Using the Method of Fibres in Mecho to Calculate Radii of Gyration

Master

PLEASE RETURN THIS TO ALAN BUNDY

USING THE METHOD OF FIBRES IN MECHO
TO CALCULATE RADII OF GYRATION

by

Alan Bundy and Lawrence Byrd

Abstract

Cohen identifies a method of solving Mechanics problems which he calls the Fibres heuristic. It consists of dividing a body into an infinite collection of subbodies and considering a typical one of these. We describe work in progress to extend the Mecho program so that it can use the method to calculate the radius of gyration of a body rotating about an axis. We show how such an infinite subdivision can be represented and how an appropriate subdivision can be chosen. The work has revealed conceptual difficulties in the processes involved, which we predict to be a source of difficulty for students. It has suggested a uniform solution to these difficulties which may find application in the teaching of Mechanics.

Acknowledgements

Harvey Cohen's analysis inspired us to implement the Fibres method. Richard O'Keefe and Bob Fisher led us to the definition of uniformity. The work was supported by SRC grant GR/A/57954.

Keywords

mathematical reasoning, mechanics, problem solving, fibres method.

1. Introduction

[Cohen 74] identifies a method of solving Mechanics problems which he calls the Fibres heuristic. It consists of dividing a body into an infinite collection of subbodies and considering a typical one of these. In this paper we describe work in progress to extend the Mecho program, [Bundy et al 79], so that it is capable of applying the method of Fibres.

The standard technique for calculating the radius of gyration of a complex body is a classic application of the method of Fibres. We chose it as a suitable domain for the study of this method. The radius of gyration is a property of a body spinning on an axis which is useful for calculating the body's moment of inertia about the axis. The moment of inertia, I, of the body is the sum over all the particles which make up the body of $m.r^2$ where m is the mass of the particle and r is its perpendicular distance from the axis. The radius of gyration, k, is chosen so that

$$M.k^2 = I$$

where M is the mass of the whole body.

The standard technique for calculating k is

- To divide the body into an infinite collection of subbodies, which we will call fibres, in such a way that the radius of gyration of each fibre with respect to the axis is easier to calculate than the radius of gyration of the whole body.

- Calculate the moment of inertia of each fibre.

- Using integration calculate the moment of inertia and hence the radius of gyration of the whole body.

The hard parts of extending Mecho to deal with these problems were:

- Representing the division of a body into an infinite collection of fibres.

- Automatically choosing a division which facilitates the solution of the problem.

The solution to these problems forms the topic of the remainder of this paper. Not surprisingly the solutions turn out to be intimately related.

2. Continuous Measure Systems

The key to the representation of a body as an infinite collection of fibres is the concept of a continuous measure system. A continuous measure system is used to measure an entity with the aid of a parameter varying between limits. It consists of 6 parts, which may be thought of as the entries in 6 slots of a frame. These parts are:

1. the entity on which the measure system is erected;

2. the parameter which constitutes the measure;

3. the origin or subentity from which the measurements are made;

4. the fibre or typical subentity to which the measurements are made;

5. the first limit or initial value of the parameter and

6. the second limit or final value of the parameter.

We will represent a continuous measure system with a 6 argument predicate, cont_meas(Entity,Parameter,Origin,Fibre,Limit1,Limit2).

Continuous measure systems are useful not just in radius of gyration problems, but wherever a measuring parameter is erected on an entity. We will give some examples from various problem areas.

Circular Disc

Let dsc be a 2 dimensional, circular disc with radius a and centre c. Let dsc be divided into an infinite collection of concentric rings with centre c and radius r0, and let typ_ring be a typical such ring (see figure 2-1).

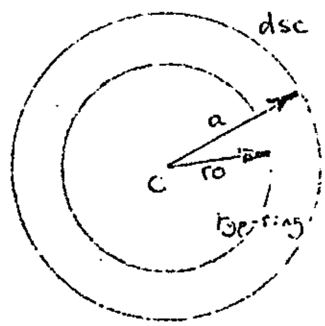

Figure 2-1: A Circular Disc Divided into an Infinite Collection of Rings

Then the situation is described by the assertion

cont_meas(dsc,r0,c,typ_ring,0,a)

together with various assertions describing the shapes of dsc, c and typ_ring and their relations to r0 and a.

point(c)	
disc(dsc)	ring(typ_ring)
centre(dsc,c)	centre(typ_ring,c)
radius(dsc,a)	radius(typ_ring,r0)

Time Period

Let 1st_session be a period of time and suppose we wish to associate times measured on a clock with each moment of the period. The conventional way to do this is to measure in hours, minutes and seconds from the previous midnight, thus the moment, midnight, forms the origin. If the 1st_session starts at 9.45 and lasts until 11.55 and if t is the time of some typical moment, typ_mom, then the situation (see figure 2-2) is described by the assertion

cont_meas(1st_session,t,midnight,typ_mom,9.45,11.55)

together with various assertions describing the nature of 1st_session, midnight and typ_mom and their relation to t, 9.45 and 11.55.

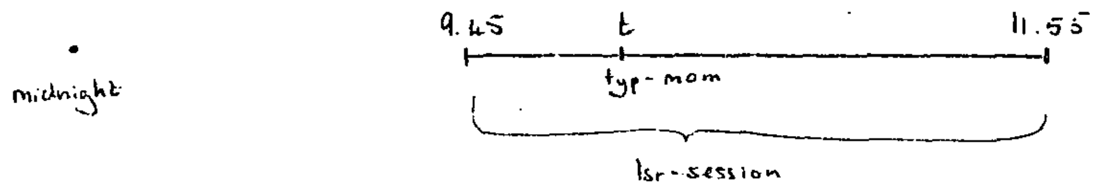

Figure 2-2: A Time Period Divided into an Infinite Collection of Moments

Trajectory of Particle

Suppose path0 is the trajectory of a particle. Let start be the initial position of the particle and posn be its position at some arbitrary moment in time. Let the distance along path0 from start to posn be x and the distance to the end of path0 be d then the situation (see figure 2-3) is partially described by the assertion

```
cont_meas(path0,x,start,posn,0,d)
```

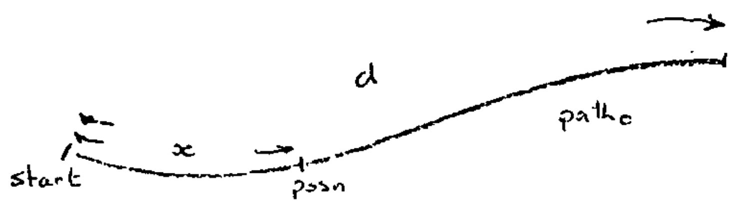

Figure 2-3: The Trajectory of a Particle showing its Typical Position

3. Choosing Continuous Measure Systems

The division of a body into an infinite collection of fibres can be represented using the cont_meas predicate. We now turn to the problem of choosing a division which facilitates the calculation of a radius of gyration.

For any given entity, especially a 2 or 3 dimensional one, there are several ways of erecting a continuous measure system on it, i.e. several ways of dividing it into an infinite collection of fibres. Figure 3-1 shows how a disc may be divided into a series of concentric rings or radial, horizontal or vertical lines.

The task of dividing a body into a collection of fibres, so that its radius of gyration can be calculated, consists of two subtasks:

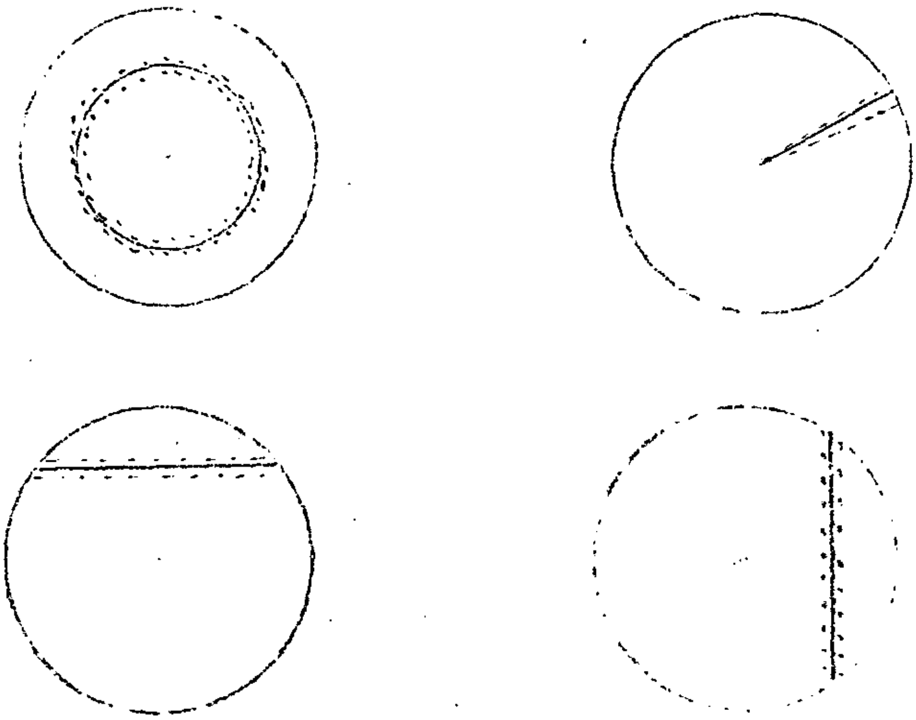

Figure 3-1: Alternative Measure Systems on a Disc

- the generation of candidate measure systems and

- the choosing of an appropriate system.

In the Mecho program these two subtasks are interleaved, but for purposes of exposition it is useful to seperate them.

Obviously, the shape of a body determines the ways in which it can be divided into fibres. The traditional way of describing the shape of a body is by using algebraic expressions in some coordinate system, for instance, a disc, of radius a, may be described with the expressions,

$0 \leq r \leq a$
$0 \leq \theta < 2.pi$
in polar coordinates

or by
$x^2 + y^2 \leq a^2$
in cartesian coordinates

Notice the central role of inequalities in these descriptions. Pairs of such inequalities can be used to define a system of fibres. Consider, for instance, the pair of inequalities,

$0 \leq r \leq a$

above. This allows r to vary between two limits, 0 and a. If we fix the value of r to,say, r0, by replacing these two inequalities with an equation between r and r0,

$$\begin{array}{l} 0 \le r \le a \\ 0 \le \theta < 2.pi \\ \quad\quad \Big\downarrow \quad 0 \le r \le a \;\Rightarrow\; r = r0 \\ r = r0 \\ 0 \le \theta < 2.pi \end{array}$$

then the resulting equations define a ring with the same centre as the disc, but radius r0. If this ring is regarded as a fibre of the disc and r0 as the parameter of a continuous measure system varying between 0 and a then we have generated the system of concentric rings in figure 2-1 above.

The system of radial lines can be generated by allowing the r coordinate to vary between 0 and a, but fixing the θ coordinate at say, θ0, that is by replacing the second pair of inequalities in the polar definition of a disc with an equation between θ and θ0,

$$\begin{array}{l} 0 \le r \le a \\ 0 \le \theta < 2.pi \\ \quad\quad \Big\downarrow \quad 0 \le \theta < 2.pi \;\Rightarrow\; \theta = \theta 0 \\ 0 \le r \le a \\ \theta = \theta 0 \end{array}$$

and using θ0 as the parameter of a continuous measure system.

The remaining systems of figure 3-1, the horizontal and vertical lines can be generated from the cartesian coordinate system. The same technique, of replacing a pair of inequalities by an equation, can be used provided the cartesian expressions are first rewritten into the equivalent

$$\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2}$$

$$\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$

Therefore, to generate a continuous measure system, Mecho must

1. first, generate the algebraic defining expressions of the body whose radius of gyration it wishes to calculate;

2. then choose a pair of inequalities of the form

   $$a \le v \le b \qquad \text{(i)}$$

   which prescribe limits, a and b, on one of the coordinates, v;

3. replace these inequalities with an equation of the form

   $$v = v0$$

   where v0 is the new parameter of the continuous measure system;

4. recognise the type of object defined by these new algebraic expressions;

5. use this object as the fibre of the continuous measure system, v0 as the parameter and a and b as the limits.

Note that the fibres generated by this technique are always of one dimension less than the original body.

We have seen that step 2 of this process may involve manipulating the algebraic description of a body so that a pair of inequalities with the form of (i) is produced. Some algebraic manipulation may also be required at step 4 to manipulate the new expressions into a form which may be recognised as the standard algebraic description of a body. Currently, Mecho can only solve problems in which the algebraic descriptions of the bodies are in the required form to start with. In extending Mecho to deal with these two bits of algebraic manipulation we will be able to call on our experience with the Press equation solving program, [Bundy and Welham 81].

Given that it is possible to generate a wide range of measure systems for a body, how can one be chosen which will facilitate the calculation of the radius of gyration? Mecho employs two tests which either weed out or postpone consideration of those measures systems that it considers are unlikely to lead to success.

The first test is designed to postpone consideration of any fibre whose radius of gyration is not already known, e.g. in the case of the disc, the system of concentric rings passes the test because the radius of gyration of a ring with respect to a perpendicular axis is known to Mecho. Mecho makes two passes at choosing the fibres: in the first it tries to find a fibre whose radius of gyration is already known; in the second it relaxes this constraint and allows a recursive process of solution. This recursive process will terminate because a fibre is always of dimension one less than the body it is part of.

The second test ensures that the thickness of the fibre in the parameter dimension is constant. This test was introduced to eliminate various conceptual difficulties in the calculation of radii of gyration reported in earlier versions of this paper. It is discussed more fully in the next section.

4. Uniformity

The investigation of radii of gyration problems uncovered two conceptual difficulties, whose resolution required a deep analysis of the assumptions underlying the solution process.

The first conceptual difficulty is the necessity to idealize a fibre in two different ways, for instance as both a 1 dimensional and a 2 dimensional object. Consider the system of concentric rings in a disc in figure 3-1 above. In order to calculate the mass of one of these rings it is necessary to regard it as having a non-zero area and hence a non-zero thickness of d(r0). Thus the ring is idealized as a 2 dimensional object. However, in order to calculate the ring's radius of gyration this thickness must be neglected and the ring seen as a 1 dimensional object. Not to do so would necessitate the calculation of the radius of gyration of an annulus - a special case of a disc - and the problem solver would descend into an infinite regress. Similar difficulties arise in all these problems. In the personal experience of both authors, as school students, this 'double thinking' was a stumbling block when learning how to solve these problems. The second conceptual difficulty concerns the generation of continuous measure systems. Consider the system of radial lines in figure 3-1. While this is one of the measure systems generated by our technique (by freezing the θ coordinate) it cannot be used in the calculation of the radius of gyration of the disc, because it would lead to the wrong answer. In fact, a continuous measure system obtained by freezing the θ coordinate seems only to be useful when the body being subdivided is 1 dimensional. The problem is that the area of the line fibre is not uniformly distributed along its length. The fibre is more accurately idealized as a narrow sector (see figure 4-1). However, since the sector is a special case of the disc, using this idealization would at best lead to a non-optimal solution and might plunge the problem solver into an infinite regress.

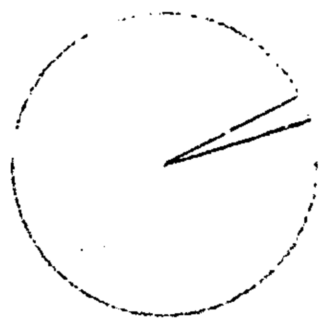

Figure 4-1: A Circular Disc Divided into an Infinite Collection of Sectors

This difficulty seems to be avoided in Mechanics textbooks by not considering this way of subdividing a body. We have found no explanation of why such subdivisions should not be used and think it likely that the unprepared student may easily try to use them. In this case (s)he will require a principled explanation of why they fail to generate the correct answers.

The explanation embodied in the current version of Mecho is that, initially, fibres have the <u>same</u> dimensionality as the body they compose, that is, in the formation of a measure system from the algebraic defining expressions of a body a pair of inequalities of the form

$$a \leq v \leq b$$

are initially replaced by the inequalities

$$v0 \leq v \leq v0 + d(v0)$$

where d(v0) is an infinitesimal quantity, rather than the equation

$$v = v0$$

(Compare the prescription on page 6.) Thus the concentric rings and radial lines are to be considered, initially, as 2 dimensional objects: annuli and sectors respectively; albeit with one infinitesimal dimension. This infinitesimal dimension can only be neglected if the fibres are of uniform thickness in the parameter dimension. The concentric rings pass this test as they have uniform thickness d(r0), but the radial lines fail the test as they have a thickness r.d(θ0), which increases with r.

A fibre is no use in the calculation of radii of gyration unless its infinitesimal thickness can be neglected. It leads to non-optimal solutions or infinite regression. Thus non-uniform measure systems are rejected by Mecho and uniform ones re-idealized as being composed of fibres of one dimension less than the body they compose, i.e. the radial lines are rejected and the concentric rings are re-idealized as 1 dimensional rings by replacing

$r0 \leq r \leq r0 + d(r0)$

with

$r = r0$

Mecho's uniformity test is based on the knowledge that freezing angular coordinates, e.g. Θ, always leads to non-uniform fibres, unless the body being subdivided is 1 dimensional, in which case the fibre is trivially uniform. Freezing distance coordinates, e.g. r, x, etc, always leads to uniform fibres.

As a result of this re-idealization process, the mass per unit length of the concentric ring fibre can now be inferred to be d(r0).mu, where mu is the mass per unit area of the disc. Similar inferences can be made for other fibres. This enables both the mass and the radius of gyration of the ring fibre to be calculated from its 1 dimensional idealization, so eliminating the necessity for double idealization.

5. A Worked Example

Let us see how this process works in a particular case. We will show how Mecho can calculate the radius of gyration of a circular disc about a perpendicular axis through its centre. The problem is described to Mecho with a series of assertions.

```
disc(dsc).          centre(dsc,c).     radius(dsc,a).
mass(dsc,m).        line(axis).
meets(axis,dsc,c).                     rad_of_gyr(dsc,axis,k).
given(a).           given(m).          sought(k).
```

The first three of these describe the disc whose radius of gyration is sought. They give the type of object, its radius and centre. The fourth gives its mass. The fifth and sixth describe the axis about which the radius of gyration is to be calculated and state (using the 'meets' predicate) that it is perpendicular to the plane of the disc, intersecting it at point c. The seventh defines the radius of gyration itself. The last three assertions say which of the various quantities mentioned in the problems are to be solved for (i.e. k) and which quantities can be involved in the solution (i.e. a and m).

The standard techniques of Mecho, as described in e.g. [Bundy et al 79], are then brought to bear. A list is made of the quantities whose value is sought, i.e. [k]. An equation is then formed for each of the quantities in this list in terms of the 'given' quantities, if possible, otherwise the equations may introduce new 'intermediate' unknowns. Currently, Mecho knows only two formula which contain a radius of gyration: the parallel axis theorem and the 'Fibre' formula given below. It chooses the 'Fibre' formula because the axis passes through the centre of gravity of the body.

```
isform(moment_of_inertia, situation(Obj,Axis,Fibre),
       M.RG^2 = integrate(Mf.RGf^2, A, B, X),
                (mass(Obj,M) &
                 rad_of_gyr(Obj,Axis,RG) &
                 cont_meas(Obj,X,Origin,Fibre,A,B) &
                 mass(Fibre,Mf) &
                 rad_of_gyr(Fibre,Axis,RGf)) ).
```

The predicate isform takes 4 arguments:

1. the name of the formula, 'moment of inertia';

2. the situation in which the formula is to be used, consisting of, the name of the object whose radius of gyration is sought, the axis about which it rotates and the fibre into which it is subdivided;

3. the formula itself, containing variables, M, RG, etc, which must be filled with the names of particular entities in order to make an equation* and

4. a conjunction of relations between these variables and the entities in the 'situation' argument slot.

Further explanation can be found in [Bundy et al 79].

Filling in the variables of this formula to make an equation involves:

- accessing the mass and radius of gyration of the disc from the initial assertions;

- erecting a continuous measure system on the disc and

- calculating the mass and radius of gyration of the fibre thus created.

The continuous measure system is chosen by inferring an algebraic description of the disc from its type and then applying the method described in section 3. The first system discovered is the system of concentric rings. These rings are found to be uniform and the rings are re-idealized as 1 dimensional bodies. Using the Mecho schema system, (see [Bundy et al 79]) various assertions about the shape of a typical such ring are then put in the database.

```
ring(typ_ring).          centre(typ_ring,c).
radius(typ_ring,r0).     meets(axis,typ_ring,c).
```

The mass of the typical ring is not known and cannot be inferred by Mecho. However, since mass is a function of an object, Mecho knows that it can create a new intermediate unknown, mf, to fill the variable slot, and this is what it does. A subsequent round of equation forming is now needed to find an equation which solves for mf in terms of a, m and k.

---

*Note that we are using the PROLOG convention where identifiers beginning with a capital letter are variables and the rest are constants

A successful attempt is then made to calculate the radius of gyration of a ring. In fact, this is prestored in Mecho in the inference rule

```
ring(R) & centre(R,C) & radius(R,A) & meets(Axis,R,C)
                                -> rad_of_gyr(R,Axis,A)
```

which is satisfied by accessing the recently asserted facts about the typical ring.

If this information had not been prestored in Mecho, a new intermediate unknown could have been created to fill the variable in the equation, as with mf above. This would have initiated a recursive attempt to calculate the radius of gyration of a ring. It is nice that Mecho has this ability, but it should only be initiated as a last resort. Fortunately, Mecho discourages the introduction of new unknowns unless there is no alternative and the effect of this is that before creating one, Mecho first backtracks through the possible continuous measure systems looking for a fibre whose radius of gyration is already known. Thus an existing, general purpose, mechanism finds an unexpected application in helping to make a sensible choice of measure system for radius of gyration problems.

The result of all this inference is the 'filled in' equation,

$$m.k^2 = \text{integrate}(mf.r0^2,\ 0,\ a,\ r0) \qquad \text{(ii)}$$

integrate is a four argument function of: the expression to be integrated; the lower and upper limits of integration and the variable to integrate with respect to.

Since equation (ii) could only be formed by introducing a new intermediate unknown, mf, an equation must now be formed which relates this unknown to the givens. The equation found by Mecho relates the mass of the ring to its mass per unit length and hence the mass per unit area of the disc.

```
isform(mass_per_length, situation(Obj),
       M = L.Mu,
       (mass(Obj,M) &
        length(Obj,L) &
        mass_per_length(Obj,Mu)) )
```

Filling in the variables in this formula to make an equation involves: accessing the mass of the ring, inferring its length and its mass per unit length.

The length of the ring is inferred using the inference rule.

```
ring(Ring) & radius(Ring,R) -> length(Ring,2.pi.R)
```

The mass per unit length of the ring is inferred to be the same as the infinitesimal thickness of the ring, d(r0), multiplied by the mass per unit area of the disc, but no further progress can be made, since this latter quantity is not known to Mecho. In consequence, the mass per unit area of the disc is introduced as a new unknown, mu.

The resulting equation is,

$$mf = 2.pi.r0.d(r0).mu \qquad \text{(iii)}$$

Finally, an equation must be formed which expresses the new unknown, mu, in terms of known quantities. Mecho decides to use the 'mass_per_area' formula, a 2 dimensional version of the 'mass_per_length' formula, with the disc playing the role of the Obj(ect). Filling in the variables in this version of the formula, involves: accessing the values of the mass and mass per unit area of the disc, and inferring the area of the disc from its type and radius. The resulting equation is,

$$m = pi.a^2.mu \qquad \text{(iv)}$$

Equations (ii), (iii) and (iv) are now passed to the algebra package Press, [Bundy and Welham 81], which is currently being extended by David Skinner to do symbolic integration.

6. Conclusion

We have shown that Mecho can be extended so that it is capable of using the method of Fibres identified by Cohen. Mecho has used the method to calculate the radius of gyration of a disc about a perpendicular axis through its centre, as described in section 5 above. It has also solved several similar problems. Mecho is currently limited in that:

- the object whose radius of gyration is sought must be a 1 or 2 dimensional, regular body;

- the axis of rotation must be perpendicular to the body and

- no special manipulation of its defining equations must be required.

Work is currently in progress to eliminate these restrictions. We are also exploring further applications of continuous measure systems and the method of Fibres. Cohen himself might well pejoratively categorise Mecho's use of the Fibre method as 'Formula Cranking'. In [Cohen 74] he describes how the method can be used, in a highly imaginative way, to solve some hard problems, which he calls dragons. Mecho is not yet capable of solving such problems, but the current paper describes some of the ground work necessary to do so.

- We have shown how the notion of dividing a body into an infinite collection of fibres can be represented using the concept of a continuous measure system (section 2).

- We have shown how such measure systems can be automatically generated and chosen (section 3).

- We have shown how the method can be smoothly integrated with the existing Mecho system (section 5) to enable the calculation of the radii of gyration of some simple bodies.

What Mecho lacks, which prevents it from solving Cohen's dragons is a sophisticated idealization mechanism. Idealization is the process whereby complex real world objects, e.g. a milk bottle, are mapped to 'ideal' objects, e.g. a cylinder, which Mecho can deal with, and in which certain properties, e.g. colour, are neglected and others, e.g. pressure, are considered.

The radius of gyration problems are input in pre-idealized form, i.e. the objects involved are already ideal ones. Thus no idealization mechanism is called for. In other problems Mecho does do some simple idealization, but there is usually little ambiguity about what the idealization should be, nor is there any need for the idealization and problem solving processes to interact. Chris Mellish is currently investigating more sophisticated idealization mechanisms.

Extending Mecho to deal with radius of gyration problems has entailed a detailed investigation of the problem solving processes involved. As we saw in section 4, this investigation has uncovered conceptual difficulties in these processes and has suggested a principled way of avoiding the difficulties employing the notion of a uniform fibre. We hope our analysis may lead to the improvement of teaching in this area. The discovery of such applications is one of the major motivations of our work.

REFERENCES

[Bundy and Welham 81]

Bundy, A. and Welham, B.
Using meta-level inference for selective application of multiple rewrite rules in algebraic manipulation.
Artificial Intelligence , in press 1981.

[Bundy et al 79]

Bundy, A., Byrd, L., Luger, G., Mellish, C., Milne, R. and Palmer, M.
Solving Mechanics Problems Using Meta-Level Inference.
In *Procs of the sixth*. IJCAI, Tokyo, 1979.
Also available from Edinburgh as DAI Research Paper No. 112.

[Cohen 74]

Cohen, H.A.
The art of snaring dragons.
Technical Report, La Trobe University, 1974.
A revised version of LOGO Working Paper No. 28 AI Lab MIT.