A Projected Inverse Dynamics Approach for Multi-arm Cartesian Impedance Control

Citation for published version:

Digital Object Identifier (DOI):
10.1109/ICRA.2018.8461202

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
2018 IEEE International Conference on Robotics and Automation (ICRA)

General rights
Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.
A Projected Inverse Dynamics Approach for Multi-arm Cartesian Impedance Control

Hsiu-Chin Lin, Joshua Smith, Keyhan Kouhkilou Babarahmati, Niels Dehio, and Michael Mistry

Abstract— We propose a model-based control framework for multi-arm manipulation of a rigid object subject to external disturbances. The control framework, based on projected inverse dynamics, decomposes the control law into constrained and unconstrained subspaces. Unconstrained components accomplish the motion task with a desired 6-DOF Cartesian impedance behaviour against external disturbances. Meanwhile, the constrained component enforces contact and friction constraints by optimising for contact forces within the constrained subspace. External disturbances are explicitly compensated for without using force/torque sensors at the contact points. The approach is evaluated on a dual-arm platform manipulating a rigid object while coping with unknown object dynamics and human interaction.

I. INTRODUCTION

Many activities in robotics can be described in terms of performing a desired task subject to physical constraints and external disturbances. For example, a dual-arm robot squeezing a rigid object (i.e. constraints), while a human interacts by pushing the object or adding unknown mass (i.e. disturbances) (Fig. 1). A controller must be aware of contributions from both types of forces in order to achieve its task in an optimal manner. For example, to counteract disturbances with a desired impedance response, while squeezing only as necessary to maintain contact of the object.

Several past works have considered multi-arm object manipulation [1][2], including control of the manipulated object’s impedance [3][4]. A particular challenge, however, is coping with external disturbance. Maintenance of contact when grasping requires dealing with unknown forces and moments applied to the object, which may include disturbances arising from motion of the robot, unknown inertial dynamics, or the forces due to gravity. One common strategy has been to regulate the internal impedance of the grasp (i.e. the impedance between the end-effectors and the object) [5][6][4][7]. However, an increase in grasp force (e.g. to compensate for a forceful push) requires a displacement between the object and contact point (which does not occur with a rigid object) or an appropriate adjustment of stiffness gains and/or desired set points (which requires contact force sensing). Rather, to compute forces to maintain a grasp, we prefer to work directly in the space of contact forces, where friction cones and external forces can be explicitly, and optimally, accounted for in a constraint optimisation framework [8]. This is the approach commonly taken in locomotion [9][10], as well as in some works on grasping [11][12][13][14].

II. BACKGROUND

Our work stems from prior literature in projected inverse dynamics control, impedance control, and grasping.
A. Projected Inverse Dynamics

The problem is formulated in the cartesian space projected inverse dynamics framework [16], such that the control law is decomposed into unconstrained and constrained subspaces.

Let \( q, \dot{q}, \ddot{q} \in \mathbb{R}^Q \) denote the joint positions, velocities, and accelerations of a \( Q \) degree-of-freedom manipulator, the dynamics can be expressed in the Lagrangian form

\[
M \ddot{q} + H = \tau
\]

where \( \tau \in \mathbb{R}^Q \) is the vector of joint torques, \( M \in \mathbb{R}^{Q \times Q} \) is the inertia matrix, and \( H \in \mathbb{R}^Q \) is the vector of centripetal, gyroscopic, and Coriolis effects, and generalised gravitational torque.

When a robot is interacting with the environment, the end-effector motion may be subject to the constraints imposed by the environment, which modifies the motion in order to accommodate the constraints. An additional term is added to describe the rigid body dynamics under constraints

\[
M \ddot{q} + H + \tau = \tau_c
\]

where \( J_c \in \mathbb{R}^K \times Q \) is the constraint Jacobian that describes \( K \) linearly independent constraints, and \( \lambda_c \) are the constraint forces due to contact that enforce the following conditions:

\[
J_c \ddot{q} = 0
\]

\[
J_c \dot{q} + J_c \dot{q} = 0
\]

C. Grasping

Following the definition in [21], the grasp matrix of the \( i^{th} \) arm in a multi-arm manipulation system is defined by the mapping between the object twist to the object centre-of-mass position, and \( \mathbb{S}(r_i) \in \mathbb{R}^{3 \times 3} \) is the skew-symmetric matrix performing the cross product

\[
\mathbb{S}(r) = \begin{bmatrix} 0 & -r_z & r_y \\
 r_z & 0 & -r_x \\
- r_y & r_x & 0 \end{bmatrix}
\]

Assuming that the robot has \( K \) manipulators, the grasp map \( G \) is the horizontal concatenation of \( K \) grasp matrices

\[
G \in \mathbb{R}^{6 \times 6K} = [G_1, G_2, \ldots, G_K]
\]

For example, the grasp map of the dual-arm system \( (K = 2) \) is defined as \( G := [G_L, G_R] \in \mathbb{R}^{6 \times 12} \) where \( G_L, G_R \) are the grasp matrix of the left and the right arm.

III. METHOD

Our main control framework is based on extending the previous work of projected inverse dynamics control framework [16] from single arm manipulation to multi-arm manipulation, together with 6-DOF Cartesian impedance controller at the object and optimisation of constraint forces at the points of contact.

Specifically, the control law with external disturbance is decomposed into unconstrained and constrained subspaces
Fig. 2: An overview of the projected inverse dynamics framework

($\text{§III-A}$). For multi-arm manipulation, we defined the constraints such that only internal forces are allowed in the constrained subspace ($\text{§III-B}$). The unconstrained subspace controller realises the underlying task and impedance behaviour ($\text{§III-C}$) while the constrained subspace controller estimates the contact force need to maintain the contact against external disturbance ($\text{§III-D}$).

### A. Projected Inverse Dynamics with External disturbance

The original projected inverse dynamics does not consider external disturbance. We extended the previous work by adding an additional term for the external disturbance force $F_x$ into the inverse dynamics equation (2), the general rigidbody dynamics can be described as

$$M\ddot{q} + h = \tau + J_c^T \lambda_c + J_x^T F_x \tag{10}$$

where $J_x$ is the Jacobian that relates the joint space to the position of interaction. As only the unconstrained component of the control torque contributes to motion and the desired impedance behaviour, we multiply both sides of (10) by $P$, and resulting in

$$PM\ddot{q} + Ph = P(\tau + J_x^T F_x) \tag{11}$$

**Remark** The contact force $\lambda_c$ vanished from the above equation since $P J_x^T = 0$. An important insight is, in order to realise our desired impedance behaviour, the unconstrained space control law does not involve the constraint force.

Detail of $P\tau$ will be provided in §III-C. For the constrained subspace, we multiply both sides of (10) by $(I - P)$, the dynamics can be described by

$$(I - P)(M\ddot{q} + h) = (I - P)(\tau + J_x^T F_x) + J_c^T \lambda_c \tag{12}$$

**Remark** The above equation aims at adding additional torque *within the constrained subspace* without any effect on the unconstrained subspace (i.e., the desired motion impedance characteristic). We can exploit this property to optimize the constraint forces required to maintain grasp of an object (see §III-D).

The notions of $P$, $J_c$, $P\tau$, and $(I - P)\tau$ are generic and can be applied to various problems of constraint systems with external disturbance. In the following sections, we will define these variables for multi-arm manipulation. For simplicity, we define $\tau_c$ as the output from the unconstrained space controller and $\tau_c$ as the output torque from the constrained space controller throughout this paper. An overview of the control framework is illustrated in Fig. 2.

### B. Projected Inverse Dynamics for Multi-arm Manipulation

The projected inverse dynamics formulation was originally used to control single arm acting on a rigid environment. In this work, it is extended to multiple arms manipulating a single rigid object via formulating the constraints such that only internal force is allowed in the constrained space.

For a multi-arm robot manipulating a single rigid object via a force-closed grasp, each end-effector is in contact with the object and may generate arbitrary wrenches upon the object (see Fig. 3). The constraints are to enforce the force-closed grasp of the object and generate no motion that might violate the underlying task.

The projected inverse dynamics was built upon analytical dynamics, and based on the study in [22], the constraint force should not produce any virtual work for any virtual displacement. From the analysis in [23], internal wrenches, or end-effector wrench acting in the null space of grasp map (§II-C), yields the same property with constraint force in grappling. For this, the multi-arm system is constrained such that only internal wrenches are allowed to enforce the contacts.

Given a grasp map $G$, the null space projection $I - G^+ G$ projects any arbitrary vector onto the null space of the grasp map. The resulting contact force satisfies $G \lambda_c = 0$, yielding no net wrench on the object and contributing to only internal force. Under this formulation, the constraint Jacobian in (16) for a multi-arm system is written as:

$$J_c \in \mathbb{R}^{K \times (P+Q)} = (I - G^+ G) \begin{bmatrix} J_1 & 0 \\ \vdots \\ 0 & J_K \end{bmatrix} \tag{13}$$

where $P$ denotes the dimensionality of the end-effector space, $J_i \in \mathbb{R}^{P \times Q}$ are the Jacobian of the $i^{th}$ arm, and $\lambda_c \in \mathbb{R}^{KP}$ is the vertical concatenation of all contact wrenches due to internal forces that apply no net wrench on the object.

We use this constraint Jacobian $J_c$ to produce the projected matrix $P = I - J_c^T J_c$. This projection $P$ decomposes the control law such that the unconstrained component resolves the underlying task and impedance behaviour by using external forces while the constraint component maintains the contact by adding internal forces.

**Remark** The concept of internal/external force has been studied for multi-arm manipulating an object, (although not using the projected inverse dynamics) such as the work in [4], which uses impedance controller to control the internal force. In our work, we explicitly model the external disturbance and optimise the internal force to maintain the contact constraints against external disturbances (see §III-D).

### C. Impedance Controller

In §III-A, we have shown that the dynamics in the unconstrained subspace can be described by (11). Let $F_a$ be actuation force needed to accomplish the task such that $\tau = J_a F_a$, the dynamics can be written as

$$PM\ddot{q} + Ph = PJ_x^T (F_a + F_x) \tag{14}$$
The above dynamics can also be expressed in operational space, yields (see Appendix A for details):
\[ \Lambda_c \ddot{x} + h_c = F_a + F_x \] (15)
where \( \Lambda_c = (J_x M_c^{-1} P J_x^T)^{-1} \) is the operational space inertia matrix, \( h_c = \Lambda_c J_x M_c^{-1} (P \dot{q} - \dot{P} \dot{q}) - \Lambda_c J_z \dot{q} \) is the operational space coriolis, centripetal, and gravity vector. As described in §II-B, assuming that the desired inertia is identical to the robot inertia, we can avoid the feedback of the external force \( F_x \), and yields
\[ \tau_u = PJ_x^T F \] (16)
where \( F = h_c + \Lambda_c \ddot{x} - D_d \dot{x} - K_\gamma \dot{x} \) is the force needed to accomplish the underlying task and desired impedance response.

D. Optimal Contact Wrenches

In the last section, we showed that the unconstrained space controller realises the desired task and impedance behaviour without involving the constraint force. In this section, we outline our approach for the constrained space controller that attempts to apply the minimum torque required to maintain the contact without use of Force/Torque sensing at the contacts.

1) Constraints: Maintenance of the contact requires dealing with unknown forces and moments applied to the object, which may include the disturbances arising from the motion of the robot, inertial forces during manipulation, or the forces due to gravity. For this, the contact wrench applied by the hands should be sufficient enough to prevent the separation or sliding of the contact. However, too much internal force may decrease the stability of the grasp or damage the object. Therefore, we incorporate optimisation strategies to seek the minimal contact force needed to maintain the grasp.

The contact wrench includes the contact force and the contact moment \( \lambda_c \in \mathbb{R}^6 = [\lambda_{f,x}, \lambda_{m,z}]^T \). Throughout the rest of this paper, we use the subscripts \( f \) and \( m \) to denote the force and moment, respectively, and we choose the \( z \)-axis as the direction normal to the contact surface. Specifically, the contact force \( \lambda_f = [\lambda_{f,x}, \lambda_{f,y}, \lambda_{f,z}]^T \) where \( \lambda_{f,z} \) is the normal force, and \( \lambda_{f,x}, \lambda_{f,y} \) are the tangential forces. The moment \( \lambda_m = [\lambda_{m,x}, \lambda_{m,y}, \lambda_{m,z}]^T \) are the moment along each axis. A dual-arm example is illustrated in Fig. 3.

- Unilateral constraints: The manipulators should only push toward the contact, but not pull, in order to maintain contacts. Hence, the contact normal should satisfy the unilateral constraint
  \[ \lambda_{f,z} \geq 0 \] (17)
- Friction Cone Constraints: If there is significant contact friction, a common way to describe the contact is by the Coulomb’s friction model [8]. By Coulomb’s Law, the magnitude of tangential force \( \lambda_f \) should not exceed the friction coefficient times the normal force to avoid slipping
  \[ \mu \lambda_{f,z} \geq \sqrt{\lambda_{f,x}^2 + \lambda_{f,y}^2} \] (18)
where \( \mu \) is the friction coefficient which depends on the material of the object. Geometrically, the set of forces which can be applied should lie in a cone centred about the direction normal to the contact surface (i.e. the grasp is more stable if the direction of the force is more orthogonal to the surface of the object).

- Moment Constraints We assume the surface friction and the contact patch are large enough to generate friction force and moment. To avoid the hand from rolling at the contact point, the constraints are imposed on the torsional moment [12] and shear moment [13]
  \[
  \gamma \lambda_{f,z} \geq |\lambda_{m,z}|
  \\
  \delta x \lambda_{f,z} \geq |\lambda_{m,x}|
  \\
  \delta y \lambda_{f,z} \geq |\lambda_{m,y}|
  \] (19)
where \( \gamma \) is the torsional friction coefficient, and \( \delta x, \delta y \) are the distance from the centre of the hand to the edge of the hand in \( x \) and \( y \) direction (assuming a rectangular contact patch). The latter two constraints ensure the contact centre of pressure remains within the contact patch of the hand.

2) Objective Function: In general, the optimisation objective is to find the minimum actuator torques needed to maintain all contacts,

\[ \text{minimise } \tau^T \tau \]

Substituting \( \tau \) with \( \tau_u + \tau_c \) from (4) and expanding the quadratic function, the objective function becomes

\[ \text{minimise } \tau_u^T \tau_u + 2 \tau_u^T \tau_c + \tau_c^T \tau_c \]

Since the unconstrained space controller \( \tau_u \) is independent of the constrained space controller \( \tau_c \) and hence is a constant during optimisation, the first term can be eliminated from the objective function.

Let \( F_c \) be the equivalent end-effector wrench corresponding to the input torque \( \tau_c \) such that \( \tau_c = J_c^T F_c \).

Remark We can see that \( J_c^T F_c \) lies within the constrained manifold, i.e., \( (I - P) J_c^T F_c = J_c^T F_c \). It is sufficient to enforce that the resulting torques satisfies \( \tau_c \perp \tau_u \). The value of the second term \( 2 \tau_u^T \tau_c \) is always 0. Therefore, we can simply minimise \( \tau_u^T \tau_c \).
Equivalently, the objective function can be reformulated in terms of the unknown variable $F_c$:

$$\text{minimise } F_c^\top J_c^\top J_c F_c$$  \hspace{1cm} (20)

3) Constrained optimisation: Finally, the optimisation problem is to find the optimal contact wrenches that minimises the objective function (20) while satisfying the unilateral constraints (17), the friction cone constraints (18), and the moment constraints (19) at the contact points, and balance out the external forces, including the forces acting on the object and the object dynamics.

Assuming that we have $K$ contacts ($K = 2$ for the dual arm example), there are $K$ contact wrenches, and constraints for all the contact wrenches need to be solved. If the contact locations are fixed, finding the minimum torques is a convex optimisation problem over contact wrenches [24].

$$\text{minimise } F_c^\top J_c^\top J_c F_c$$

subject to $\lambda^{i}_{f,z} \geq 0$

$$\mu \lambda^{i}_{f,z} \geq \sqrt{(\lambda^{i}_{f,x})^2 + (\lambda^{i}_{f,y})^2}$$

$$\gamma \lambda^{i}_{f,z} \geq |\lambda^{i}_{m,z}|$$

$$\delta x \lambda^{i}_{f,z} \geq |\lambda^{i}_{m,x}|$$

$$\delta y \lambda^{i}_{f,z} \geq |\lambda^{i}_{m,y}|$$

where the superscript $i$ denotes the $i^{th}$ contact.

**Remark** By setting $\tau_c = J_c^\top F_c$, the resulting torque satisfies $\tau_c \perp \tau_u$. We can simplify the objective function by removing $2\tau_u^\top \tau_c$ and relax the constraint $\mathbf{F}_c = 0$, as comparing to the optimisation problem proposed in [17].

To ensure that the acceleration generated from the constrained space controller $\tau_c$ is consistent with the unconstrained space controller $\tau_u$, the joint-acceleration in (12) is replaced by $\ddot{\mathbf{q}} = M_c^{-1}(\tau_u - \mathbf{Ph} + \dot{\mathbf{P}}\dot{\mathbf{q}})$ (See Appendix B)

$$(I - \mathbf{P}) \left[ (J_c^\top)^\top M_c^{-1}(\tau_u - \mathbf{Ph} + \dot{\mathbf{P}}\dot{\mathbf{q}}) + \mathbf{h} - J_c^\top \mathbf{F}_c \right]$$

$$= J_c^\top \mathbf{F}_c + J_c^\top \lambda_c$$

We multiply both sides of (22) by $(J_c^\top)^+$ resulting in

$$(J_c^\top)^+ (I - \mathbf{P}) \left[ (J_c^\top)^\top M_c^{-1}(\tau_u - \mathbf{Ph} + \dot{\mathbf{P}}\dot{\mathbf{q}}) + \mathbf{h} - J_c^\top \mathbf{F}_c \right]$$

$$= (J_c^\top)^+ J_c^\top \mathbf{F}_c + \lambda_c$$

(23)

The left hand side can be interpreted as the sum of all external wrenches (i.e., from robot/object dynamics or human interactions) in the constrained space. Let $\eta = (J_c^\top)^+ (I - \mathbf{P}) \left[ (J_c^\top)^\top M_c^{-1}(\tau_u - \mathbf{Ph} + \dot{\mathbf{P}}\dot{\mathbf{q}}) + \mathbf{h} - J_c^\top \mathbf{F}_c \right]$ and $\rho = (J_c^\top)^+ J_c^\top \mathbf{F}_c$ the relationship between the contact wrench, the commanded wrench, and the external wrench can be described by a compact equality expression $\eta = \rho + \lambda_c$.

Each element of the contact force can be described as

$$\lambda^{i}_{f,x} = \eta^{i}_{f,x} - \rho^{i}_{f,x}$$

$$\lambda^{i}_{f,y} = \eta^{i}_{f,y} - \rho^{i}_{f,y}$$

$$\lambda^{i}_{f,z} = \eta^{i}_{f,z} - \rho^{i}_{f,z}$$

(24)

Finally, the optimisation problem (25) requires knowledge of the external disturbances $\mathbf{F}_x$ (included in the $\eta$ term). Because $\mathbf{F}_x$ is estimated using the displacement of object from (7), the result of the optimisation can resist the external disturbance without using force/torque sensors.

Finally, as the friction cone constraints are quadratic (and therefore not realistic for real-time control) we approximate the constraints with linearised friction cones of 8 edges, resulting a quadratic problem with linear constraints. The constraints optimisation problem was then solved using quadratic programming [25]. During the experiment, we are able to find a solution within 1 milli-second.

**Remark** By decomposing the control law into two subspaces, we can impose impedance behaviour only in the unconstrained subspace without breaking the constraint (e.g., dropping the object). We can also perform constrained optimisation only within the constrained subspace, which can reduce the complexity of the optimisation problem.

In summary, we first compute $\tau_u$, the torques needed for the desired task and impedance characteristics (16). Then, we calculate the sum of all force acting in the constrained space $\eta$, and then find $\tau_c$, the minimum torque needed to maintain the contacts (25). The final output of our controller is the sum of these two $\tau = \tau_u + \tau_c$. The overview of the control framework is illustrated in Fig. 4.

IV. EVALUATION

We conduct experiments using our dual KUKA LWR platform Boris. Although the robots are equipped with force/torque sensors at the end-effector, these are only used for recording forces and not in the controller.
A. Holding an object

In this experiment, we would like to evaluate how well the robot can resist external forces. For this, the task of robot is to hold an object at a static posture while external disturbances are supplied. The robot has no knowledge about the weight of the object nor the magnitude of the external forces.

At the beginning of the experiment, as shown in Fig. 5 (a), the robot is holding a rigid box at a position in front of its torso. The size of the box is approximately 20 cm × 30 cm × 40 cm (known to the controller) and the weight is 700 grams (unknown to the controller). A human subject pushes the box about 40 cm downward, stays for a few seconds (Fig. 5 (b)), and releases it (Fig. 5 (c)).

This process is repeated a few times, and the norm of the contact force is shown in Fig. 6 (top), where the colours denote the expected contact force (blue) and the measured contact force (red). Note that the majority of force is due to the end-effectors pushing toward each other. When the robot is at the static position, it squeezes the box with 70 N from both arms. As the person pushes the box down, the robot squeezes the box with a higher force (110 N) to prevent the box from slipping. Note that the robot does not need to measure the external forces in order to know to push harder. The external force is estimated using the displacement of the box relative to its desired position (7).

In the second half of the experiment, a human subject continuously adds extra weights on top of the box, 500 grams at a time, until a total of 2500 grams are add (see Fig. 5 (d)). The corresponding contact forces are plot in Fig. 6 (bottom). We can clearly see that the contact force increases as the total weight of the object gets heavier.

B. Manipulating an object

In our final experiment, we would like to see how well the robot reacts to external forces while performing some task. For this, the robot moves the box in a periodic trajectory, and a human attempts to interrupt the robot by holding the box at a given position (see Fig. 7).

The norm of the applied force is shown in Fig. 8 (bottom). We can see that the force needed to move in a circular motion varies depends on the direction of the motion, i.e. when the robot moves downward, the direction of motion is with gravity and hence less force is needed to maintain the grasp.

V. Conclusion

In this paper, a method for multi-arm manipulation with external disturbance is proposed. The problem is formulated in a projected inverse dynamics framework, such that the unconstrained (motion) controller accomplishes the task with desired impedance behaviour, and the constrained component enforces the contact in an optimal manner. The technique
Fig. 7: Experiment of a dual-arm robot moving a box in a circular trajectory. A human attempts to break the trajectory by holding the box.

Fig. 8: A dual-arm robot moves a box in circular trajectory with human interactions. The top two figures show the trajectory tracking in y-axis and z-axis, and the bottom figure shows magnitude of expected contact force (blue) and measured contact force (red). Human interaction occurs roughly between 3 and 4 seconds.

is evaluated on a dual-arm platform, showing the proposed method’s robustness to unknown disturbances. The proposed theory is generic and can be extended to multi-arms, and one of our future work is to carry out robot experiment on a multi-arm [26] and multi-leg [27] platform.

Note that throughout this work, the manipulated object is always assumed to be massless. As a consequence, any inertial or gravity forces due to mass of the object are treated as external disturbances by the impedance controller. The present work demonstrates our controller’s robustness and ability to maintain a grasp, subject to unknown human interactions and unknown object inertia. However, in future work we plan to include on-line estimation of the objects mass/inertia, such that the controller may compensate for these during manipulation.

Furthermore, we have demonstrated that external disturbance forces do not need direct measurement, and may be estimated based on the displacement of the object relative to our desired impedance behaviour. This enables us to compute optimal constraint forces without direct force measurement. In future work, however, we plan to incorporate F/T contact sensors to allow for inertia shaping in the impedance controller.

ACKNOWLEDGEMENT

This work was funded by the European Commission Horizon 2020 Work Programme: CogIMon ICT-23-2014 644727 and THING ICT-2017-1 780883, as well as EPSRCs RAI Hubs for Extreme and Challenging Environments: ORCA EPR026173/1 and NCNR EPR02572X/1

APPENDIX

A. Desired end-effector Force with external disturbance

From (11), the dynamic of the unconstrained controller is described by $PM\ddot{q} + Ph = P(\tau + J_x^T F_x)$. Adding $\dot{P}\ddot{q} - \dot{P}\dot{q}$ to the left side of (11)

$$PM\ddot{q} + Ph = PM\ddot{q} + Ph + \dot{P}\ddot{q} - \dot{P}\dot{q}$$

Using the projection matrix $P$, the constraints in (3) can also be described as $(I - P)\dot{q} = 0$. By taking the derivative, $(I - P)\dot{q} - \dot{P}\dot{q} = 0$. Replace the first $\dot{P}\ddot{q}$ in the above equation by $(I - P)\ddot{q}$

$$PM\ddot{q} + Ph = PM\ddot{q} + Ph + (I - P)\ddot{q} - \dot{P}\dot{q}$$

Let $M_c = PM + I - P$, the dynamics equation in (14) can be written as

$$M_c\ddot{q} + Ph - \dot{P}\dot{q} = PJ_x^T (F_a + F_x)$$

(26)

Multiply (26) by $J_x M_c^{-1}$

$$J_x \ddot{q} + J_x M_c^{-1} (Ph - \dot{P}\dot{q}) = J_x M_c^{-1} PJ_x^T (F_a + F_x)$$

Since $\ddot{x} = J_x \ddot{q} + \dot{J}_x \dot{q}$, we replace $J_x \ddot{q}$ with $\ddot{x} - \dot{J}_x \dot{q}$

$$\ddot{x} - \dot{J}_x \dot{q} + J_x M_c^{-1} (Ph - \dot{P}\dot{q}) = J_x M_c^{-1} PJ_x^T (F_a + F_x)$$
Replacing $\tau$ with $J^\top x F_a$ where $F_a$ is the actuation force needed to accomplish the task,

$$\ddot{x} - J \dot{q} + J x M^{-1} (Ph - \dot{P}q) = J x M^{-1} P J^\top x (F_a + F_x)$$

Multiply by $A_c = (J x M^{-1} P J^\top x)^{-1}$

$$A_c \left[ \ddot{x} - J \dot{q} + J x M^{-1} (Ph - \dot{P}q) \right] = F_a + F_x$$

Let $h_c$ denotes all gravity and velocity terms such that $h_c = A_c J x M^{-1} (Ph - \dot{P}q) - A_c J x \dot{q}$, the operational space configuration is

$$A_c \ddot{x} + h_c = F_a + F_x$$

If inertia shaping is avoided (see §II-B), the operational space control force is

$$F = h_c + A_c \ddot{x}_d - D_c \dot{x} - K_c \dot{x}$$

(27)

B. Constraint-consistent desired acceleration

To ensure that the joint accelerations in (12) is consistent with the desired joint accelerations in (16), we need to solve $\ddot{q}$ in (16). However, $P$ is rank deficient, and the term $PM\dot{q}$ is not invertible.

From (26), the dynamics of the unconstrained space is $M_c \ddot{q} + Ph - \dot{P}q = PJ^\top x (F_a + F_x)$. The left-hand side is the torque needed in the unconstrained space controller. Substitute with the definition from (16),

$$M_c \ddot{q} + Ph - \dot{P}q = \tau_u$$

Since $M_c$ is invertible, $\ddot{q}$ can be solved by

$$\ddot{q} = M_c^{-1} (\tau_u - Ph + \dot{P}q)$$

(28)

REFERENCES


