Uniqueness of Equilibrium in Two-sided Matching

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Abstract

This paper analyses a sufficient condition for uniqueness of equilibrium in two-sided matching with non-transferable utility. The condition is easy to interpret, being based on the notion that a person’s characteristics both form the basis of their attraction to the opposite sex, and determine their own sexual preferences.

Keywords: Uniqueness; matching; marriage.

JEL classification: C7.

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1. Introduction

In 1962 Gale and Shapley posed and solved “the stable marriage problem”, which asks whether it is possible to pair the members of one set (men) with members of another, disjoint, set (women), in such a way that no man and woman who are not paired with each other would both prefer to leave their partners and marry each other. Gale and Shapley proved that such an equilibrium, called a stable matching, exists and they showed how to find it.

This paper is concerned with the uniqueness of equilibrium. I propose and analyse a primitive condition on preferences that ensures a unique stable matching. The central idea is based on the notion that a person’s characteristics, e.g. their physical appearance, their personal qualities, or their productive capabilities, both form the basis of their attraction to the opposite sex and determine who they are attracted to. The condition I propose is called the No Crossing Condition.
(NCC). It has two components: firstly, we must be able to order completely any group of men, $M$, and any group of women, $W$, the implication being that such orderings are based on one or more personal characteristics which make up their type; secondly, men further along the ordering of $M$ tend to prefer women further along the ordering of $W$, and vice versa. The exact sense of “tend to prefer” is made clear in the next section, but the NCC encompasses two special cases: when all members of one sex agree on their preferences for the other sex, and when each person would prefer a partner who is similar to themselves.$^1$

The next section of the paper sets up the formal matching framework, defines the NCC, and proves the central theorem of the paper. Section 3 considers the relationships between the No Crossing Condition and a condition recently proposed by Eeckhout (2000). Section 4 concludes.

2. **Uniqueness of Stable Matching**

2.1. *The Matching Framework*

The standard matching framework considers two finite and disjoint sets, both with $n$ elements: a set of men $M$ and a set of women $W$. We refer to $P = M \cup W$ as the population. Each man has complete, reflexive, and transitive preferences over the set $W$. We assume that these preferences are strict (so that no man is indifferent between two women), and are such that each man would rather be married to any woman than remain single. The preferences of a man $x \in M$ can thus be described by a binary relation $\succ_x$ defined on the set $W$, the statement $y \succ_x y'$ denoting that $x$ prefers $y$ to $y'$. Similar assumptions are made for women’s preferences, *mutatis mutandis*, with the preferences of a woman $y \in W$ described
by a binary relation \( \succ_y \) defined on the set \( M \). Let \( \Phi = \{ \succ_i, i \in P \} \) be the preference profile (or set of preference relations) of the population \( M \cup W \). The triple \((M, W, \Phi)\) constitutes a marriage market.

**Definition 1** A matching \( \mu \) is a one-to-one function from \( P \) onto itself such that
(i) \( x = \mu(y) \) if and only if \( y = \mu(x) \); (ii) if \( x \in M \) then \( \mu(x) \in W \) and if \( y \in W \) then \( \mu(y) \in M \).

**Definition 2** A matching \( \mu \) can be blocked by a pair \((x, y) \in M \times W\) for whom \( x \neq \mu(y) \) if \( y \succ_x \mu(x) \) and \( x \succ_y \mu(y) \). A matching \( \mu \) is stable if it cannot be blocked by any pair.

Then we have:

**Theorem 1** A stable matching exists for every marriage market.

**Proof.** See Gale and Shapley, 1962.

2.2. Ordering the sets \( M \) and \( W \)

The No Crossing Condition requires \( M \) and \( W \) to be ordered. The most straightforward way to approach this is to consider \( M \) and \( W \) when ordered as vectors (i.e. as ordered lists of the elements of \( M \) and \( W \)) with different orderings represented by different vectors. Let \( I_n = \{1, 2, ..., n\} \).

**Definition 3** The vector \( m = (m_i) \) is an ordering of \( M \) if (i) \( m \) has \( n \) elements; (ii) for all \( i \in I_n, m_i \in M \); (iii) for all \( x \in M, x = m_i \) for some \( i \in I_n \); similarly the vector \( w = (w_i) \) is an ordering of \( W \) if (i) \( w \) has \( n \) elements; (ii) for all \( i \in I_n, w_i \in W \); (iii) for all \( y \in W, y = w_i \) for some \( i \in I_n \).
We can now move easily from one ordering of $M$ or $W$ to another; this will be useful when we analyse the relationship between the No Crossing Condition and the condition proposed by Eeckhout (2000).

2.3. The No Crossing Condition

This may now be stated quite simply:

**Definition 4** A population $P = M \cup W$ with preference profile $\Phi$ satisfies the No Crossing Condition (NCC) if there exists an ordering $m$ of $M$ and an ordering $w$ of $W$ such that if $i < j$ and $k < l$ then

(i) it is not the case that both $m_l \succ_w m_k$ and $m_k \succ_w m_l$;

(ii) it is not the case that both $w_j \succ_m w_i$ and $w_i \succ_m w_j$.

It is sometimes convenient to refer to the orderings $m$ and $w$ themselves as satisfying the NCC. Part (i) of the definition may be interpreted as saying that it cannot be the case that the woman further back in the female ordering ($w_i$) prefers the man further forward in the male ordering ($m_l$) and at the same time woman further forward in the female ordering ($w_j$) prefers the man further back in the male ordering ($m_k$). Diagrammatically this rules out the preferences depicted in Figure 1, where the sexes are ordered along the two horizontal lines, and the arrow from each woman points to the man she prefers out of the two shown. If the NCC is satisfied there exist orderings $m$ and $w$ such that for any pair of women and any pair of men the two arrows representing the women’s preferences do not cross, nor do the two arrows representing the men’s preferences.

The condition does not rule out the possibility that both women prefer the same man; more generally it allows for all members of one sex to have the same
Figure 1: Preferences ruled out by the No Crossing Condition

preferences. Nor does it necessarily imply 'single-peakedness' of preferences”, given
the orderings $m$ and $w$. For example if $n = 3$ the NCC does not forbid preferences
for $w_1$ such that $m_3 \succ_w m_1 \succ_w m_2$. However in this case the NCC rules out
either $m_1 \succ_w m_3$ or $m_2 \succ_w m$, although it has nothing to say about $w_2$’s ranking
of $m_1$ and $m_2$.

A very important property of the NCC is that if it holds for a population $P$
then it also holds for any sub-population of $P$ with equal numbers of men and
women. Formally:

**Lemma 1** Let the population $P = M \cup W$ satisfy the No Crossing Condition and
let $M' \subset M$ and $W' \subset W$, where $\#(M') = \#(W') = n'$; then the population
$P' = M' \cup W'$ satisfies the No Crossing Condition.

**Proof.** See Appendix.

2.3.1. No Crossing and the Existence of Fixed Pairs

We now develop two lemmas that lie at the heart of the main theorem on
uniqueness. If we can find any couple who love each other (where “love” means
“prefer, out of the members of the opposite sex) then each such couple, called a fixed pair, must be matched in equilibrium. The main theorem then uses the No Crossing Condition to identify a sequence of $n$ fixed pairs, thus generating a unique equilibrium matching.

**Definition 5** A couple $(x, y) \in M \times W$ is a fixed pair of the population $P = M \cup W$ if $y \succ_x y'$ for all $y' \in W \setminus y$ and $x \succ_y x'$ for all $x' \in M \setminus x$.

The advantage of being able to identify fixed pairs is that any stable matching of $P$ must consist of the partnerships of the fixed pairs of $P$ plus a stable matching of the remainder of the population. Formally.

**Lemma 2** Let $\mu$ be a stable matching of the population $P = M \cup W$ which has $p$ fixed pairs $(x_i, y_i), i \in I_p$; let $P' = M' \cup W'$, where $M' = M \setminus \{x_1, x_2, \ldots, x_p\}$ and $W' = W \setminus \{y_1, y_2, \ldots, y_p\}$; and let $\mu'$ be a matching of the population $P'$ defined by $\mu'(z) = \mu(z)$ for all $z \in P'$. Then (i) $\mu(x_i) = y_i$ for all $i \in I_p$; (ii) $\mu'$ is a stable matching of $P'$.

**Proof.** See Appendix.

The problem of finding a unique stable matching of the population $P$ can thus be broken down into finding the fixed pairs of $P$, and then finding a unique stable matching of $P'$. But this requires that $P$ does indeed have at least one fixed pair: enter the No Crossing Condition.

**Lemma 3** If a population $P = M \cup W$ satisfies the No Crossing Condition, then it has a fixed pair.

**Lemma 4** **Proof.** See Appendix.
The proof considers the function that for each man $x$ gives $x$’s rival, the preferred man of $x$’s preferred woman. Given orderings $m$ and $w$ satisfying the NCC, this function is non-decreasing in the sense that if $x$ is further along the ordering $m$ then $x$’s rival can be no further back in the ordering. The existence of a fixed point, and hence of a fixed pair, is almost immediate. But there is no reason to suppose that if $m_i$ and $w_k$ are a fixed pair then $i = k$. For example, the shortest man and the tallest woman form a fixed pair if everyone prefers a partner as tall as themselves and all men are taller than all women. Of course, the fixed point is not necessarily unique.

2.4. The Main Theorem

Bringing together Lemmas 4, 5, and 6, we now have the main uniqueness result:

**Theorem 2** If a population $P = M \cup W$ satisfies the No Crossing Condition then there exists a unique stable matching

**Proof.** See Appendix

The proof shows how to construct the unique stable matching: first match the fixed pairs of $P$; take the remaining population $P_2$, match the fixed pairs of $P_2$; and so on, until the population is exhausted (perhaps literally). To illustrate how the successive identification of fixed pairs leads to a unique stable matching, consider the following example of a population of women with heights 1.50, 1.64, 1.69, 1.78, and men with heights 1.60, 1.67, 1.72, 1.80. Each individual would prefer to be matched with someone as near to their own height as possible i.e. someone of height $h_1$ matched with someone of height $h_2$ has utility that is a negative function of $|h_1 - h_2|$. Such preferences satisfy the No Crossing Condition,
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Figure 2: Preferences to illustrate Theorem 2

and are illustrated in Figure 2 (the arrows from each person point to her/his most preferred partner.

The fixed pairs of this population are \((w_3, m_2)\) and \((w_4, m_4)\), so they are matched in any stable matching. The remaining population, \(P_2 = W_2 \cup M_2\), equals \(\{w_1, w_2, m_1, m_3\}\), with heights 1.50, 1.64, 1.60, and 1.72 respectively. \(P_2\) also satisfies the No Crossing Condition and the couple \((w_2, m_1)\) are a fixed pair; note that \(w_2\) would have preferred \(m_2\) but he is already matched with \(w_3\). Finally, \(w_1\) and \(m_3\) are left and must be matched. This process by which new fixed pairs emerge as others are taken out of the population is illustrated in Fig 3, where the bold double arrows denote a fixed pair of the population or sub-population under consideration.
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Figure 3: The emergence of fixed pairs, denoted by bold double arrows.

\[ w_1 = 1.50 \]
\[ m_1 = 1.60 \]
\[ w_2 = 1.64 \]
\[ m_2 = 1.67 \]
\[ w_3 = 1.69 \]
\[ m_3 = 1.72 \]
\[ w_4 = 1.78 \]
\[ m_4 = 1.80 \]
3. No Crossing and Sequential Preferences

Recently, Eeckhout (2000) has suggested a sufficient condition for uniqueness. Let $m$ and $w$ be orderings of $M$ and $W$ respectively, not necessarily satisfying the NCC. Suppose that, for $i < n$, $m_i$ prefers $w_i$ to all the women from $w_{i+1}$ to $w_n$, and $w_i$ prefers $m_i$ to all the men from $m_{i+1}$ to $m_n$. I call this the Sequential Preference Condition (SPC), and for convenience refer to the orderings $m$ and $w$ as satisfying the SPC. Then there is a unique stable matching in which $m_i$ is matched with $w_i$, for all $i$ (Theorem 1 in Eeckhout, 2000). The equilibrium can be constructed by a sequential process: $m_1$ and $w_1$ prefer each other above all others and so must be paired in any stable matching (since they could block any matching in which they were not paired); $m_2$ and $w_2$ prefer each other to anyone else in $W \setminus w_1$ and $M \setminus m_1$ respectively and hence must also be paired in equilibrium (since they could block any matching in which they were not paired but $m_1$ and $w_1$ were paired); and so on, until we are left with $m_n$ and $w_n$, who would rather marry each other than remain single.

A drawback of the Sequential Preference Condition is that it does not indicate when or why it might be satisfied in any particular population. It applies a test to the preference orderings of a given population, but gives no clue about the underlying structure of tastes that might result in the test being satisfied. Consequently whether the SPC holds or not depends critically on the exact membership of the sets $W$ and $M$. For example, if we reduce both the number of men and the number of women by one, the condition may no longer hold (unless we take out the $i^{th}$ man and the $i^{th}$ woman). This possibility is illustrated in Table 1, which gives the preferences of a population of three men and three women:
The SPC is satisfied and \( m_i \) is matched with \( w_i \), \( i = 1, 2, 3 \). But if we take out \( m_1 \) and \( w_2 \) and consider the population of \( M' = \{m_2, m_3\} \) and \( W' = \{w_1, w_3\} \) then \( w_1 \) prefers \( m_2 \) who prefers \( w_3 \) who prefers \( m_3 \) who prefers \( w_1 \), with the result that both of the possible matchings are stable.

What is the relationship between the NCC and Eeckhout’s SPC? Let \( m \) and \( w \) be orderings satisfying the NCC for a population \( P = M \cup W \). Recall that \( P \) satisfies the SPC if there exists orderings \( m' \) and \( w' \) such that for \( i < n \), \( m'_i \) prefers \( w'_i \) to all the women from \( w'_{i+1} \) to \( w'_n \), and \( w'_i \) prefers \( m'_i \) to all the men from \( m'_{i+1} \) to \( m'_n \). If \( P \) satisfies the NCC, orderings \( m' \) and \( w' \) satisfying the SPC can be derived from the order in which the fixed pairs of \( P \) and its subpopulations are generated in constructing the unique stable matching of \( P \). In essence, the \( k^{th} \) elements of \( m' \) and \( w' \) must be the \( k^{th} \) fixed pair in that sequence. More precisely, consider the sequence of populations \( \{P_s\}, s = 1, ..., S \) such that \( P_{s+1} = P_s \setminus F_s \), with \( P_1 = P \), where \( F_s \) is the set of individuals who constitute the fixed pairs of \( P_s \). Each man in \( M \) and each woman in \( W \) is an element of only one set in the sequence \( \{F_s\} \). Let \( n_s = \#F_s/2 \) and \( N_s = \sum_{t=1}^s n_t \) (so that \( N_S = n \)) and set \( N_0 = 0 \). Then the vectors \( m' \) and \( w' \) satisfy the SPC if (i) they are orderings of

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Table 1
$M$ and $W$ respectively and (ii) for any $k \leq n$, $m'_k = m_i$ and $w'_k = w_j$ for some $i, j$ such that $m_i$ and $w_j$ are a fixed pair of $P_s$, where $s$ is uniquely defined by the condition $N_{s-1} < k \leq N_s$.

In short, the NCC implies the SPC, for a given population $P$. The reverse is not true, as the example of Table 1 shows; the population $P = M \cup W$ satisfies the SPC, but not the NCC; if $P$ did satisfy the NCC then so would its subset $P' = M' \cup W'$, which would ensure a unique stable matching of $P'$.

An alternative view of the relationship between the NCC and the SPC is to regard $M$ and $W$ as the sets of all possible men and women. In any particular instance, we are therefore dealing with subsets $M'$ and $W'$. From Lemma 1, if $P = M \cup W$ satisfies the NCC, so does $P' = M' \cup W'$, so it is sufficient, when analysing the population $P'$, to show or assume that $P$ satisfies the NCC. Moreover, when considering the orderings $m'$ and $w'$ of $M'$ and $W'$ that satisfy the NCC, the order in which any two men or two women appear in $m'$ or $w'$ is independent of the other elements in those vectors. In effect, the orderings $m$ and $w$ of $M$ and $W$ that satisfy the NCC may be treated as “master orderings”. But consider now the emergence of fixed pairs as the stable matching is constructed (for example, as in Fig.3). The positions in $m'$ and $w'$ of the first fixed pair, the second fixed pair, and so on, may bear no resemblance to the order in which they emerge, and will typically vary with the precise membership of $M'$ and $W'$. But it is the order in which they emerge as fixed pairs that gives the orderings that satisfy the SPC. Both theoretically and from an applied perspective, it seems an advantage that the ordering satisfying the NCC is invariant to the particular groups $M'$ and $W'$ being considered.
4. Conclusion

Uniqueness of equilibrium is typically regarded as a desirable characteristic of an economic model. It helps to make prediction and comparative statics sharp and unambiguous. This paper shows that the standard model of two-sided matching has a unique stable matching if agents’ preferences satisfy a condition, that is both intuitively reasonable and easy to interpret, being based on the notion that a person’s characteristics both form the basis of their attraction to the opposite sex, and determine their own sexual preferences. If we are prepared to assume that men and women can be ordered on the basis of their characteristics, and that men further along the male ordering tend to prefer women further along the female ordering, then the No Crossing Condition is satisfied and equilibrium is unique.

One application of the results of this paper may be found in Clark and Kanbur (2002), which looks at the question of assortment in two-sided matching. One possible interpretation of the NCC is that agents tend to prefer partners who are similar to themselves. It might therefore be thought that in the equilibrium matching like will match with like, resulting in positive assortment. Clark and Kanbur show that this line of reasoning is incorrect and, using the NCC, show how the degree of assortment depends on, *inter alia*, the distribution of agents characteristics. They thus demonstrate how information about utility functions can be used to see if the NCC is satisfied, regardless of the precise membership of the two groups to be matched.
Appendix

Proof of Lemma 1. Take the orderings $m$ and $w$ that satisfy the NCC for the population $P$, delete those elements corresponding to $M \setminus M'$ and $W \setminus W'$ to form the $n'$ dimensional vectors $m'$ and $w'$. Then since $m'$ and $w'$ must continue to satisfy conditions (i) and (ii) in Definition 4 they are orderings that satisfy the NCC for the population $P'$.

Proof of Lemma 2. If (i) is not satisfied then $\mu$ can be blocked by one of the fixed pairs $(x_i, y_i)$ $i \in I_{n'}$ and hence cannot be stable, a contradiction. If (i) is satisfied but not (ii), then there exists a pair $(x', y') \in M' \times W'$, with $x' \neq \mu'(y)$, who can block the matching $\mu'$ i.e. $y' \succ_{x'} \mu'(x')$ and $x' \succ_{w'} \mu'(y')$. The definition of $\mu'$ ($\mu'(z) = \mu(z)$ for all $z \in P'$) implies that $\mu'(x') = \mu(x')$ and $\mu'(y') = \mu(y')$ for all $y' \in W'$, so that $y' \succ_{x'} \mu(x')$ and $x' \succ_{w'} \mu(y')$. This means that the pair $(x', y')$ can block the matching $\mu$, and hence $\mu$ cannot be stable, a contradiction.

Proof of Lemma 3. For any man $x \in M$, let $f(x) \in W$ denote his preferred woman in $W$; i.e. $f(x) \succ_x y$ for all $y \in W \setminus f(x)$. Since preferences are complete and strict, $f(x)$ exists and is unique. Similarly, for any woman $y \in W$, let $g(y) \in M$ denote her preferred man in $M$; i.e. $g(y) \succ_y x$ for all $x \in W \setminus g(y)$. $g(y)$ also exists and is unique. Let $m$ and $w$ be orderings of $M$ and $W$ satisfying the NCC for $P = M \cup W$. For each element of $m$ the function $f$ specifies an element of $w$; this in turn defines a function $\phi : I_n \rightarrow I_n$ as follows: if $f(m_k) = w_i$ then $\phi(k) = i$, which may be read as “the $k^{th}$ man prefers the $i^{th}$ woman”. Compare the preferences of $m_k$ and $m_l$, where $k < l$. If $f(m_k) = f(m_l)$ then $\phi(k) = \phi(l)$. If $f(m_k) \neq f(m_l)$, then $\phi(k) \neq \phi(l)$ and $w_{\phi(k)} \succ_{m_k} w_{\phi(l)}$; but if $\phi(l) < \phi(k)$ then part (ii) of Definition 4 of the NCC, with $i = \phi(l)$ and $j = \phi(k)$, implies that $w_{\phi(k)} \succ_{m_l} w_{\phi(l)}$, a clear contradiction. Hence if $k < l$ then $\phi(k) \leq \phi(l)$ i.e. the function $\phi$ is non-
decreasing. A similar argument applies to the function $\gamma : I_n \rightarrow I_n$, defined as follows: if $g(w_i) = m_k$ then $\gamma(i) = k$; if $i < j$ then $\gamma(i) \leq \gamma(j)$ i.e. the function $\gamma$ is non-decreasing. Now, consider the composition of $\phi$ and $\gamma$, the function $\rho(k) = \gamma(\phi(k))$; this gives the position, in the ordering $m$, of the preferred man of the $k^{th}$ man’s preferred woman (the $k^{th}$ man’s rival); i.e. $m_{\rho(k)} = g(f(m_k))$.

Since $\phi$ and $\gamma$ both map $I_n$ into $I_n$ and are non-decreasing the function $\rho$ also maps $I_n$ into $I_n$ and is non-decreasing. It therefore has a fixed point $k^* = \rho(k^*)$. Let $i^* = \phi(k^*)$; then $k^* = \gamma(i^*)$; thus $w_{i^*} = f(m_{k^*})$ and $m_{k^*} = g(w_{i^*})$; i.e. $m_{k^*}$ and $w_{i^*}$ are a fixed pair. 

Remark. The proof of Lemma 3 could have proceeded by considering the composition of $\gamma$ and $\phi$, the function $\kappa(i) = \phi(\gamma(i))$ (the $i^{th}$ woman’s rival). Clearly if $k^*$ is a fixed point of $\rho$, then $i^* = \phi(k^*)$ is a fixed point of $\kappa$.

**Proof of main theorem.** We consider a sequence of populations $\{P_s\}, s = 1, \ldots, S$ such that $P_{s+1} = P_s \setminus F_s$, with $P_1 = P$, where $F_s$ is the set of individuals who constitute the fixed pairs of $P_s$; i.e. if $(x, y)$ is a fixed pair of $P_s$ then $\{x, y\} \subseteq F_s$. $S$ is defined by the condition $P_S = F_S$. Since $F_s$ is unique given $P_s$, the sequence $\{P_s\}$ is uniquely defined. By the repeated application of Lemmas 1 and 3, each element in the sequence $\{P_s\}$ satisfies the No Crossing Condition and has at least one fixed pair. Thus $P_{s+1}$ is a proper subset of $P_s$, and since $P$ is finite $S$ exists and is finite.

Let $\mu$ be any stable matching of $P$, and for all $s \in I_S$, let $\mu_s$ be a matching of $P_s$ defined by $\mu_s(z) = \mu(z)$ for all $z \in P_s$; i.e. $\mu_s$ is the matching $\mu$ as it applies to the population $P_s$. Lemma 2, part (ii), says that if $\mu_s$ is a stable matching of $P_s$ then $\mu_{s+1}$ is a stable matching of $P_{s+1}$. But since $\mu = \mu_1$ is a stable matching
of $P = P_1$, this implies that for all $s \in I_S$, $\mu_s$ is stable. Then by Lemma 2, part (i), for all $s \in I_S$, $x = \mu_s(y)$ for any fixed pair $(x, y)$ of $P_s$; and hence, given the definition of $\mu_s$, we have $x = \mu(y)$ for any fixed pair $(x, y)$ of $P_s$ and for all $s \in I_S$. But since $P = \cup_{s=1}^{S} F_s$ every individual in $P$ is a member of some fixed pair of some population $P_s$ and is therefore matched by $\mu$ with the other member of the fixed pair. Since the sequence $\{P_s\}$ and the associated sequence $\{F_s\}$ are independent of the choice of which stable matching $\mu$ of $P$ we consider, the matching $\mu$ is uniquely determined. ■

References


Notes

1 Clark and Kanbur (2002) analyse these two special cases based on preferences for local public goods. A central issue is whether the equilibrium matching displays positive assortment. It does when all members of one sex agree on their preferences
for the other sex, but when each person would prefer a partner who is similar to themselves positive assortment is only one of many possibilities