, X^{a_n}), \]
and, by Lemma 2.6, we have
\[ T^n_i(f) \leq T^{n-1}_{i-1}(f(X_1, \ldots, X_{n-1}, 1)), \quad i = 1, \ldots, n - 1. \]

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Applying the same procedure to the polynomial \( f(X_1, \ldots, X_{n-1}, 1) \) and so on, we will remove \( n - r \) variables and get the desired polynomial \( f^* \).

### 4.2 \( \mathcal{L}(f) \) of rank \( n \), \( \mathcal{L}(f) \not\subseteq \mathbb{Z}^n \)

**Lemma 4.2.** Let \( f \in \mathbb{C}[X_1, \ldots, X_n] \), \( n \geq 2 \), be an irreducible polynomial of degree \( d \). Suppose that \( \mathcal{L}(f) \) has rank \( n \) and \( \mathcal{L}(f) \not\subseteq \mathbb{Z}^n \). Then there exists an irreducible polynomial \( f^* \in \mathbb{C}[X_1, \ldots, X_n] \) of degree at most \( c_1(n, d) = n^2(n+1)!d \) such that \( \mathcal{L}(f^*) = \mathbb{Z}^n \) and

\[
T_0^\mathcal{L}(f) = \det(\mathcal{L}(f))T_0^\mathcal{L}(f^*),
\]

\[
T_i^\mathcal{L}(f) \leq \det(\mathcal{L}(f))T_i^\mathcal{L}(f^*), \quad i = 1, \ldots, n-1.
\]

**Proof.** Since \( S_f \subseteq dB^1 \), we have \( D(S_f) \subseteq dD(B^1) = 2dB^1 \). Thus, multiplying \( f \) by a monomial, we may assume that \( f \) is a Laurent polynomial with \( S_f \subseteq \mathcal{L}(f) \cap 2dB^1 \). Let \( L^*(f) \) be the polar lattice for the lattice \( \mathcal{L}(f) \) and let \( \mathbf{A}^* = (a_1^*, \ldots, a_n^*) \) be a basis of \( L^*(f) \). Consider the map \( \psi : \mathcal{L}(f) \rightarrow \mathbb{Z}^n \) defined by

\[
\psi(u) = (\langle u, a_1^* \rangle, \ldots, \langle u, a_n^* \rangle).
\]

The Laurent polynomial

\[
f^*(X) = \sum_{u \in S_f} a_u X^{\psi(u)}
\]

has \( \mathcal{L}(f^*) = \mathbb{Z}^n \). Observe that we have

\[
f = f^*(X^{a_1}, \ldots, X^{a_n}).
\]

Therefore the polynomial \( f^* \) is irreducible and, by Lemma 2.6, the inequalities (23) hold. Note also that the equality (22) follows from (17).

Let us estimate the size of \( S_f \). Recall that \( B^n_\infty \) is the polar reciprocal body of \( B^n_1 \) – see Theorem III of Ch. IV in Cassels [8]. Thus, by Theorem VI of Ch. VIII ibid., we have

\[
\lambda_i(B^n_1, \mathcal{L}(f)) \lambda_{n+1-i}(B^n_\infty, L^*(f)) \leq n!.
\]

Noting that \( \lambda_i(B^n_1, \mathcal{L}(f)) \geq 1 \), we get the inequality

\[
\lambda_n(B^n_\infty, L^*(f)) \leq n!.
\]

Next, by Corollary of Theorem VII, Ch. VIII of Cassels [8], there exists a basis \( \mathbf{A}^* = (a_1^*, \ldots, a_n^*) \) of the lattice \( L^*(f) \) such that

\[
a_j^* \in \max\{1, j/2\} \lambda_j(B^n_\infty, L^*(f))B^n_\infty.
\]
Combining the inequalities (25) and (26) we get the bound

\[ \|a_j^*\|_\infty \leq \frac{n \cdot n!}{2}. \]

Then, by the definition of the Laurent polynomial \( f^* \), we have

\[ S_{f^*} \subset (\max_{1 \leq j \leq n} \|a_j^*\|_\infty)2ndB_1^n \subset n^2n!dB_1^n. \]

Thus, multiplying \( f^* \) by a monomial, we may assume that \( f^* \in \mathbb{C}[X_1, \ldots, X_n] \) and

\[ \deg(f^*) \leq n^2(n + 1)!d = c_1(n, d). \]

\[ \square \]

### 4.3 The case \( L(f) = \mathbb{Z}^n \)

Let

\[ T(i, n, d) = \max_{f \in \mathbb{C}[X_1, \ldots, X_n], \deg f \leq d} T_i^n(f), \quad i = 0, \ldots, n - 1 \]

be the maximum number of maximal torsion \( i \)-dimensional cosets lying on a subvariety of \( \mathbb{G}_m^n \) defined by a polynomial of degree at most \( d \).

**Lemma 4.3.** Let \( f \in \mathbb{C}[X_1, \ldots, X_n], n \geq 2, \) be an irreducible polynomial of degree at most \( d \) with \( L(f) = \mathbb{Z}^n \). Then

\[
T_0^n(f) \leq (2^{n+1} - 1)(T(0, n - 1, c_2(n, d)) \sum_{s=1}^{n-2} T(s, n - 1, 2d^2)) + dT(0, n - 1, 2d^2)),
\]

\[ (27) \]

\[
T_1^n(f) \leq (2^{n+1} - 1) (T(1, n - 1, c_2(n, d)) \sum_{s=1}^{n-2} T(s, n - 1, 2d^2)) + T(0, n - 1, 2d^2)),
\]

\[ (28) \]

\[
T_i^n(f) \leq (2^{n+1} - 1) T(i, n - 1, c_2(n, d)) \sum_{s=i-1}^{n-2} T(s, n - 1, 2d^2),
\]

\[ i = 2, \ldots, n - 2, \]

\[ (29) \]

\[
T_{n-1}^n(f) \leq 1,
\]

\[ (30) \]

where \( c_2(n, d) = n(n + 1)d + 2(n - 1)(n^2 - 1)n!d^3. \)
Proof. By Lemma 2.7, we immediately get the inequality (30). Assume now that $\mathcal{H}(f)$ contains no $(n-1)$-dimensional cosets. Applying Theorem 1.3 to the polynomial $f$, we obtain $m \leq 2^{n+1} - 1$ polynomials $f_1, f_2, \ldots, f_m$ satisfying conditions (i)–(iii) of this theorem. For $1 \leq k \leq m$, put $g_k = \text{Res}(f, f_k, X_n)$. By Theorem 1.3 (ii), the polynomials $f$ and $f_k$ have no common factor and thus $g_k \neq 0$. Recall also that $g_k$ lies in the elimination ideal $\langle f, f_k \rangle \cap \mathbb{C}[X_1, \ldots, X_{n-1}]$ and $\deg(g_k) \leq \deg(f) \deg(f_k) \leq 2d^2$.

Given a maximal $i$–dimensional torsion coset $C$ on $\mathcal{H}(f)$, $i \leq n - 2$, its orthogonal projection $\pi(C)$ into the coordinate subspace corresponding to the indeterminates $X_1, \ldots, X_{n-1}$ is a torsion coset in $\mathbb{G}^{n-1}_m$. Note that the coset $\pi(C)$ is either $i$ or $i - 1$ dimensional. The proof of inequalities (27)–(29) is based on the following observation.

Lemma 4.4. Suppose that $1 \leq k \leq m$, $1 \leq s \leq n - 2$ and $0 \leq i \leq s + 1$. Then for any maximal torsion $s$-dimensional coset $D$ on the hypersurface $\mathcal{H}(g_k)$ of $\mathbb{G}^{n-1}_m$, the number of maximal torsion $i$-dimensional cosets $C$ on $\mathcal{H}(f)$ with $\pi(C) \subset D$ is at most $T(i, n - 1, c_2(n, d))$.

Proof. Let $D = \omega H_B$, where $B$ is a primitive sublattice of $\mathbb{Z}^{n-1}$ with rank $(B) = n - 1 - s$. By Corollary 2.1, applied to the subspace $\text{span}_\mathbb{Q}(B)$, there exists a basis $A = (a_1, \ldots, a_{n-1})$ of the lattice $\mathbb{Z}^{n-1}$ such that $a_1 \in B$ and its polar basis $A^* = (a_1^*, \ldots, a_{n-1}^*)$ satisfies the inequality (9). Let $C$ be a maximal torsion $i$-dimensional coset on $\mathcal{H}(f)$ with $\pi(C) \subset D$. Observe that the coset $D$ and, consequently, the coset $C$ satisfy the equation

$$(X_1, \ldots, X_{n-1})^{a_1} = \omega, \tag{31}$$

with the root of unity $\omega = \omega^{a_1}$. The basis $A$ of $\mathbb{Z}^{n-1}$ can be extended to the basis $B = ((a_1, 0), \ldots, (a_{n-1}, 0), e_n)$ of $\mathbb{Z}^n$, where $(a_i, 0)$ denotes the vector $(a_i, \ldots, a_{n-1}, 0)$ and $e_n = (0, \ldots, 0, 1)$. Let $(Y_1, \ldots, Y_n)$ be the coordinates associated with $B$. By Lemma 2.5, the coset $C_B$ is a maximal $i$–dimensional torsion coset on $\mathcal{H}(f^B)$ and, by (31), it lies on the subvariety of $\mathcal{H}(f^B)$ defined by the equation $Y_1 = \omega$. Therefore, the orthogonal projection of the coset $C_B$ into the coordinate subspace corresponding to the indeterminates $Y_2, \ldots, Y_n$ is a maximal $i$-dimensional torsion coset on the hypersurface $\mathcal{H}(f^B(\omega, Y_2, \ldots, Y_n))$ of $\mathbb{G}^{n-1}_m$. Here the polynomial $f^B(\omega, Y_2, \ldots, Y_n)$ is not identically zero. Otherwise the $(n-1)$-dimensional coset defined by (31) would lie on the hypersurface $\mathcal{H}(f)$.

The $(n - 1 - s)$–dimensional subspace $\text{span}_\mathbb{R}(B)$ is generated by $n - 1 - s$ vectors of the difference set $D(S_{nk})$ (see for instance the proof of Theorem 8 in [21] for details). Therefore,

$$\det(B) \leq (\text{diam}(S_{nk}))^{n-1-s} < (4d^2)^{n-1-s},$$

where $\text{diam}(\cdot)$ denotes the diameter of the set. It is well known (see e. g. Bombieri and Vaaler [5], pp. 27–28) that $\det(B) = \det(\text{span}_\mathbb{R}(B) \cap \mathbb{Z}^{n-1})$. Hence, by (9),
we have
\[ S_{j}B \subset (n \max_{1 \leq j \leq n-1} ||\alpha_{j}||_{\infty})dB_{1}^{n} \not\subset (nd + 2n(n-1)\gamma_{n-1} \gamma_{n-1}^{\frac{1}{2}}d^{3})B_{1}^{n}. \]

Multiplying \( f^{B} \) by a monomial, we may assume that \( f^{B} \in \mathbb{C}[Y_{1}, \ldots, Y_{n}] \). Now, observing that \( \gamma_{k}^{3/2} \leq k! \), we get
\[ \deg(f^{B}) < c_{2}(n, d). \]

Therefore, we have shown that the maximal torsion coset \( D \) can contain projections of at most \( T_{i}^{n-1}(f^{B}(\omega, Y_{2}, \ldots, Y_{n})) \leq T(i, n-1, c_{2}(n, d)) \) maximal torsion \( i \)-dimensional cosets of \( H(f) \).

\[ \square \]

By part (iii) of Theorem 1.3, given a maximal torsion \( i \)-dimensional coset \( C \) on \( H(f) \), its projection \( \pi(C) \) lies on \( H(g_{k}) \) for some \( 1 \leq k \leq m \). If \( i \geq 2 \) then the coset \( \pi(C) \) has positive dimension, and Lemma 4.4 implies the inequality (29). Suppose now that \( i \leq 1 \). Let \( C \) be a maximal \( i \)-dimensional coset on \( H(f) \). The case when \( \pi(C) \) lies in a torsion coset of positive dimension of one of the hypersurfaces \( H(g_{k}) \) is settled by Lemma 4.4. It remains only to consider the case when \( \pi(C) \) is an isolated torsion point. The number of isolated torsion points \( u \) on \( H(f) \) whose projection \( \pi(u) \) is an isolated torsion point on \( H(g_{k}) \) is at most \( dT_{i}^{n-1}(g_{k}) \leq dT(0, n-1, 2d^{2}) \). Now, each isolated torsion point on \( H(g_{k}) \) is the \( \pi \)-projection of at most one torsion \( 1 \)-dimensional coset on \( H(f) \). These observations together with Lemma 4.4 imply the inequalities (27)–(28).

\[ \square \]

### 4.4 Completion of the proof

Put \( T(n, d) = \sum_{i=0}^{n-1} T(i, n, d) \). We will show that for \( n \geq 2 \)
\[ T(n, d) \leq (2nd)^{n+1}T(n-1, n^{8+4n}d^{2})T(n-1, n^{8+4n}d^{3}). \tag{32} \]

This inequality implies Theorem 1.1. Indeed, noting that, by (5), we have \( T(2, d) \leq 11d^{2} + d \) and \( N_{tor}(H(f)) \leq T(n, d) \), we get from (32) the inequality (3).

Let \( f \in \mathbb{C}[X_{1}, \ldots, X_{n}] \) be a polynomial of degree \( d \). The lattice \( L(f) \) clearly has \( n \) linearly independent points in the difference set \( D(S_{f}) \) and \( D(S_{f}) \subset dD(B_{1}^{n}) = 2dB_{1}^{n} \). Therefore, by Lemma 8 in Cassels [8], Ch. V, the lattice \( L(f) \) has a basis lying in \( ndB_{1}^{n} \). Since \( B_{1}^{n} \subset B_{2}^{n} \), for each irreducible factor \( f' \) of \( f \) the inequality
\[ \det(L(f')) \leq (nd)^{n} \]
holds. Then, by Lemmas 4.1–4.3 applied to all irreducible factors of \( f \), we have for all \( 0 \leq i \leq n-1 \)
\[ T_{i}^{n}(f) \leq d(2n+1-1)(nd)^{n} \times T(i, n-1, c_{2}(n, c_{1}(n, d)))T(n-1, 2(c_{1}(n, d))^{2}). \tag{33} \]

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To avoid painstaking estimates we simply observe that for \(n \geq 3\) and for all \(d\) we have
\[
n^{8+4n}d^2 > 2(c_1(n,d))^2 \quad \text{and} \quad n^{8+4n}d^3 > c_2(n,c_1(n,d)).
\]
Then the inequality (33) implies (32).

5 \hspace{1em} \textbf{Proof of Theorem 1.2}

\textbf{Lemma 5.1.} For \(n \geq 2\) the inequality
\[
N_{\text{tor}}(n,d) \leq T(n,d)N_{\text{tor}}(n-1,n^{2+n}d^2)
\]
holds.

\textit{Proof.} Suppose that the variety \(\mathcal{V}\) is defined by the polynomials \(f = f_1, f_2, \ldots, f_t\). Then any maximal torsion coset \(\omega H\) on \(\mathcal{V}\) is contained in a maximal torsion coset \(\omega H'\) on the hypersurface \(\mathcal{H}(f)\). Now, let \(C = \omega H_A\) with \(\omega = (\omega_1, \ldots, \omega_n)\) be a maximal \(i\)-dimensional torsion coset on \(\mathcal{H}(f)\) and suppose \(C\) does not lie on \(\mathcal{V}\). By Corollary 2.1, applied to the subspace \(\text{span}_{\mathbb{R}}(A)\), there exists a basis \(A = (a_1, a_2, \ldots, a_n)\) of the lattice \(\mathbb{Z}^n\) such that \(a_1 \in A\) and its polar basis \(A^* = (a_1^*, a_2^*, \ldots, a_n^*)\) satisfies the inequality (9). Let \((Y_1, \ldots, Y_n)\) be the coordinates associated with the basis \(A\). By (11), the coset \(C^A\) lies on the hypersurface of \(\mathbb{G}_m^n\) defined by the equation
\[
Y_1 = \omega, \quad \text{(35)}
\]
with \(\omega = \omega^{a_1}\). Observe that for any torsion coset \(\zeta H_B \subset \omega H_A\), the lattice \(A\) is a sublattice of the lattice \(B\) and \(\zeta = (\omega_1 x_1, \ldots, \omega_n x_n)\) for some \((x_1, \ldots, x_n) \in H_A\). Consequently, \(\zeta H_B\) also satisfies (35). Then the number of maximal torsion cosets on \(\mathcal{V}\) that are subcosets of \(C\) is at most the number of maximal torsion cosets on the subvariety of \(\mathbb{G}_m^n\) defined by the equations
\[
\begin{align*}
f_2^A(\omega, Y_2, \ldots, Y_n) &= 0, \\
&\vdots \\
f_t^A(\omega, Y_2, \ldots, Y_n) &= 0.
\end{align*}
\]
Note that since \(C \not\subset \mathcal{V}\), not all Laurent polynomials \(f_i^A(\omega, Y_2, \ldots, Y_n)\) are identically zero. The \((n-i)\)-dimensional subspace \(\text{span}_{\mathbb{R}}(A)\) is spanned by \(n-i\) vectors of the difference set \(D(S_f)\). Therefore,
\[
\det(A) \leq (\text{diam}(S_f))^{n-i} < (2d)^{n-i}.
\]
Note that \(\det(A) = \det(\text{span}_{\mathbb{R}}(A) \cap \mathbb{Z}^n)\). Hence, by (9), we have
\[
S_{f_i^A} \subset d(n \max_{1 \leq j \leq n} ||a_j^*||_\infty)B_1^n \subseteq (nd + n(n-1)\gamma_{n-1}^{\frac{n+1}{2}} \gamma_{n-1}^2 d^2)B_1^n
\]
for \(j = 2, \ldots, t\). Multiplying the Laurent polynomials \(f_j^A\) by a monomial, we may assume that \(f_j^A \in \mathbb{C}[Y_2, \ldots, Y_n]\). Noting that \(\gamma_k^{k/2} \leq k!\), we get the inequalities
\[
\deg(f_j^A) < n(n+1)d + (n-1)(n^2-1)n!d^2, \quad j = 2, \ldots, t.
\]


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Finally, observe that for \( n \geq 2, 1 \leq i \leq n - 1 \) and for all \( d \), we have
\[
 n^{2+n}d^2 > n(n+1)d + (n-1)(n^2-1)n!d^2.
\]

By Theorem 1.1, \( T(n, d) \leq c_1(n)d^{2(n)} \) and, consequently,
\[
 N_{\text{tor}}(n, d) \leq c_1(n)d^{2(n)} N_{\text{tor}}(n-1, n^{2+n}d^2).
\]
Noting that \( N_{\text{tor}}(1, d) = T(1, d) = d \) we obtain the inequality (4).

6 The algorithm

Let \( \mathcal{V} \) be an algebraic subvariety of \( \mathbb{G}^n_m \). In this section we will describe a new recursive algorithm that finds all maximal torsion cosets on \( \mathcal{V} \). The algorithm consists of several steps that reduce the problem to finding maximal torsion cosets of a finite number of subvarieties of \( \mathbb{G}^{n-1}_m \). When \( n = 2 \) we can apply the algorithm of Beukers and Smyth [3].

6.1 Hypersurfaces

We first consider a hypersurface \( \mathcal{H} \) defined by a polynomial \( f \in \mathbb{C}[X_1, \ldots, X_n] \) with \( f = \prod h_i \), where \( h_i \) are irreducible polynomials. By Lemma 2.7, the \((n-1)-dimensional torsion cosets on \( \mathcal{H} \) will precisely correspond to the factors \( h_j \) of the form \( X^{u_j} - \omega^j X^{v_j} \), where \( \omega \) is a root of unity. Now we will assume without loss of generality that \( f \) is irreducible and \( \mathcal{H} \) contains no torsion cosets of dimension \( n - 1 \). Then we proceed as follows.

H1. The proofs of Lemmas 4.1, 4.2 and Theorem 1.3 are effective. Consequently, applying Lemmas 4.1 and 4.2, we may assume without loss of generality that \( L(f) = \mathbb{Z}^n \). Next, applying Theorem 1.3, we get \( m < 2^{n+1} \) polynomials \( f_1, \ldots, f_m \) satisfying conditions (i)–(iii) of this theorem.

H2. For \( 1 \leq k \leq m \), calculate \( g_k = \text{Res}(f, f_k, X_n) \). Find all isolated torsion points \( \zeta_1, \zeta_2, \ldots \) and all maximal torsion cosets \( D_1, D_2, \ldots \) of positive dimension on the hypersurfaces \( \mathcal{H}(g_k) \) of \( \mathbb{G}^{n-1}_m \). For each coset \( D_i = \eta_i H_{B_i} \), take a primitive vector \( a_i \in B_i \) and put \( \omega_i = \eta_i^{a_i} \).

H3. For each torsion point \( \zeta_i = (\zeta_{i1}, \ldots, \zeta_{in-1}) \), if \( f(\zeta_{i1}, \ldots, \zeta_{in-1}, X_n) \) is identically zero then the coset
\[
 (\zeta_{i1}, \ldots, \zeta_{in-1}, t)
\]
lies on \( \mathcal{H} \). Otherwise, solving the polynomial equation \( f(\zeta_{i1}, \ldots, \zeta_{in-1}, X_n) \) in \( X_n \), we will find all torsion points \( \zeta \) on \( \mathcal{H} \) with \( \pi(\zeta) = \zeta_i \). When all
torsion cosets of positive dimension on \( \mathcal{H} \) are found, we can easily determine which of the torsion points \( \zeta \) are isolated.

H4. For each \( D_i \), extend the vector \( a_i \) to a basis \( B_i = ((a_i, 0), z_2, \ldots, z_n) \) of \( \mathbb{Z}^n \). Find all maximal torsion cosets \( E_1, E_2, \ldots \) on the hypersurface in \( \mathbb{G}^{n-1}_n \) defined by the polynomial \( f^B_i(\omega, Y_2, \ldots, Y_n) \). For each \( E_j = \rho_j H_{P_j} \) say with \( \rho_j = (\rho_j, \ldots, \rho_j) \) put \( \omega_j = (\omega, \rho_j, \ldots, \rho_j) \) and \( A_j = \{(z, p_2, \ldots, p_n) : z \in \mathbb{Z}, (p_2, \ldots, p_n) \in P_j \} \). Now the cosets \( (\omega_j H_{A_j})^{B_i} \) are the maximal torsion cosets on \( \mathcal{H} \).

6.2 General subvarieties

Suppose now that \( V \) is defined by the polynomials \( f_1, \ldots, f_t \in \mathbb{C}[X_1, \ldots, X_n] \), when \( t \geq 2 \).

V1. Find all isolated torsion points \( \zeta_1, \zeta_2, \ldots \) and all maximal torsion cosets \( D_1, D_2, \ldots \) of positive dimension on the hypersurface \( \mathcal{H}(f_1) \). Then \( \zeta_1, \zeta_2, \ldots \), if on \( V \), are isolated torsion points on \( V \) as well.

V2. For each coset \( D_i = \eta_i H_{B_i} \), take a primitive vector \( a_i \in B_i \), put \( \omega_i = \eta_i a_i \) and extend the vector \( a_i \) to a basis \( B_i = (a_i, z_2, \ldots, z_n) \) of \( \mathbb{Z}^n \). Find all maximal torsion cosets \( E_1, E_2, \ldots \) on the subvariety of \( \mathbb{G}^{n-1}_n \) defined by the polynomials \( f^B_k(\omega, Y_2, \ldots, Y_n) \), \( k = 2, \ldots, t \). For each \( E_j = \rho_j H_{P_j} \) with \( \rho_j = (\rho_j, \ldots, \rho_j) \) put \( \omega_j = (\omega, \rho_j, \ldots, \rho_j) \) and \( A_j = \{(z, p_2, \ldots, p_n) : z \in \mathbb{Z}, (p_2, \ldots, p_n) \in P_j \} \). Now the cosets \( (\omega_j H_{A_j})^{B_i} \), along with the isolated torsion points found in step V1, are the maximal torsion cosets on \( V \).

The described algorithm clearly stops after a finite number of steps and the proofs of Theorems 1.1 and 1.2 show that the algorithm finds all maximal torsion cosets on \( V \). Furthermore, the constants \( c_i(n, d) \) give explicit bounds for the degrees of the polynomials generated at each step.

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References


School of Mathematics and Wales Institute of Mathematical and Computational Sciences, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, UK
E-mail address: alievi@cf.ac.uk

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Kings Buildings, Edinburgh EH9 3JZ, UK
E-mail address: C.Smyth@ed.ac.uk