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Citation for published version:

Digital Object Identifier (DOI):
10.1103/PhysRevD.73.074502

Link:
Link to publication record in Edinburgh Research Explorer

Published In:
Physical Review D

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B Meson Semileptonic Form Factors from Unquenched Lattice QCD

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The semileptonic process, \( B \to \pi l\nu \), is studied via full QCD lattice simulations. We use unquenched gauge configurations generated by the MILC collaboration. These include the effect of vacuum polarization from three quark flavors: the \( s \) quark and two very light flavors (\( u/d \)) of variable mass allowing extrapolations to the physical chiral limit. We employ Nonrelativistic QCD to simulate the \( b \) quark and a highly improved staggered quark action for the light sea and valence quarks. We calculate the form factors \( f_+(q^2) \) and \( f_0(q^2) \) in the chiral limit for the range \( 16 \text{GeV}^2 \leq q^2 < q^2_{\text{max}} \) and obtain \( f_{+}(q^{2}) \big|_{\text{MILC}}^{\text{MILC}} \big|_{\text{HPQCD}} < \frac{d\Gamma}{dq^2} \big|_{\text{MILC}}^{\text{MILC}} f_{0}(q^{2}) \big|_{\text{MILC}}^{\text{MILC}} < \frac{d\Gamma}{dq^2} \big|_{\text{MILC}}^{\text{MILC}} \). Combining this with a preliminary average by the Heavy Flavor Averaging Group (HFAG’05) of recent branching fraction data for exclusive \( B \) semileptonic decays from the BaBar, Belle and CLEO collaborations, leads to \( |V_{ub}| = 3.55(25)(50) \times 10^{-3} \).

I. INTRODUCTION

A major achievement of the \( B \) factories in recent years has been the observation of CP violation in the neutral \( B \) system [1, 2]. The emphasis since then has been on overconstraining the unitarity triangle and checking for consistency with, or deviations from, the three family Standard Model. The goal is to independently measure not only the three angles but also the lengths of the sides of the triangle and thereby determine (7, 7), the apex of the unitarity triangle, in as many ways as possible. In order to fix the sides of the unitarity triangle, the magnitudes of several Cabibbo-Kobayashi-Maskawa (CKM) matrix elements are required and the accuracy is currently limited mainly by theoretical uncertainties in two of them, \( |V_{cb}| \) and \( |V_{ub}| \). Theory input for these quantities involves hadronic matrix elements of several electroweak operators and these in turn require good control over nonperturbative QCD. This article reports on significant recent progress in lattice QCD determinations of form factors relevant for the CKM matrix element \( |V_{ub}| \).

\( |V_{ub}| \) can be determined from studies of either inclusive or exclusive \( B \) meson semileptonic decays. The first determinations relied on inclusive measurements. However, recent impressive progress in measurements of branching fractions for exclusive decays by CLEO [3], Belle [4, 5] and BaBar [6, 7, 8] have started to make exclusive determinations competitive [6, 10]. In either approach errors are now dominated by theory errors and which method will eventually win out depends on how well the theoretical uncertainties in shape functions (for the inclusive approach) or form factors (in the case of the exclusive approach) can be brought under control. Lattice QCD provides a first principles nonperturbative QCD method for calculating form factors in exclusive semileptonic decays. The first lattice calculations were carried out in the quenched approximation that ignored vacuum polarization effects [11, 12, 13, 14, 15]. For some period these pioneering results were the only ones available and experimentalists, e.g. the CLEO collaboration, used them to extract some of the earliest exclusive \( |V_{ub}| \) results [3]. In the summer of 2004 the first preliminary unquenched results for the \( B \to \pi l\nu \) form factors were presented by the Fermilab/MILC [16] and HPQCD [17] collaborations. These calculations employed the MILC collaboration \( N_f = 2 + 1 \) unquenched configurations, the most realistic gauge configurations to date with vacuum polarization from 2 flavors of very light quarks and from strange quarks [15]. Furthermore, the good chiral properties of the improved staggered quark action used for the light sea and valence quarks, allowed for investigations much closer to the chiral limit than in the earlier calculations. Hence, references [16, 17] constitute a major step forward in lattice QCD determinations of semileptonic form factors. These preliminary results have been incorporated by Belle and BaBar into their recent \( B \) semileptonic analysis [3, 6, 7, 8, 4, 5] and used by other theorists...
in their extractions of $|V_{ub}|$ \cite{19, 20, 21}.

References \cite{16, 17} are part of a growing list of recent lattice calculations that use the MILC unquenched configurations. The creation of these configurations became feasible on present day computers due to the development of highly improved staggered light quark actions \cite{22}. There is one well known drawback of staggered actions, namely that each flavor comes in four types, called “tastes”. To simulate just one taste of sea quark per flavor a fourth root of the quark determinant is used and the validity of this procedure has not yet been rigorously proven. All tests undertaken to date to study the fourth root procedure, however, have led to encouraging results \cite{23}. There are no indications of problems or any deviations from continuum QCD behavior beyond small and expected discretisation errors that can be systematically improved upon. A recent review of the fourth root issue is given in \cite{24}. The MILC unquenched configurations and the heavy and light quark actions currently in use have also been tested repeatedly by calculating a large set of well measured quantities such as light hadron spectroscopy, light meson decay constants, quarkonium and B meson spectroscopy. Agreement between experiment and lattice calculations is found to be excellent within the few percent errors in the lattice results \cite{25, 26, 27, 28, 29}. More impressively, these same gauge configurations have led to some recent predictions from Lattice QCD that have been confirmed subsequently by experiment \cite{30, 31, 32}. Most relevant for B semileptonic decays, is the successful calculation by the Fermilab/MILC collaboration of D meson semileptonic form factors and the agreement of their $q^2$ dependence with experiment \cite{32}. All this should boost confidence in results for B meson semileptonic decays that are now emerging from Lattice QCD calculations.

In this article we make several further improvements on the preliminary calculations of ref. \cite{17} and finalize our form factor results. Some of these improvements have been reported on in \cite{33}. Among the improvements, we have now included all the dimension 4 current corrections through $O(\alpha_s/M)$ and $O(\alpha_s^2)$ to the temporal and spatial components of the heavy-light electroweak currents relevant for $B \rightarrow \pi \ell \nu$ semileptonic decays. Our previous results included currents only at lowest order in $1/M$ and through $O(\alpha_s)$. We now have simulation data from several MILC ensembles, whereas in ref. \cite{17} results were at a single fixed light sea quark mass and only the valence quark mass was varied. Another development since ref. \cite{17} is that we now employ the formulas of staggered chiral perturbation theory \cite{34, 35, 36} simultaneously to both partially quenched and full QCD results to extrapolate in the light quark mass to the physical pion. Previously we used linear chiral extrapolations to our partially quenched data.

So, the list of improvements in our semileptonic decay studies since the work of ref. \cite{17} is quite substantial. Nevertheless, we will see that changes in the final results for the form factors $f_+(q^2)$ and $f_0(q^2)$ are almost negligible. The individual contribution from each higher dimension current correction is found to be small, mainly due to small one-loop matching coefficients. There is also some cancellation between the many terms. The difference between linear chiral extrapolations of our previous partially quenched data and results from staggered chiral perturbation theory fits to the new full data set also turns out to be a small effect. With all the improvements now in place, our semileptonic form factor calculations are at the same level as recent $B$ meson decay constant determinations by the HPQCD collaboration \cite{37, 38}. Those latter calculations are crucial for determinations of the CKM matrix elements $|V_{td}|$ and $|V_{ts}|$.

In the next section we provide some details of the simulation parameters and of the lattice actions employed. We also summarize formulas for the relevant form factors.

In section III. we discuss matching of the lattice heavy-light currents used in our simulations to their continuum QCD counter parts. One-loop matching coefficients for the temporal components $V_0$ and $A_0$ were published in ref. \cite{39} and have been used already in our decay constant determinations \cite{39, 38}. Matching of the spatial components $V_\perp$ was completed as part of the current project. Section IV. describes how we extract the form factors $f_\parallel$ and $f_\perp$ from numerical simulations. The more commonly used form factors $f_+$ and $f_0$ can be expressed as simple linear combinations of $f_\parallel$ and $f_\perp$. Section V. focuses on chiral extrapolations of $f_\parallel$ and $f_\perp$. In section VI. we summarize our final results for $f_+(q^2)$ and $f_0(q^2)$ in the physical chiral limit. We also present Tables of partially integrated differential decay rates (divided by $|V_{ub}|^2$) in several $q^2$ bins. In section VII. we combine the results of section VI. with experimental data on $B \rightarrow \pi \ell \nu$ branching fractions to estimate $|V_{ub}|$. We then conclude with a summary section.

II. SIMULATION DETAILS AND FORM FACTOR FORMULAS

Most of our simulations were performed on the MILC collaboration “coarse” ensembles with lattice spacing around 0.12fm and with light sea quark masses in units of the strange quark mass, $m_f/m_s$, ranging between 0.125 and 0.5. We have also carried out some checks using MILC lattices with finer lattice spacings. We refer to the original papers by the MILC collaboration for details of how their lattices were generated \cite{18}. Some specifications for the ensembles are given in Table I. The highly improved staggered action, the AsqTad action, is used for both the sea and valence light quarks. The leading discretisation errors in this formalism are $O(a^2 \alpha_s)$. To simulate the $b$ quark inside the $B$ meson we use the same highly improved nonrelativistic (NRQCD) action employed in recent studies of the $\Upsilon$ system on the same MILC ensembles \cite{28, 40}. In terms of the two-component
the lattice spacing was determined through discretization of the continuum and ∆(4)c is the symmetric lattice derivative and ∆(4)c

\[ \delta H = -c_1 \frac{(\Delta(2))^2}{8M_0^3} + c_2 \frac{ig}{8M_0^2} \left( \nabla \cdot \vec{E} - \vec{E} \cdot \nabla \right) \]

\[ -c_3 \frac{g}{8M_0^2} \sigma \cdot \left( \nabla \times \vec{E} - \vec{E} \times \nabla \right) \]

\[ -c_4 \frac{g}{2M_0^2} \sigma \cdot \vec{B} + c_5 \frac{a^2(D(4))}{24M_0} - c_6 \frac{a(\Delta(2))^2}{16nM_0^2} \]  

\( M_0 \) is the bare b-quark mass, \( \Delta(2) \) the lattice Laplacian, \( \nabla \) the symmetric lattice derivative and \( \Delta(4c) \) the lattice discretization of the continuum \( \sum D_i^4 \). Expressions for the improved \( \vec{E} \) and \( \vec{B} \) fields are given in [28]. All derivatives are tadpole improved. As in [28] we set the \( c_i \)'s to their tree level value, \( c_i = 1 \). These coefficients will be modified by radiative corrections at higher order. For heavy-light systems the most important radiative corrections will be to \( c_4 \). All other \( c_i \)'s are multiplied by additional factors of \( 1/M_0 \) or \( a/M_0 \). Based on the discussions in [28] we estimate the effect from radiative corrections to \( c_4 \) on \( B \) semileptonic form factor calculations to be less than 1%.

Just as in continuum QCD, our lattice actions include a small number of parameters that can only be fixed via experimental input. These are the bare quark mass parameters and the scale (or coupling). For the present case of heavy-light simulations the action parameters have already been fixed for us via simulations of quarkonium and light quark systems using the same MILC configurations. We use lattice spacings determined by the \( \Upsilon \) 2S-1S splitting [28]. For two of the MILC ensembles (denoted by \( \Upsilon \) in Table I), \( \Upsilon \) simulations have not yet been carried out. On the other hand, the MILC collaboration has measured the heavy quark potential parameter \( r_1/a \) (in lattice units) for these ensembles [26]. In reference [28], using MILC ensembles on which both \( r_1/a \) and the \( \Upsilon \) 2S-1S splittings were calculated, we could use the experimental 2S-1S splitting to determine \( r_1 = 0.321(5) \) fm. This physical value for \( r_1 \) was combined with the MILC collaboration \( r_1/a \) to fix the scale for the above two ensembles. The \( b \) quark mass is also fixed by our \( \Upsilon \) studies. Studies of pions and kaons have fixed the \( u \) and \( d \) masses (which we take to be equal to each other) and the \( s \) quark mass respectively [26,27]. Hence by the time one gets to the \( B \) system there are no adjustable parameters left in our QCD action.

To study the process \( B \to \pi \ell \nu \), one needs to evaluate the matrix element of the charged electroweak current between the \( B \) and the \( \pi \) states, \( (\pi |J^\mu|B) \). Only the vector current \( J^\mu \) contributes to the pseudoscalar-to-pseudoscalar amplitude and the matrix element can be written in terms of two form factors \( f_+(q^2) \) and \( f_0(q^2) \). These depend only on the square of the momentum transferred between the \( B \) and the \( \pi \), \( q^\mu = p_B^\mu - p_\pi^\mu \).

\[ \langle \pi(p_\pi)|J^\mu|B(p_B) \rangle = f_+(q^2) \left[ p_B^\mu + p_{\pi}^\mu - \frac{M_B^2 - m_{\pi}^2}{q^2} q^\mu \right] \]

\[ + f_0(q^2) \frac{M_B^2 - m_{\pi}^2}{q^2} q^\mu . \]  

If one neglects the mass of the charged lepton in the final state, only the form factor \( f_+(q^2) \) contributes to the decay rate \( \Gamma(B \to \pi \ell \nu) \). Nevertheless, it is useful to keep track of the form factor \( f_0(q^2) \) as well since, as we shall see, it helps in our interpolations and extrapolations of simulation data. In our data analysis another pair of form factors, \( f_{\parallel} \) and \( f_{\perp} \), turn out to be more convenient.

\[ \langle \pi(p_\pi)|V^{\mu}|B(p_B) \rangle = \sqrt{2M_B} |v^\mu f_{\parallel} + p_{\perp}^\mu f_{\perp}| , \]

with

\[ v^\mu = \frac{p_B^\mu}{M_B} \], \hspace{1cm} p_{\parallel}^\mu = p_B^\mu - (p_\pi \cdot v) v^\mu . \]  

In the \( B \) rest frame (in this article we only consider \( B \) mesons decaying at rest) the temporal and spatial parts of (5) become,

\[ \langle \pi|V^0|B \rangle = \sqrt{2M_B} f_{\parallel} \]

\[ \langle \pi|V^k|B \rangle = \sqrt{2M_B} p_{\parallel}^k f_{\perp} . \]
Hence, one sees that one can separately determine \( f_\parallel \) or \( f_\perp \) simply by looking at either the temporal or spatial component of \( V^\mu \). These two form factors have the additional advantage that they have simpler HQET scaling properties and chiral perturbation theory is carried out in terms of them rather than for \( f_+ \) and \( f_0 \). After carrying out the chiral extrapolations for \( f_\parallel \) and \( f_\perp \), we convert back to obtain \( f_+ \) and \( f_0 \) for the physical theory using,

\[
f_+ = \frac{1}{\sqrt{2M_B}} f_\parallel + \frac{1}{\sqrt{2M_B}} (M_B - E_\pi) f_\perp
\]

\[
f_0 = \frac{\sqrt{2M_B}}{(M_B^2 - m_\pi^2)} [(M_B - E_\pi) f_\parallel + (E_\pi^2 - m_\pi^2) f_\perp],
\]

where \( E_\pi \) is the pion energy in the \( B \) rest frame. From these formulas one sees that \( f_+ \) will be dominated by \( f_\perp \), i.e. by the matrix element of \( V_\perp \), and \( f_0 \) by \( f_\parallel \) or the matrix element of \( V_0 \).

Our goal is to evaluate the hadronic matrix elements \( \langle \pi|V^0|B \rangle \) and \( \langle \pi|V^k|B \rangle \) via lattice simulations. There are several steps in the calculation. First, one must relate the continuum electroweak currents, \( V^0 \) and \( V^k \), to lattice operators written in terms of the heavy and light quark fields in our lattice actions. In the second step the matrix elements of these lattice current operators must be evaluated numerically and the relevant amplitude, i.e. the matrix element between the ground state \( B \) meson and the ground state pion with appropriate momenta must be extracted. This will give us, via eqns.(7), the form factors one sees that

\[
J_0^{(0)}(x) = \bar{q}(x) \Gamma_0 Q(x),
\]

\[
J_0^{(1)}(x) = -\frac{1}{2M_0} \bar{q}(x) \Gamma_0 \gamma \cdot \nabla Q(x),
\]

\[
J_0^{(2)}(x) = -\frac{1}{2M_0} \bar{q}(x) \gamma \cdot \nabla \gamma_0 \Gamma_0 Q(x).
\]

where \( \Gamma_0 \) can be either \( \gamma_\mu \) or \( \gamma_5 \gamma_\mu \), and \( M_0 \) is the bare heavy quark mass in the NRQCD action. One sees that there are two dimension 4 current corrections to the temporal component and four such corrections to the spatial components. To the order that we are working, one has

\[
\langle V_0 \rangle = (1 + \alpha_s \rho_0^{(0)}) \langle J_0^{(0)} \rangle + (1 + \alpha_s \rho_0^{(1)}) \langle J_0^{(1),\text{sub}} \rangle + \alpha_s \rho_0^{(2)} \langle J_0^{(2)} \rangle
\]

and

\[
\langle V_\perp \rangle = (1 + \alpha_s \rho_\perp^{(0)}) \langle J_\perp^{(0)} \rangle + (1 + \alpha_s \rho_\perp^{(1)}) \langle J_\perp^{(1),\text{sub}} \rangle + \alpha_s \rho_\perp^{(2)} \langle J_\perp^{(2)} \rangle + \alpha_s \rho_\perp^{(3)} \langle J_\perp^{(3)} \rangle + \alpha_s \rho_\perp^{(4)} \langle J_\perp^{(4)} \rangle.
\]

We introduce the combination \( J_\mu^{(1),\text{sub}} = J_\mu^{(1)} - \alpha_s \zeta_{10,\mu} j_\mu^{(0)} \). This subtracts out power law contributions to the matrix elements of the higher dimension operator \( J_\mu^{(1)} \) through \( \mathcal{O}(\alpha_s/(aM)) \). \( J_\mu^{(1)} \) enters the matching already at tree level and after the subtraction one is left with the physical \( \mathcal{O}(\Lambda_{QCD}/M) \) contribution that is a relativistic correction to the leading order term. Power law subtractions of the other dimension 4 current corrections come in as \( \mathcal{O}(\alpha_s^2/(aM)) \) effects and are only partially included here. The one-loop coefficients \( \rho_\mu^{(n)} \) and \( \zeta_{10,\mu} \) for \( \mu = 0 \) are given in ref([17]). The results for \( \mu = k \) have not been published before and are summarized in Table II. In ref([17]) only the contributions from the first lines in eqns.(12) and (13) were taken into account, i.e. \( J_0^{(0)} \) and \( J_0^{(1)} \) matched through \( \mathcal{O}(\alpha_s) \).

As mentioned in the introduction, the effects of all the dimension 4 current corrections turn out to be very small. In Fig.1 we show results for \( \langle J_0^{(1)}/J_0^{(0)} \rangle \) for one of our ensembles with and without the power law subtraction. One sees that although the unsubtracted \( \langle J_0^{(1)}/J_0^{(0)} \rangle \) is at the \( \sim 6\% \) level, the physical \( \langle J_0^{(1),\text{sub}}/J_0^{(0)} \rangle \) is \( \leq 1\% \). In Figs.2 - 4 we give further examples of \( \langle J_0^{(1)}/J_0^{(0)} \rangle \) for \( j > 1 \). These get multiplied by factors of \( \rho_\mu^{(j,\alpha_s)} \) in eq.(13). Using \( \alpha_s \approx \alpha_V/(2a) = 0.32 \) and Table II, one finds \( \rho_\mu \alpha_s \) factors between 0.01 and 0.11, which leads to contributions from higher order currents that are at most 1%. For instance the largest current correction is

III. MATCHING OF HEAVY-LIGHT CURRENTS

Matching of heavy-light currents between continuum QCD and a lattice effective theory with two-component nonrelativistic heavy quark fields \( \Phi \) and four-component light quarks \( q(x) \) is discussed in ref([41]). Since staggered light quarks can be written in terms of four-component “naive” AsqTad quark fields the formalism developed there carries over unchanged to the present calculation. Introducing also a four-component notation for the heavy field, \( Q(x) \equiv (\Phi,0) \), one finds that through \( \mathcal{O}(\alpha_s\Lambda_{QCD}/M, \alpha_s/(aM), \alpha_s\alpha_s\Lambda_{QCD}) \) the following current operators in the effective theory are required.

**temporal:**

\[
J_0^{(0)}(x) = \bar{q}(x) \Gamma_0 Q(x),
\]

\[
J_0^{(1)}(x) = -\frac{1}{2M_0} \bar{q}(x) \Gamma_0 \gamma \cdot \nabla Q(x),
\]

\[
J_0^{(2)}(x) = -\frac{1}{2M_0} \bar{q}(x) \gamma \cdot \nabla \gamma_0 \Gamma_0 Q(x).
\]
TABLE II: Matching coefficients for the spatial currents $V_k$. Where errors are not indicated explicitly, they are of order one or less in the last digit. $aM_0$ is the bare heavy quark mass in lattice units and $n$ a parameter in the NRQCD action. The three selected values for $aM_0$ correspond to the $b$ quark on the MILC extra-coarse, coarse and fine lattices respectively [28].

<table>
<thead>
<tr>
<th>$aM_0$</th>
<th>$n$</th>
<th>$\rho^{(0)}_k$</th>
<th>$\rho^{(1)}_k$</th>
<th>$\rho^{(2)}_k$</th>
<th>$\rho^{(3)}_k$</th>
<th>$\rho^{(4)}_k$</th>
<th>$\zeta_{10,k}$</th>
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</thead>
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<td>4.00</td>
<td>2</td>
<td>0.256</td>
<td>0.484(3)</td>
<td>0.340(6)</td>
<td>0.244(3)</td>
<td>-0.137(3)</td>
<td>0.041</td>
</tr>
<tr>
<td>2.80</td>
<td>2</td>
<td>0.270</td>
<td>0.349(3)</td>
<td>0.169(6)</td>
<td>0.218(4)</td>
<td>-0.029(4)</td>
<td>0.055</td>
</tr>
<tr>
<td>1.95</td>
<td>2</td>
<td>0.332</td>
<td>0.232(3)</td>
<td>0.121(8)</td>
<td>0.161(4)</td>
<td>0.063(3)</td>
<td>0.073</td>
</tr>
</tbody>
</table>

IV. SIMULATION RESULTS FOR FORM FACTORS $f_\parallel$ AND $f_\perp$

The starting point for calculations of the hadronic matrix elements $\langle \pi|J_{\mu}^{(j)}|B \rangle$ is the 3-point correlator,

$$C^{(3)}(\vec{p}_\pi, \vec{p}_B, t, T_B) = \sum_{\vec{x}} \sum_{\vec{y}} \langle \Phi_\pi(0) | J_{\mu}^{(j)}(\vec{x}, t) \Phi_B^{†}(\vec{y}, -T_B) \rangle e^{i\vec{p}_B \cdot \vec{y}} e^{i(\vec{p}_\pi - \vec{p}_B) \cdot \vec{x}},$$

(14)
where $\Phi_B$ and $\Phi_\pi$ are interpolating operators for the $B$ meson and the pion respectively. All results here have the $B$ meson three momentum, $\vec{p}_B$, set equal to zero. For simplicity, the pion operator $\Phi_\pi$ was always placed at the origin. The $B$ meson was then created at time slice $-T_B$ and the electroweak current, $J_{\mu}^{(k)}$, that converts the $b$ quark into an $u$ quark was inserted at times $0 \leq t \leq -T_B$. We have also simulated the time-reversed process, which then has the electroweak current inserted between $+T_B \geq t \geq 0$ and $\Phi_B$ acting on time slice $+T_B$. By looking at both forward and time-reversed processes and verifying that they lead to consistent results (within statistical errors), we were able to increase statistics and at the same time provide some check on our codes. For most of our simulations we used $T_B/a = 16$ on the coarse MILC lattices and $T_B/a = 24$ on the fine lattices. On one of the coarse lattices we also ran with $T_B/a = 20$ and verified that results for form factors were independent of $T_B$. Making $T_B$ too large is not helpful since statistical errors grow with $T_B$. On the other hand making $T_B$ too small limits the number of data points available and gives us less flexibility in our fits.

In constructing the interpolating operators $\Phi_B$ and $\Phi_\pi$ we have found it convenient, just as in the currents of eqs.\,(10) and (11), to work with four component naive fields $q(x)$ rather than one component staggered fields $\chi(x)$. Hence in eq.\,(14) we use,

$$\Phi_B^\dagger = \bar{Q} \gamma_5 q, \quad \Phi_\pi = \bar{q} \gamma_5 q.$$  \hfill (15)

The relation between naive and staggered fermion propagators is given by \[
G_q(x,y) = \Omega(x) \Omega^\dagger(y) G_\chi(x,y), \]

with

$$\Omega(x) = \prod_{\mu=0}^3 (\gamma_\mu)^{\pi_\mu}.$$ \hfill (16)

In our simulations we first calculate staggered propagators $G_\chi$, since they are cheaper, and then convert to naive propagators $G_q$ using eq.\,(19) before evaluating 3-point correlators. The naive AsqTad theory has 16 tastes of quark per flavor and hence one could form 16 different heavy-light pseudoscalar boundstates. However, as discussed in ref.\,[14] these exactly degenerate 16 $B$ mesons do not mix and the 2-point correlator $\langle \Phi_B(x) \Phi_B^\dagger(y) \rangle$ for instance, receives contributions from only one of these possible $B$ mesons. Similarly one can argue that in eq.\,(14) only one type of $B$ meson is involved and that it connects to only one of the 16 true Goldstone pions of the naive light quark theory (out of the total number of 256 pions). In other words both eq.\,(14) and the $\Phi_B$ two-point correlator have the same normalization as in a theory with undoubled light and heavy quarks, such as continuum QCD. The one correlator where adjustment of normalization is required is the $\Phi_\pi - \Phi_\pi$ correlator. Using naive fields as in eq.\,(15) brings in an extra factor of 16 due to

the trace over a 16 $\times$ 16 taste matrix and this factor of 16 must be divided out. If one works with conventional staggered light quarks one would have an extra factor of 4 rather than 16 relative to a pion correlator in a theory with undoubled fermions.

To extract the matrix elements $\langle \pi | J_{\mu}^{(j)}(t) | B \rangle$, the 3-point correlators must be fitted to:

$$C^{(3)}(\vec{p}_B, \vec{p}_B, t, T_B) \rightarrow \sum_{k=0}^{N_N-1} \sum_{l=0}^{N_B-1} (-1)^{k+l} \langle T_B-t \rangle \langle J_{\mu}^{(j)}(t) \rangle \sum_{\pi} \epsilon_{\pi}^{E_{\mu}^{(j)}(t)} \epsilon_{T_B-t}^{E_{\pi}^{(1)}}.$$  \hfill (18)

With this ansatz every second exponential ($k$ or $l$ odd) corresponds to an oscillatory (in time) contribution to the correlator, a characteristic feature of staggered fermions. We use Bayesian fitting methods \cite{15} and in most of our fits we kept $N_N = 1$ and let $N_B$ vary between 3 ~ 9. In order to avoid contamination from excited pions we dropped 5 to 8 points close to the pion source and made certain that within errors fit results did not depend on the number of points omitted. We have also tried some fits with $N_N = 2$ or 3. For some light quark masses and pion momenta one obtained results consistent with the $N_N = 1$ fits, in other cases, however, it was not possible to get very stable (with respect to $N_B$) fits. Hence, for our final results we rely on the $N_N = 1$ fits. Since oscillatory contributions are much more significant in the $B$ channel than for pions, it was more important to allow $N_B$ to increase at fixed $N_N$ rather than the other way around. For priors in our Bayesian fits we use central values corresponding to energy splittings of about 400MeV allowing, however for large ~100% widths. As priors for the amplitudes we allow for ranges typically between $-10 \times A_{00}$ and $+10 \times A_{00}$, where $A_{00}$ is the ground state amplitude. Fit results for groundstate energies and amplitudes were very insensitive to choices for priors. In Figs.5 & 6 we show examples of fit results for $\langle J_{\mu}^{(0)}(t) \rangle$ for pion momentum $\frac{2\pi}{a} \mu (0,1,1)$ on one of the coarse ensembles. For comparison we show in Fig.7 results for a fit to the $B$ correlator that was done simultaneously with the fit to the 3-point correlator.

The main goal in all our fits is to extract the ground state amplitudes $A_{00}^{(j)}$, which lead directly to the form factors $f_\parallel$ and $f_\perp$ via eq.\,(7).

$$f_\parallel = \frac{A_{00}(V_0)}{\sqrt{\zeta_\pi \zeta_B}} \sqrt{2E_\pi}$$

$$f_\perp = \frac{A_{00}(V_k)}{\sqrt{\zeta_\pi \zeta_B} p_{k}^\parallel} \sqrt{2E_\pi}.$$ \hfill (19)

Here $A_{00}(V_0)$ includes contributions from all currents with appropriate matching factors as dictated by eqs.\,(12) and (13). $\zeta_\pi$ and $\zeta_B$ are the ground state amplitudes from the pion and $B$ meson correlators, respectively. $\zeta_\pi$ is correctly normalized as in continuum QCD.
FIG. 5: Fit result for \( \langle J^{(0)}_0(t) \rangle \times e^{M_B t} \times e^{E\pi(T-t)} \) versus \( t \) for pion momentum (0, 1, 1)\( \frac{2\pi}{N_s} \). The horizontal time axis has been rearranged so that the \( B \) meson source is at \( t/a = 1 \) and the pion source at \( t/a = T = 16 \).

Results for the form factors \( f_\parallel \) and \( f_\perp \) for the different ensembles and pion momenta are summarized in Tables III & IV.

V. CHIRAL EXTRAPOLATIONS

The form factors listed in Tables III and IV are for an unphysical world with \( u \) and \( d \) quark masses larger than in reality. They need to be extrapolated to the physical chiral limit. In ref. 17 linear extrapolations were per-

\[
\begin{array}{cccc}
\text{Table III: Results for the form factor } f_\parallel \text{ in GeV}^{1/2} \text{ from coarse MILC lattices. Errors are statistical errors coming from Bayesian bootstrap fits.} \\
\hline
u0 am_q & u0 am_q & p_x = (000) & (001) & (011) & (111) \\
0.005 & 0.005 & 1.486(34) & 1.326(36) & 1.221(38) & 1.127(78) \\
0.007 & 0.007 & 1.538(32) & 1.328(33) & 1.185(49) & 1.013(115) \\
0.010 & 0.005 & 1.582(37) & 1.322(37) & 1.201(60) & 1.053(121) \\
0.010 & 0.010 & 1.584(44) & 1.322(39) & 1.121(52) & 1.064(138) \\
0.010 & 0.020 & 1.581(30) & 1.372(29) & 1.253(37) & 1.117(72) \\
0.020 & 0.020 & 1.508(48) & 1.378(43) & 1.264(47) & 1.121(83) \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Table IV: Same as Table III for the form factor } f_\perp \text{ in GeV}^{-1/2}. \\
\hline
u0 am_q & u0 am_q & p_x = (001) & (011) & (111) \\
0.005 & 0.005 & 1.543(205) & 0.703(44) & 0.515(39) \\
0.007 & 0.007 & 1.082(31) & 0.606(31) & 0.433(48) \\
0.010 & 0.005 & 1.328(37) & 0.662(36) & 0.413(38) \\
0.010 & 0.010 & 1.235(90) & 0.657(37) & 0.449(43) \\
0.010 & 0.020 & 1.029(21) & 0.611(19) & 0.423(23) \\
0.020 & 0.020 & 1.097(140) & 0.597(24) & 0.397(20) \\
\end{array}
\]
formed on the three points with $u_0 a m_f = 0.01$, the only simulation data available at that time. Here we carry out this important step in several different ways:

1) linear extrapolation with only the full QCD ($m_f = m_q$) data, 2) staggered chiral perturbation theory (SChPT) with full QCD data, 3) continuum ChPT with full QCD data, and 4) SChPT simultaneously to both full QCD and partially quenched data. We use the last, most involved chiral extrapolation for our final answer, but make certain that the other methods give agreement within quoted errors.

The formulas of chiral perturbation theory for heavy-light form factors have the general form,

$$f_{\parallel/\perp} = c_0[1 + \delta f_{\parallel/\perp} + c_1 m_q + c_2(2m_f + m_q) + \ldots].$$

The discussion in the previous paragraph shows that chiral extrapolations are most conveniently carried out at fixed values of $E_\pi$. In order to do so, one needs to interpolate the data of Tables III & IV to fixed common values of $E_\pi$ in our fits to SChPT and to continuum ChPT formulas we will let $g_{B\pi}$ vary as one goes from one $E_\pi$ value to another, together with $c_0$, $c_1$ and $c_2$.

In Figs. 10 & 11 we show chiral extrapolations for $f_\parallel$ and $f_\perp$ respectively for several values of $E_\pi$. These are then extrapolated to the chiral limit using the SChPT formulas described above. One advantage of carrying out interpolations for $f_\parallel$ and $f_\perp$ rather than directly in $f_\parallel$ and $f_\perp$, is that the kinematic constraint $f_0(0) = f_+(0)$ is easily incorporated into the ansaetze and one can use the $f_0$ data to constrain the normalization of $f_+$ as well.
possible to disentangle statistical and chiral extrapolation errors. $f_\perp$ at small $E_\pi$ translates into $f_\parallel$ at large $q^2$. One can speculate that the large statistical errors we are finding in this kinematic region may be due to the proximity of the $M_{B^*}$ pole. This, however, for reasons we do not understand, was not evident in the partially quenched data of reference [17].

For the SChPT extrapolations, as mentioned above, we have let $g_{B\pi}$ float and be one of the fit parameters. We find that the “effective” $g_{B\pi}^2$ ranges between 0.0 and 0.2 and decreases with $E_\pi$, although within large errors. In Fig.12 we show a comparison of fits with and without the $\mathcal{O}(a^2)$ corrections in the ChPT theory formulas, i.e. a comparison between continuum and staggered full QCD ChPT. Again the differences are very small. In summary, for most of our data points, different ways of carrying out the chiral extrapolation, including using no input from ChPT at all, lead to a spread in extrapolated results of only 2.5% or less. This indicates that contributions we have neglected in the ChPT formulas, such as $1/M$ corrections, higher order (in pion momentum) terms, finite volume effects etc. are not important. We take as central values for the form factors $f_\parallel$ & $f_\perp$ in the chiral limit, the results coming from the SChPT extrapolation using both full QCD and partially quenched data. The combined statistical and chiral extrapolation errors are discussed in the next section.

In Fig.13 we compare results from one of the MILC fine lattice ensembles with the coarse lattice data discussed so far. For this comparison both the coarse and fine data points include only the $(J_\mu^{(0)})$ contributions through $\mathcal{O}(\alpha_s)$, i.e. the first lines in eqns. [12] or [13] respectively. Since we have shown that the higher order currents have minimal effect, we believe meaningful scaling tests can be carried out with just $(J_\mu^{(0)})$. For $f_\perp$, which is the main contributor to the phenomenologically relevant form factor $f_\parallel(q^2)$, one sees that the fine lattice point falls nicely on the fixed $E_\pi$ curve determined by coarse lattice data. Our statistical error is large currently for the $f_\parallel$ point from the fine lattice. Within these large errors there is consistency between the coarse and fine lattices for $f_\parallel$ as well. We conclude from this exercise that there are
no indications of large discretization effects in the form factor calculations on MILC coarse lattices. Such errors are smaller than current statistical errors. Eventually it would be desirable to carry out a more thorough scaling test, once more data on fine lattices at several light sea quark masses become available.

VI. RESULTS FOR FORM FACTORS $f_+(q^2)$ AND $f_0(q^2)$ IN THE CHIRAL LIMIT

We convert the chirally extrapolated $f_+(E_\pi)$ and $f_\perp(E_\pi)$ to the form factors $f_+(q^2)$ and $f_0(q^2)$ in the physical limit. These are shown in Fig.14 and tabulated in Table V. For comparison we also plot in Fig.14 the data presented in ref.17. One sees that changes are minimal in spite of all the improvements included in our new results. This indicates that the approximations that were made previously and that we are systematically improving upon, such as partial quenching, linear chiral extrapolations, working with currents at lowest order in $1/M$, did not drastically affect the theory. The solid curves in Fig.14 are fits to our new results using the Ball-Zwicky (BZ) parametrization of $f_+$ and $f_0$. We have also tried fits to other parametrizations, described in the Appendix, including the Becirevic-Kaidalov (BK) [49], Richard Hill (RH) [20] and a series expansion (SE) [19, 21, 51, 52] parametrization. The RH parametrization fit is essentially indistinguishable from the BZ fit. The BK fit is also a good fit to our data although not quite as good as the first two. This should not be surprising, since the BK fit has only three parameters to tune whereas the BZ and RH fits are both four parameter fits. Any further parameters, however, are very poorly determined and do not help in the fit. Another class of fit ansatze, the series expansion (SE) fits, are discussed in the Appendix. The main reason we are interested in obtaining a good analytic parametrization of the form factors, is to facilitate partial integration of differential decay rates, as discussed below. These parametrizations can also be used to try and extrapolate to lower $q^2$ where lattice data are currently not available.

The statistical plus chiral extrapolation errors for $f_+(q^2)$ lie between 7 ~ 10% depending on $q^2$. They are smaller for the form factor $f_0(q^2)$. For $q^2 \geq 16 GeV^2$, the range we will be focusing on, the average error for $f_+(q^2)$ comes out to be ~ 8%. In Table VI we list this average statistical plus chiral extrapolation error together with estimates of systematic errors from other sources. These other systematic errors are dominated by the ~ 9% uncertainty in higher order matching of the heavy-light currents.

The differential partial decay rate for $B \rightarrow \pi \ell \nu$, ignoring the charged lepton mass, is given by,

$$\frac{dT}{dq^2} = \frac{G_F^2}{24\pi^3} p_\pi^3 |V_{ub}|^2 |f_+(q^2)|^2$$

(21)

where $G_F$ is the Fermi constant and $p_\pi$ the magnitude of the pion three momentum in the $B$ rest frame. Knowing $f_+(q^2)$ then allows us to evaluate $\frac{1}{|V_{ub}|^2} \frac{dT}{dq^2}$ and also integrate this quantity over different $q^2$ bins. We take our best fit, the BZ fit shown in Fig.14, and integrate to obtain,

$$\frac{1}{|V_{ub}|^2} \int_{16 GeV^2}^{q_{max}^2} \frac{dT}{dq^2} dq^2 = 1.46(23)(27) ps^{-1}$$

(22)
FIG. 14: Form factors $f_+(q^2)$ and $f_0(q^2)$ in the chiral limit. The black squares and triangles are the new and final results for $f_+$ and $f_0$ respectively. For comparison, the data from ref. [17] are also shown as circles. The full black curves follow a BZ parametrization fit (see text) to the new data. Errors are combined statistical and chiral extrapolation errors.

The first error is the combined statistical plus chiral extrapolation error and the second the sum of all other systematic errors added in quadrature. Eq. (22) is the main result of this article. It serves as basis for determinations of the CKM matrix element $|V_{ub}|$. Similar integrals using other parametrizations and over other $q^2$ ranges are summarized in Table VII together with $f_+(0)$ from the different fits. The second error for $f_+(0)$ and for the integrated rate over the entire $q^2$ range includes an additional 10% systematic uncertainty in $f_+(q^2)$ which is not part of Table VI and which comes from the extrapolation into the low $q^2$ region.

<table>
<thead>
<tr>
<th>$q^2$ [GeV$^2$]</th>
<th>$f_+(q^2)$</th>
<th>$f_0(q^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.23</td>
<td>0.649(63)</td>
<td>0.475(26)</td>
</tr>
<tr>
<td>16.28</td>
<td>0.727(64)</td>
<td>0.508(25)</td>
</tr>
<tr>
<td>17.34</td>
<td>0.815(65)</td>
<td>0.527(25)</td>
</tr>
<tr>
<td>18.39</td>
<td>0.944(66)</td>
<td>0.568(24)</td>
</tr>
<tr>
<td>19.45</td>
<td>1.098(67)</td>
<td>0.610(24)</td>
</tr>
<tr>
<td>20.51</td>
<td>1.248(97)</td>
<td>0.651(25)</td>
</tr>
<tr>
<td>21.56</td>
<td>1.554(156)</td>
<td>0.703(26)</td>
</tr>
</tbody>
</table>

TABLE V: Form factors $f_+(q^2)$ and $f_0(q^2)$ in the chiral limit. Errors shown are combined statistical and chiral extrapolation errors.

<table>
<thead>
<tr>
<th>source of error</th>
<th>size of error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistics + chiral extrapolations</td>
<td>8</td>
</tr>
<tr>
<td>two-loop matching</td>
<td>9</td>
</tr>
<tr>
<td>discretization</td>
<td>3</td>
</tr>
<tr>
<td>relativistic</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>12</td>
</tr>
</tbody>
</table>

TABLE VI: Estimate of percentage errors in $f_+(q^2)$ for $q^2 > 16$GeV$^2$.

| Fit | $f_+(q^2 = 0)$ | $\frac{\int dq^2 |V_{ub}|^2}{16\text{GeV}^2}$ [ps$^{-1}$] |
|-----|----------------|-----------------------------------------------|
| BZ  | 0.27(2)(4)     | 6.00(96)(1.68)                               |
| BK  | 0.26(2)(4)     | 6.03(96)(1.69)                               |
| RH  | 0.27(2)(4)     | 5.99(96)(1.68)                               |

TABLE VII: Partially integrated differential decay rates and $f_+(0)$ using several parametrizations. The first error reflects statistical and chiral extrapolation uncertainties. The second error is due to remaining systematic errors.

VII. ESTIMATING $|V_{ub}|$

In this section we combine the lattice results of the previous section with experimental input for $B \to \pi l\nu$ branching fractions and extract an estimate for $|V_{ub}|$. In order to do so, we use results from the Heavy Flavor Averaging Group (HFAG) [53] and rely on its analysis of the current experimental uncertainties. The HFAG gives preliminary averages of BaBar, Belle and CLEO results (as of the conferences of Summer 2005) for the integrated branching fraction $B\{B^0 \to \pi^- l^+\nu\}$. They quote $[1.35 \pm 0.08 \pm 0.08] \times 10^{-4}$ for $0 \leq q^2 \leq q^2_{\text{max}}$ and $[0.40 \pm 0.04 \pm 0.04] \times 10^{-4}$ for $q^2 \geq 16$GeV$^2$. Combining this with eq. (22) and a $B^0$ lifetime of 1.536 ps [54] leads to,

$$|V_{ub}| = 4.22(30)(51) \times 10^{-3}, \quad q^2 \geq 16\text{GeV}^2$$

where the first error is experimental (7%) and the second is the total lattice error (12%). The result (23) is consistent, at the one $\sigma$ level, with the preliminary value 3.78(25)(52) $\times 10^{-3}$ obtained by the Fermilab/MILC collaboration using the same HFAG branching fraction averages [53].

VIII. SUMMARY

We have completed a determination of the $B$ meson semileptonic form factors $f_+(q^2)$ and $f_0(q^2)$ using state-of-the-art Lattice QCD methods. Our calculations employ unquenched gauge configurations, created by the MILC collaboration, that incorporate vacuum polarization effects from two very light flavors plus strange sea...
quarks. Both the sea and the valence light quarks are simulated using a highly improved staggered quark action. This action allows us to work close enough to the chiral limit, so that chiral extrapolations to physical pions are mild and do not introduce large uncertainties. Our results for $f_+(q^2)$ can be combined with experimental branching fraction data to extract the CKM matrix element $|V_{ub}|$. This quantity is a crucial ingredient in tests of the unitarity triangle and in solidifying our understanding of CP violation in the Standard Model.

The total lattice error in the $f_+(q^2)$ form factor presented here, and hence also in $|V_{ub}|$, is at the $\sim 12\%$ level. This error is dominated by uncertainty in higher order perturbative matching of heavy-light, light-light and heavy-heavy currents and four-fermion operators. It has become increasingly obvious that such calculations are necessary for accurate lattice determinations of form factors, decay constants and mixing parameters at the $\sim 5\%$ level. For form factor calculations, more work is also required to reduce statistical errors. One approach that helped significantly in the HPQCD collaboration is to carry out higher order matching for heavy-light, light-light and heavy-heavy currents and four-fermion operators. It is to carry out higher order matching for heavy-light, light-light and heavy-heavy currents and four-fermion operators. W e refer to the literature for further discussion of the merits of this transformation. The form factor $f_+(q^2)$ can then be expanded as a power series in $z(q^2, t_0)$, where $q^2 \equiv q^2/M_{B^*}^2$ and $t_0 \equiv (M_B + m_\pi)^2$. The kinematic constraint $f_+(0) = f_0(0)$ is automatically satisfied. This is a 4 parameter parametrization of $f_+$ and $f_0$, sometimes called the “4 Parameter BK Parametrization” (Becirevic & Kaidalov) [49]. Examples of generalizations and special cases developed in the literature are given below. They differ only in the parametrization of $f_+$.

### APPENDIX A: PARAMETRIZATION OF FORM FACTORS

Most parametrizations start from a dispersive representation of the form factors.

\begin{align}
  f_+(q^2) &= \frac{r_1}{1 - \frac{q^2}{\alpha_0}^2} + \frac{1}{\pi} \int_1^\infty dt \frac{\text{Im}[f_+(t)]}{t - q^2 - i\epsilon} \\
  f_0(q^2) &= \frac{r_1 + r_2}{1 - \frac{q^2}{\beta_0}^2} \\
  f_+(q^2) &\rightarrow \frac{r_1}{1 - \frac{q^2}{\alpha_0}^2} + \frac{r_2}{1 - \frac{q^2}{\beta_0}^2} \\
  f_0(q^2) &\rightarrow \frac{r_1 + r_2}{1 - \frac{q^2}{\beta_0}^2}
\end{align}

where $q^2 \equiv q^2/M_{B^*}^2$ and $t_0 \equiv (M_B + m_\pi)^2$. The kinematic constraint $f_+(0) = f_0(0)$ is automatically satisfied. This is a 4 parameter parametrization of $f_+$ and $f_0$, sometimes called the “4 Parameter BK Parametrization” (Becirevic & Kaidalov) [49]. Examples of generalizations and special cases developed in the literature are given below. They differ only in the parametrization of $f_+$.

1. **3 Parameter Becirevic-Kaidalov (BK) [49]**

\begin{align}
  f_+(q^2) &= \frac{f_+(0)}{1 - \frac{q^2}{\alpha_0}^2} \\
  f_0(q^2) &= \frac{f_0(0)}{1 - \frac{q^2}{\beta_0}^2}
\end{align}

2. **4 Parameter Ball-Zwicky (BZ) [50]**

\begin{align}
  f_+(q^2) &= \frac{f_+(0)}{1 - \frac{q^2}{\alpha_0}^2} + \frac{r_2 q^2}{1 - \frac{q^2}{\beta_0}^2}
\end{align}

3. **4 Parameter R.Hill (RH) [20]**

\begin{align}
  f_+(q^2) &= \frac{f_+(0)(1 - \delta \cdot q^2)}{1 - \frac{q^2}{\gamma_0}^2}
\end{align}

### Acknowledgements

This work was supported by the DOE and NSF (USA) and by PPARC (UK). A.G., J.S. and M.W. thank the KITP U.C. Santa Barbara for support during the workshop, “Modern Challenges in Lattice Field Theory” when part of the present research was carried out. Simulations were done at NERSC and on the Fermilab LQCD cluster. We thank the MILC collaboration for making their unquenched gauge configurations available and the Fermilab collaboration for use of their light propagators on the fine lattices. We are grateful to Claude Bernard for sending us his notes on SChPT for heavy-light form factors. We would also like to acknowledge useful conversations with Richard Hill, Masataka Okamoto and Iain Stewart.
The “Blaschke” factor $P(q^2)$ must take into account any isolated poles below the $B\pi$ threshold at $q^2 = t_+$. We set $P(q^2) = z(q^2, M_{B}^2)$ to take care of the $B^*$ pole. For $\Phi(q^2, t_0)$ we take the expression given in ref. [19] (with simplified $\chi^{(0)}$). We combine the ansatz (A8) for $f_+(q^2)$ with $k_{\text{max}} = 2$ together with (A4) for $f_0(q^2)$ to get another (and our last) 4 parameter ansatz,

\begin{equation}
\tilde{f}_+(q^2) = \frac{a_0 + a_1 z(q^2, t_0) + a_2 z^2}{z(q^2, M_{B}^2, t_0)} \Phi(q^2, t_0)
\end{equation}

We have explored the SE parametrization but not because we needed a better analytic expression to cover the range $q^2 \geq 16 \text{GeV}^2$ of our lattice results. Rather, we did so to assess how reliable is the information on the shape of the form factors at lower $q^2$ that we are getting from the BK/BZ/RH type parametrizations. We find that good fits to the simulation data can be obtained with the SE parametrizations as well and that they are insensitive to the value of $t_0$. Table VIII gives results for integrated differential decay rates. Not surprisingly, agreement is found with Table VII for $q^2 \geq 16 \text{GeV}^2$. Results are systematically slightly lower than in Table VII if one includes the entire $q^2$ range. One can summarize by saying that we do not have evidence for any strong dependence on the choice of ansatz employed to extrapolate lattice data to lower $q^2$ values. Nevertheless, we believe it is important to look for additional information on the form factors at $q^2 < 16 \text{GeV}^2$, either from experimental or from theoretical models, if one wants to discuss the entire $q^2$ range at the present time. This has already been done in the recent literature [19, 21, 50]. Alternatively, getting lattice results directly in the low $q^2$ regime using for instance Moving NRQCD would also solve this problem.

**APPENDIX B: ERRATUM: B MESON SEMILEPTONIC FORM FACTORS FROM UNQUENCHED LATTICE QCD [PHYS.REV.D 73, 074502 (2006)]**

Due to a normalization error in one of our analysis codes many results for the form factor $f_\perp$ were evaluated incorrectly. In particular, Table IV should be replaced by,

<table>
<thead>
<tr>
<th>$q^2$ [GeV$^2$]</th>
<th>$f_+(q^2)$</th>
<th>$f_0(q^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.34</td>
<td>1.101(53)</td>
<td>0.561(26)</td>
</tr>
<tr>
<td>18.39</td>
<td>1.273(99)</td>
<td>0.600(21)</td>
</tr>
<tr>
<td>19.45</td>
<td>1.458(142)</td>
<td>0.639(23)</td>
</tr>
<tr>
<td>20.51</td>
<td>1.627(185)</td>
<td>0.676(41)</td>
</tr>
<tr>
<td>21.56</td>
<td>1.816(126)</td>
<td>0.714(56)</td>
</tr>
</tbody>
</table>

The corrected form factors $f_+(q^2)$ and $f_0(q^2)$ in the chiral limit become (previous Table V):

We restrict the range in $q^2$ to values where simulation data before chiral extrapolation are available for all light quark masses and omit two lower $q^2$ points that were obtained previously through small extrapolations in $E_\pi$.

The new error budget is given by (previous Table VI):

<table>
<thead>
<tr>
<th>source of error</th>
<th>size of error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistics + chiral extrapolations</td>
<td>10</td>
</tr>
<tr>
<td>two-loop matching</td>
<td>9</td>
</tr>
<tr>
<td>discretization</td>
<td>3</td>
</tr>
<tr>
<td>relativistic</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>14</td>
</tr>
</tbody>
</table>

and the partially integrated differential decay rates become (previous Table VII):

| Fit $f_+(q^2 = 0)$ | $\int \frac{dt}{t} / |V_{ub}|^2$ [ps$^{-1}$] |
|---------------------|-----------------------------------|
| 0 $\leq q^2 \leq q_{\text{max}}^2$ | $16 \text{GeV}^2 \leq q^2 \leq q_{\text{max}}^2$ |
| BZ 0.31(5)(4)       | 9.10(1.82)(2.55)             | 2.07(41)(39) |
| BK 0.31(5)(4)       | 9.30(1.86)(2.60)             | 2.13(43)(40) |
| SE 0.30(5)(4)       | 9.35(1.87)(2.62)             | 2.02(40)(38) |

Equation (22) should be replaced by,
\[ \frac{1}{|V_{ub}|^2} \int_{16 GeV^2}^{q_{max}^2} \frac{dT}{dq^2} dq^2 = 2.07(41)(39) \text{ps}^{-1}, \]

which signifies an increase on our previous incorrect value (1.46(23)(27) \text{ps}^{-1}) by 1.7 times the previously quoted total error.

The new eq.(22) leads to a new eq.(23),

\[ |V_{ub}| = 3.55(25)(50) \times 10^{-3}, \quad q^2 \geq 16 GeV^2. \]

Accordingly, the last two sentences in the Abstract should be changed to,

“..., We calculate the form factors \( f_+(q^2) \) and \( f_0(q^2) \) in the chiral limit for the range 16 GeV^2 \( \leq q^2 < q_{max}^2 \) and obtain \( \int_{16 GeV^2}^{q_{max}^2} \frac{dT}{dq^2} dq^2 / |V_{ub}|^2 = 2.07(57) \text{ps}^{-1} \). Combining this with a preliminary average by the Heavy Flavor Averaging Group (HFAG’05) of recent branching fraction data for exclusive B semileptonic decays from the BaBar, Belle and CLEO collaborations, leads to \( |V_{ub}| = 3.55(25)(50) \times 10^{-3} \).”

[6] B.Aubert et al. [BaBar Collaboration]; [hep-ex/0506064].
[7] B.Aubert et al. [BaBar Collaboration]; [hep-ex/0507003].
[8] B.Aubert et al. [BaBar Collaboration]; [hep-ex/0507085].
[24] For a recent review see: S.Dürr; Plenary talk at LAT 2005; [hep-lat/0509026].
[36] C.Aubin and C.Bernard; [hep-lat/0510088].
[38] A.Gray et al.; Phys. Rev. Lett. 95, 212001 (2005) [hep-lat/0507015].
[41] The NRQCD action used in this article is indeed the same as in [23], which, however, included some typos in the \( c_2 \) and \( c_3 \) terms of \( \delta H \). Those typos have been corrected here in eq.(3).


[53] Heavy Flavor Averaging Group; http://www.slac.stanford.edu/xorg/hfag/


