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On stochastic gradient Langevin dynamics with dependent data streams in the logconcave case *

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Abstract

Stochastic Gradient Langevin Dynamics (SGLD) is a combination of a Robbins-Monro type algorithm with Langevin dynamics in order to perform data-driven stochastic optimization. In this paper, the SGLD method with fixed step size $\lambda$ is considered in order to sample from a logconcave target distribution $\pi$, known up to a normalisation factor. We assume that unbiased estimates of the gradient from possibly dependent observations are available. It is shown that, for all $\varepsilon > 0$, the Wasserstein-2 distance of the $n$th iterate of the SGLD algorithm from $\pi$ is dominated by $c_1(\varepsilon)[\lambda^{1/2-\varepsilon} + e^{-a\lambda n}]$ with appropriate constants $c_1(\varepsilon), a > 0$.

1 Introduction

Sampling target distributions is an important topic in statistics and applied probability. In this paper, we are concerned with sampling from the distribution $\pi$ defined by

$$\pi(A) := \int_A e^{-U(x)} dx / \int_{\mathbb{R}^d} e^{-U(x)} dx, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}^d$ and $U : \mathbb{R}^d \to \mathbb{R}_+$ is continuously differentiable.

One of the recursive schemes considered in this paper is the unadjusted Langevin algorithm. The idea is to construct a Markov chain which is the Euler discretization of a continuous-time diffusion process whose invariant distribution is $\pi$. More precisely, we consider the overdamped Langevin stochastic differential equation

$$d\theta_t = -h(\theta_t) dt + \sqrt{2} dB_t, \quad (1)$$

with a (possibly random) initial condition $\theta_0$ (independent of $(B_t)_{t \geq 0}$) where $h := \nabla U$ and $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. It is well-known that, under appropriate conditions, the Markov semigroup associated with the Langevin diffusion $(1)$ is reversible with respect to $\pi$, and the rate of convergence to $\pi$ is geometric in the total variation norm (see [20], [25] and Theorem 1.2, Theorem 1.6 in [1]). The Euler-Maruyama discretization scheme for $(1)$ is given by

$$\theta^\lambda_n := \theta_0, \quad \theta^\lambda_{n+1} := \theta^\lambda_n - \lambda h(\theta^\lambda_n) + \sqrt{2\lambda} \xi_{n+1}, \quad (2)$$

where $(\xi_n)_{n \in \mathbb{N}}$ is an independent sequence of standard Gaussian $d$-dimensional random variables (independent of $\theta_0$), $0 < \lambda \leq 1$ is a fixed step size, and $\theta_0$ is the $\mathbb{R}^d$-valued random variable representing the initial values of both $(2)$ and $(1)$. When the step size $\lambda$ is fixed, the homogeneous Markov chain $(\theta^\lambda_n)_{n \in \mathbb{N}}$ converges to a distribution $\pi_\lambda$ (under suitable assumptions) which differs from $\pi$ but, for small $\lambda$, it is close to $\pi$ in an appropriate sense (see [11]).

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We now adopt a framework where the exact gradient $h$ is unknown, but one can observe its unbiased estimates. Let $H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be a measurable function, let $(X_n)_{n \in \mathbb{N}}$ be an $\mathbb{R}^m$-valued (strict sense) stationary process. Furthermore, we suppose that $h(\theta) = E[H(\theta, X_0)]$, $\theta \in \mathbb{R}^d$ (we implicitly assume the existence of the expectation). It is technically convenient to assume also that

$$X_n = g(\varepsilon_n, \varepsilon_{n-1}, \ldots), \quad n \in \mathbb{Z},$$

for some i.i.d sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with values in a Polish space $X$ and for a measurable function $g : X^\mathbb{N} \to \mathbb{R}^m$.

We assume in the sequel that $\theta_0$, $(\varepsilon_n)_{n \in \mathbb{N}}$, $(\xi_n)_{n \in \mathbb{N}}$ are independent.

For each $0 < \lambda \leq 1$, define the $\mathbb{R}^d$-valued random process $(\theta_n^\lambda)_{n \in \mathbb{N}}$ by recursion:

$$\theta_0^\lambda := \theta_0, \quad \theta_{n+1}^\lambda := \theta_n^\lambda - \lambda H(\theta_n^\lambda, X_{n+1}) + \sqrt{2} \xi_{n+1}.$$

The goal of this work is to establish an upper bound on the Wasserstein distance between the target distribution $\pi$ and its approximations $(\text{Law}(\theta_n^\lambda))_{n \in \mathbb{N}}$. We improve the rate of convergence with respect to [23], see also [29].

Data sequences are, in general, not necessarily i.i.d. or even Markovian. They may exhibit long memory as in the case of many models in finance and queueing theory, see e.g. [28, 2]. It is thus crucial to ensure the validity of sampling procedures like (1) in such circumstances, too.

The paper is organized as follows. Section 2 introduces the theoretical concept of conditional $L$-mixing which we will require for the process $X$. This notion accommodates a large class of (possibly non-Markovian) processes. In Section 3 assumptions and main results are presented. Section 4 discusses the contributions of our work. In Sections 5, 6, 7 we analyze the properties of (1), (2), and (4), respectively. Certain proofs and auxiliary results are contained in Section 8.

Notation and conventions. Scalar product in $\mathbb{R}^d$ is denoted by $(\cdot, \cdot)$. We use $\| \cdot \|$ to denote the Euclidean norm (where the dimension of the space may vary). $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel $\sigma$- field of $\mathbb{R}^d$. For each $R \geq 0$ we denote $B(R) := \{x \in \mathbb{R}^d : \|x\| \leq R\}$, the closed ball of radius $R$ around the origin. We are working on a probability space $(\Omega, \mathcal{F}, P)$. For two sigma algebras $\mathcal{F}_1, \mathcal{F}_2$, we define $\mathcal{F}_1 \vee \mathcal{F}_2 := \sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$. Expectation of a random variable $X$ will be denoted by $EX$. For any $m \geq 1$, for any $\mathbb{R}^m$-valued random variable $X$ and for any $1 \leq p < \infty$, let us set $\|X\|_p := E^{1/p}\|X\|^p$. We denote by $L^p$ the set of $X$ with $\|X\|_p < \infty$. The indicator function of a set $A$ will be denoted by $\mathbb{1}_A$. The Wasserstein distance of order $p \in [1, \infty]$ between two probability measures $\mu$ and $\nu$ on $\mathcal{B}(\mathbb{R}^d)$ is defined by

$$W_p(\mu, \nu) = \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_X \|x - y\|^p d\pi(x, y) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of couplings of $(\mu, \nu)$, see e.g. [27] for more information about this distance.

## 2 Conditional $L$-mixing

$L$-mixing processes and random fields were introduced in [11]. They proved to be useful in tackling difficult problems of system identification, see e.g. [12, 13, 14, 15, 24]. In [1], in the context of stochastic gradient methods, the related concept of conditional $L$-mixing was introduced. We now recall its definition below.

We consider a filtered probability space equipped with a discrete-time filtration $\mathcal{H}_n$ as well as with a decreasing sequence of sigma-fields $\mathcal{H}_n$ such that $\mathcal{H}_n$ is independent of $\mathcal{H}_n$, for all $n$.

Fix an integer $d \geq 1$ and let $D \subset \mathbb{R}^d$ be a set of parameters. A measurable function $X : \mathbb{N} \times D \times \Omega \to \mathbb{R}^m$ is called a random field. We will drop dependence on $\omega \in \Omega$ in the notation and write $(X_n(\theta))_{n \in \mathbb{N}, \theta \in D}$. A random process $(X_n)_{n \in \mathbb{N}}$ corresponds to a random field where $D$ is a singleton. A random field is $L^r$-bounded for some $r \geq 1$ if

$$\sup_{n \in \mathbb{N}} \|X_n(\theta)\|_r < \infty.$$

For a family $(Z_i)_{i \in I}$ of real-valued random variables, there exists one and (up to a.s. equality) only one random variable $g = \text{ess sup}_{i \in I} Z_i$ such that:

(i) $g \geq Z_i$, a.s. for all $i \in I$,
(ii) if \( g' \) is a random variable, \( g' \geq Z_i, \) a.s. for all \( i \in I \) then \( g' \geq g \) \( \mathbb{P} \) - a.s.,

see e.g. [22] Proposition VI.1.1.

Now we define conditional \( L \)-mixing. For some \( r \geq 1, \) let \( (X_n(\theta))_{n \in \mathbb{N}, \theta \in D} \) be a random field bounded in \( L^r. \) Define, for each \( n \in \mathbb{N}, \)

\[
M_n^r(X) := \text{ess sup} \sup_{\theta \in D} E_1^{1/r}[\|X_{n+m}(\theta)\|^{r} |H_n],
\]

\[
\gamma_n^r(\tau, X) := \text{ess sup} \sup_{\theta \in D} E_1^{1/r}[\|X_{n+m}(\theta) - E[X_{n+m}(\theta)|H_{n+m-\tau} \vee H_n]\|^{r} |H_n], \quad \tau \geq 1,
\]

\[
\Gamma_n^r(X) := \sum_{\tau=1}^{\infty} \gamma_n^r(\tau, X).
\]

When necessary, \( M_n^r(X, D), \gamma_n^r(\tau, X, D) \) and \( \Gamma_n^r(X, D) \) are used to emphasize dependence of these quantities on the domain \( D \) which may vary.

**Definition 2.1** (Conditional \( L \)-mixing). We call \( (X_n(\theta))_{n \in \mathbb{N}, \theta \in D} \) uniformly conditionally \( L \)-mixing (UCLM) with respect to \( (H_n, H_{n}^+)_{n \in \mathbb{N}} \) if \( (X_n(\theta))_{n \in \mathbb{N}} \) is adapted to \( (H_n)_{n \in \mathbb{N}} \) for all \( \theta \in D; \) for all \( r \geq 1, \) it is \( L^r \)-bounded; and the sequences \( (M_n^r(X))_{n \in \mathbb{N}}, (\Gamma_n^r(X))_{n \in \mathbb{N}} \) are also \( L^r \)-bounded for all \( r \geq 1. \) In the case of stochastic processes (when \( D \) is a singleton) the terminology “conditionally \( L \)-mixing process” will be used.

**Example 2.2.** Let us consider, for example, a linear process \( X_n = X_n(\theta), \theta \in D, \) such that

\[ X_n := \sum_{k=0}^{\infty} a_k \varepsilon_{n-k}, \quad n \in \mathbb{Z}, \]

with singleton set \( D, \) scalars \( a_k, k \in \mathbb{Z} \) and some sequence \((\varepsilon_k)_{k \in \mathbb{Z}}\) of i.i.d. \( \mathbb{R} \)-valued random variables satisfying \( \|\varepsilon_0\|_p < \infty \) for all \( p \geq 1. \) Let \( H_n = \sigma(\varepsilon_j, j \leq n), \) \( H_n^+ = \sigma(\varepsilon_j, j > n) \) for \( n \in \mathbb{N}. \) If we further assume that \( \|a_k\| \leq c(1 + k)^{-\beta}, k \in \mathbb{N} \) for some \( c > 0, \beta > 3/2 \) then [2] Lemma 4.3] shows that \((X_n)_{n \in \mathbb{N}} \) is a conditionally \( L \)-mixing process with respect to \((H_n, H_n^+)_{n \in \mathbb{N}}. \) If \((X_n)_{n \in \mathbb{N}} \) is conditionally \( L \)-mixing with respect to \((H_n, H_n^+)_{n \in \mathbb{N}} \) then so is \((F(X_n))_{n \in \mathbb{N}} \) for a Lipschitz-continuous function \( F, \) see Remark 2.3 of [3].

Finally, we know from Remark 7.3 of [10] that a broad class of functionals of geometrically ergodic Markov chains have the \( L \)-mixing property. It is possible to show, along the same lines, the conditional \( L \)-mixing property of these functionals, too.

## 3 Assumptions and main results

Let us define

\[ \mathcal{G}_n := \sigma(\varepsilon_j, j \leq n), \quad \mathcal{G}_n^+ := \sigma(\varepsilon_j, j > n + 1), \quad \forall n \in \mathbb{N}, \]

where \((\varepsilon_n)_{n \in \mathbb{Z}}\) is the noise sequence generating \((X_n)_{n \in \mathbb{Z}}, \) see (3) above.

**Assumption 3.1.** The process \((X_n)_{n \in \mathbb{N}}\) is conditionally \( L \)-mixing with respect to \((\mathcal{G}_n, \mathcal{G}_n^+)_{n \in \mathbb{N}}. \) Assume also \( \|\theta_0\|_p < \infty \) for all \( p \geq 1. \)

**Assumption 3.2.** There exist constants \( L_1, L_2 > 0 \) such that for all \( \theta_1, \theta_2 \in \mathbb{R}^d, x_1, x_2 \in \mathbb{R}^m \)

\[ \|H(\theta_1, x_1) - H(\theta_2, x_2)\| \leq L_1\|\theta_1 - \theta_2\| + L_2\|x_1 - x_2\|, \]

Assumption 3.1 implies, in particular, that \( \|X_0\| \in L^r, \) for any \( r \geq 1, \) thus, under Assumptions 3.1 and 3.2

\[ h(\theta) := E[H(\theta, X_0)], \quad \theta \in \mathbb{R}^d \]

is indeed well-defined.

**Assumption 3.3.** There is a constant \( a > 0 \) such that for all \( \theta_1, \theta_2 \in \mathbb{R}^d \) and \( x \in \mathbb{R}^m, \)

\[ \langle \theta_1 - \theta_2, H(\theta_1, x) - H(\theta_2, x) \rangle \geq a\|\theta_1 - \theta_2\|^2. \]  (6)
Two important properties immediately follow from Assumptions 3.2 and 3.3.

**B1** The function $h$ is $L$-smooth: there exists a non-negative constant $L$ such that
\[
\|h(\theta_1) - h(\theta_2)\| \leq L_1\|\theta_1 - \theta_2\|, \quad \forall \theta_1, \theta_2 \in \mathbb{R}^d. \tag{7}
\]

**B2** The function $h$ is strongly convex: there exists a constant $a > 0$ such that
\[
\langle \theta_1 - \theta_2, h(\theta_1) - h(\theta_2) \rangle \geq a\|\theta_1 - \theta_2\|^2, \quad \forall \theta_1, \theta_2 \in \mathbb{R}^d, \tag{8}
\]
which implies due to Theorem 2.1.12 in [21],
\[
\langle \theta_1 - \theta_2, h(\theta_1) - h(\theta_2) \rangle \geq \bar{a}\|\theta_1 - \theta_2\|^2 + \frac{1}{a + L_1}\|h(\theta_1) - h(\theta_2)\|, \quad \forall \theta_1, \theta_2 \in \mathbb{R}^d,
\]
where $\bar{a} = \frac{aL_1}{a + L_1}$.

Our aim is to estimate $\|\theta_n^\lambda - \overline{\theta}_n\|_2$, uniformly in $n \in \mathbb{N}$. To begin with, we present an example where explicit calculations are possible.

**Example 3.4.** Let $H(\theta, x) := \theta + x$ and let $(X_n)_{n \in \mathbb{Z}}$ be a sequence of independent standard Gaussian random variables, independent of $(\xi_n)_{n \in \mathbb{N}}$. We observe that the function $H$ satisfies Assumptions 3.2 and 3.3. Take $\theta_0 := 0$. It is straightforward to check that
\[
\overline{\theta}_n^\lambda - \theta_n^\lambda = \sum_{j=0}^{n-1} (1 - \lambda)^j \lambda X_{n-j},
\]
which clearly has variance
\[
\sum_{j=0}^{n-1} (1 - \lambda)^j \lambda^2 = \frac{\lambda(1 - (1 - \lambda)^2n)}{2 - \lambda}.
\]
It follows that
\[
\sup_{n \in \mathbb{N}} \|\overline{\theta}_n^\lambda - \theta_n^\lambda\|_2 = \sqrt{\frac{\lambda}{2 - \lambda}}.
\]
This shows that the best estimate we may hope to get is of the order $\sqrt{\lambda}$. Our Theorem 3.5 below achieves this bound asymptotically as $p \to \infty$.

Let $\lambda' = \min \{1/(a + L_1), 1/(2a), a^{1/2p}/(8pL_1^2)\}$. Then, one obtains the main results as follows.

**Theorem 3.5.** Let Assumptions 3.1, 3.2, and 3.3 hold. For every even number $p \geq \max\{8, (a + 1)/a\}$ and $\lambda < \lambda'$, there exists $C^\circ(p) > 0$ (not depending on $d$) such that
\[
\|\theta_n^\lambda - \overline{\theta}_n^\lambda\|_2 \leq C^\circ(p)\lambda^{\frac{d}{2} - \frac{d}{p}}, \quad n \in \mathbb{N}. \tag{9}
\]

The proof of this theorem is postponed to Section 8.2.

**Theorem 3.6.** Let Assumptions 3.1, 3.2, and 3.3 hold and let $\lambda < 1/(a + L_1)$ (due to Lemma 6.3). Then there is a probability $\pi_\lambda$ such that
\[
W_2(\text{Law}(\overline{\theta}_n^\lambda), \pi_\lambda) \leq c_1 6^{-a\lambda n}, \quad n \in \mathbb{N},
\]
and
\[
W_2(\pi, \pi_\lambda) \leq c\sqrt{\lambda},
\]
where $c_1$ is given explicitly in Lemma 6.3 and $c$ is given explicitly in the proof.

The proof of Theorem 3.6 is provided in Section 5. The next corollary relates our findings in Theorems 3.5 and 3.6 to the problem of sampling from $\pi$. 

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Corollary 3.7. Let Assumptions 3.1, 3.2 and 3.3 hold. For each \( \kappa > 0 \), there exist constants \( c_1(\kappa), c_2(\kappa) > 0 \) such that, for each \( 0 < \epsilon \leq 1/2 \) one has

\[
W_2(\text{Law}(\theta^\lambda_n), \pi) \leq \epsilon
\]

whenever \( \lambda < \lambda' \) satisfies

\[
\lambda \leq c_1(\kappa)e^{2+\kappa} \quad \text{and} \quad n \geq \frac{c_2(\kappa)}{e^{2+\kappa}} \ln(1/\epsilon), \tag{10}
\]

where \( c_1(\kappa), c_2(\kappa) \) depend only on \( d, a, L_1 \) and \( L_2 \).

Proof. Denote by \( \tilde{C} = \max\{C^\alpha(p), c_1, c\} \). Take \( p \) large enough so that \( \kappa > 6/(p-3) \) and thus \( \chi := 3/p \leq \kappa/(\kappa+2) \) holds. Theorems 3.5 and 3.6 imply that

\[
W_2(\text{Law}(\theta^\lambda_n), \pi) \leq W_2(\text{Law}(\theta^\lambda_n), \text{Law}(\theta^\lambda_n)) + W_2(\text{Law}(\theta^\lambda_n), \pi_\lambda) + W_2(\pi_\lambda, \pi)
\]

\[
\leq \tilde{C}[\lambda^{1-\chi} + e^{-a\lambda n} + \lambda^{1/2}]
\]

\[
\leq \tilde{C}[\lambda^{1/2} + e^{-a\lambda n}].
\]

Choosing \( \lambda \leq e^{2+\kappa}/(2\tilde{C})^{2+\kappa} \), \( \tilde{C}\lambda^{1/2} \leq \epsilon/2 \) holds. Now it remains to choose \( n \) large enough to have \( \tilde{C}e^{-a\lambda n} \leq \epsilon/2 \) or, equivalently, \( a\lambda n \geq \ln(2\tilde{C}/\epsilon) \). Noting the choice of \( \lambda \) and \( \ln(1/\epsilon) \geq \ln 2 > 0 \) this is possible if

\[
n \geq \frac{\tilde{C}}{e^{2+\kappa}} \ln(1/\epsilon). \]

Assumption 3.8. The process \( (X_n)_{n \in \mathbb{N}} \) is i.i.d. with \( \|X_0\|_p \) and also \( \|\theta_0\|_p \) being finite for any \( p \geq 1 \).

Assumption 3.9. There exists a real-valued function \( \alpha : \mathbb{R}^m \to \mathbb{R} \) such that for all \( \theta_1, \theta_2 \in \mathbb{R}^d \) and \( x \in \mathbb{R}^m \),

\[
\langle \theta_1 - \theta_2, H(\theta_1, x) - H(\theta_2, x) \rangle \geq \alpha(x)\|\theta_1 - \theta_2\|^2,
\]

and \( E[\alpha(X_0)] < \infty \).

Corollary 3.10. Let Assumptions 3.2, 3.3 and 3.4 hold. For each \( \kappa > 0 \), there exist constants \( c_1(\kappa), c_2(\kappa) > 0 \) such that, for each \( 0 < \epsilon \leq 1/2 \), one obtains

\[
W_2(\text{Law}(\theta^\lambda_n), \pi) \leq \epsilon
\]

whenever \( \lambda < \lambda' \) satisfies

\[
\lambda \leq c_1(\kappa)e^{2+\kappa} \quad \text{and} \quad n \geq \frac{c_2(\kappa)}{e^{2+\kappa}} \ln(1/\epsilon), \tag{11}
\]

where \( c_1(\kappa), c_2(\kappa) \) depend only on \( d, E[\alpha(X_0)], L_1 \) and \( L_2 \).

Proof. One notes that (B1) is still valid and (B2) holds with \( \alpha = \alpha(X_0) \). Consequently, Theorems 3.5 and 3.6 are still true. Hence the desired result is obtained.

4 Related work and discussion

Rate of convergence. Corollary 3.7 significantly improves on some of the results in [23] in certain cases, compare also to [29]. In [23] the monotonicity assumption (40) is not imposed, only a dissipativity condition is required and a more general recursive scheme is investigated. However, the input sequence \((X_n)_{n \in \mathbb{N}}\) is assumed i.i.d. In that setting, Theorem 2.1 of [23] applies to (41) (with the choice \( \delta = 0, \beta = 1, d \) fixed, see also the last paragraph of Subsection 1.1 of [23]), and we get that

\[
W_2(\text{Law}(\theta^\lambda_n), \pi) \leq \epsilon
\]

holds whenever \( \lambda \leq c_3(\epsilon/\ln(1/\epsilon))^2 \) and \( n \geq \frac{c_4}{\epsilon^2} \ln^2(1/\epsilon) \) with some \( c_3, c_4 > 0 \). Our results provide the sharper estimates (11) in a setting where \((X_n)_{n \in \mathbb{N}}\) may have dependences. For the case of i.i.d. \((X_n)_{n \in \mathbb{N}}\) see also the very recent [17].
Choice of step size. It is pointed out in [23] that the ergodicity property of (2) is sensitive to the step size $\lambda$. Lemma 6.3 of [18] gives an example in which the Euler-Maruyama discretization is transient. As pointed out in [18], under discretization, the minorization condition is insensitive with appropriate sampling rate while the Lyapunov condition may be lost. An invariant measure exists if the two conditions hold simultaneously, see Theorem 7.3 of [18] and also Theorem 3.2 of [23] for similar discussions. In our paper we follow the footsteps of [23] in imposing strongly convexity of $U$ together with Lipschitzness of its gradient and thus we do obtain ergodicity of (2).

More on exponential rates. The convergence rate of the Euler-Maruyama scheme (2) to its stationary distribution is comparable to that of the Langevin SDE (1), see [16, Section 5.4.C, Theorem 4.20] or the discussion after Proposition 3.2 of [14]. Note also that the dimension is of order $n^2$ included in [23], however, it requires global convexity. Let us consider the same setting with strongly convex $V$ satisfies $\beta > 0$ and $\lambda > 0$, see Proposition 3.2 of [14]. By Theorem 2.1.8 of [21], $U$ has a unique minimum at some point $x^*$. Consider the infinitesimal generator $A$ associated with the SDE (1)

$$\lambda \geq -\lambda_n \exp(0(d)) \text{ in } [23] \text{ and } \exp(-\lambda_n \sqrt{d}) \text{ in our case, see Lemma 6.3.}$$

Horizon dependence. For the convergence of the Euler-Maruyama scheme (2) to the stationary distribution of the Langevin dynamics (1), [6] uses the total variation distance and Kullback-Leibler divergence to obtain a bound (up to constants in the exponent and in front of the expression) like $e^{-\lambda_n} + \sqrt{n}\lambda$, where $n$ is the time horizon. As explained in their Remark 1, this bound is not sharp and improving it is a challenging question. Theorem 5.6 or see also [9], provides

$$W_2(\delta_x R^A, \pi) \leq W_2(\delta_x R^A, \pi) + W_2(\pi, \pi) \leq \exp(-a\lambda_n + \sqrt{\lambda})$$

with some $\lambda > 0$, see Section 4 for the definition of the operator $R^A$. This provides a bound that is independent of $n$. Note also that the dependence on dimension is of order $\sqrt{d}$ as in the result of [6], see Lemma 6.3 below.

5 Analysis for the Langevin diffusion (1)

By Theorem 2.1.8 of [21], $U$ has a unique minimum at some point $x^* \in \mathbb{R}^d$. Note that due to the Lipschitz condition (B1), the SDE (1) has a unique strong solution. By [16, Section 5.4.C, Theorem 4.20], one constructs the associated strongly Markovian semigroup $(P_t)_{t \geq 0}$ given for all $t \geq 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ by $P_t(x, A) = P(\theta_t \in A | \theta_0 = x)$. Consider the infinitesimal generator $A$ associated with the SDE (1) defined for all $f \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ by

$$Af(x) = -\langle h(x), \nabla f(x) \rangle + \Delta f(x),$$

where $\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$ is the Laplacian.

Lemma 5.1. Let Assumptions (B1), (B2) hold. Consider the Lyapunov function $V : \mathbb{R}^d \rightarrow [0, \infty)$ defined by $V(x) = \|x - x^*\|^{2p}$, $p \geq 1$. Then the drift condition

$$AV(x) \leq -cV(x) + c\beta$$

is satisfied with $c = ap$ and $\beta = (2d + 4(p - 1))^{p} / a^p$. Moreover, one obtains

$$\sup_{t \geq 0} PV(x) \leq V(x) + \beta.$$  \hspace{1cm} (14)$$

Proof. For all $x \in \mathbb{R}^d$, by B2, we have

$$AV(x) = (-2p(x - x^*, h(x)) + 2pd + 4p(p - 1)) \|x - x^*\|^{2p - 2} \leq -2ap \|x - x^*\|^{2p} + (2pd + 4p(p - 1)) \|x - x^*\|^{2p - 2}.$$
Proof. of the SDE started at \( \theta \).

For all \( \|x - x^*\| \geq \sqrt{(2d + 4(p - 1))/a} \), one obtains

\[
\mathcal{A}V(x) \leq -apV(x)
\]

For all \( \|x - x^*\| \leq \sqrt{(2d + 4(p - 1))/a} \), \( \mathcal{A}V(x) + apV(x) \leq p(2d + 4(p - 1))\rho/a^{p-1} \). Finally, by Theorem 1.1 in [19], one obtains [14]. \( \square \)

Lemma 5.2. Let Assumptions 3.2 and 3.3 hold ((B1), (B2) are thereby implied).

(i) For all \( t \geq 0 \) and \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \|y - x^*\|^2 P_t(x, dy) \leq \|x - x^*\|^2 e^{-2at} + (d/a)(1 - e^{-2at}).
\]

(ii) The stationary distribution \( \pi \) satisfies

\[
\int_{\mathbb{R}^d} \|x - x^*\|^2 \pi(dx) \leq d/a.
\]

Proof. See Proposition 1 of [9]. \( \square \)

Lemma 5.3. Let Assumption 3.2 and 3.3 hold ((B1), (B2) are thereby implied). Let \( (\theta_t)_{t \geq 0} \) be the solution of the SDE started at \( x \in \mathbb{R}^d \). For all \( t \geq 0 \) and \( x \in \mathbb{R}^d \),

\[
E \left( \|\theta_t - x\|^2 | \theta_0 = x \right) \leq dt(2 + L_1^2t^2/3) + \frac{d}{2}t^2 L_1^2 \|x - x^*\|^2.
\]

Proof. See Lemma 19 of [9]. \( \square \)

Theorem 3.6, which gives the convergence rate of the Euler-Maruyama scheme (2) to the Langevin dynamics [1], is proven below. In order to have explicit constants, the argument in [9] is adopted in our model setting. The discrete time process \((\theta^\lambda_n)_{n \in \mathbb{N}}\) is lifted to a continuous time process \((\theta_t^\lambda)_{t \in \mathbb{R}^+}\) such that on a grid of size \( \lambda \) the distributions of the two processes coincide. To do so, let us define the grid \( t_n = n\lambda \) for each \( n \in \mathbb{N} \) and for \( t \in [t_n, t_{n+1}] \) set

\[
\begin{aligned}
\theta_t &= \theta_{t_n} - \int_{t_n}^t h(\theta_s)ds + \sqrt{\lambda}(B_s - B_{t_n}), \\
\theta_t^\lambda &= \theta_{t_n} - \int_{t_n}^t h(\theta_s^\lambda)ds + \sqrt{2\lambda}(B_s - B_{t_n}).
\end{aligned}
\]

Note that at the grid points \( \theta_{t_{n+1}} = \theta_{t_n} - \lambda h(\theta_{t_n}^\lambda) + \sqrt{2\lambda}\xi_{n+1} \) where \( \xi_{n+1} = (B_{t_{n+1}} - B_{t_n})/\sqrt{\lambda} \) is an i.i.d sequence of Gaussian random variables.

Proof of Theorem 3.6 Let \( \theta_0 \) have law \( \pi \) and \( \theta_0^\lambda := x \) for some fixed \( x \in \mathbb{R}^d \). Let \( \zeta_0 = \pi \otimes \delta_x \). Note that \( t_0 = 0 \) and \( (\theta_0, \theta_0^\lambda) \) are distributed according to \( \zeta_0 \). Fix \( n \geq 1 \). Then using (B2), we obtain

\[
\begin{align*}
\|\theta_{t_{n+1}} - \theta_{t_n}^\lambda\|^2 &= \|\theta_{t_n} - \theta_{t_n}^\lambda\|^2 + \left\| \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \right\|^2 - 2\langle\theta_{t_n} - \theta_{t_n}^\lambda, \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \rangle \\
&= \|\theta_{t_n} - \theta_{t_n}^\lambda\|^2 + \left\| \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \right\|^2 \\
&\quad - 2\lambda(\theta_{t_n} - \theta_{t_n}^\lambda, h(\theta_{t_n}) - h(\theta_{t_n}^\lambda)) - 2\langle\theta_{t_n} - \theta_{t_n}^\lambda, \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \rangle \\
&\leq \|\theta_{t_n} - \theta_{t_n}^\lambda\|^2 + \left\| \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \right\|^2 - 2\lambda\|\theta_{t_n} - \theta_{t_n}^\lambda\|^2 - \frac{2\lambda}{a + L_1}\|h(\theta_{t_n}) - h(\theta_{t_n}^\lambda)\|^2 \\
&\quad - 2\langle\theta_{t_n} - \theta_{t_n}^\lambda, \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_s^\lambda))ds \rangle.
\end{align*}
\]
Moreover,
\[
\left\| \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\overline{\theta}_{t_n}^\lambda)) \, ds \right\|^2 = \left\| \lambda (h(\theta_{t_n}) - h(\overline{\theta}_{t_n}^\lambda)) + \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_{t_n})) \, ds \right\|^2 \\
\leq 2 \left( \lambda^2 \left\| h(\theta_{t_n}) - h(\overline{\theta}_{t_n}^\lambda) \right\|^2 + \left\| \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_{t_n})) \, ds \right\|^2 \right) \\
\leq 2 \left( \lambda^2 \left\| h(\theta_{t_n}) - h(\overline{\theta}_{t_n}^\lambda) \right\|^2 + \lambda \int_{t_n}^{t_{n+1}} \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \, ds \right),
\]
where the last line follows from the Cauchy-Schwarz inequality. Thus, since \( \lambda < 1/(a + L_1) \), we have that
\[
\left\| \theta_{t_{n+1}} - \overline{\theta}_{t_{n+1}} \right\| \leq (1 - 2\tilde{a}\lambda) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + 2\lambda \int_{t_n}^{t_{n+1}} \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \, ds \\
- 2(\theta_{t_n} - \overline{\theta}_{t_n}^\lambda) \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_{t_n})) \, ds \\
\leq (1 - 2\tilde{a}\lambda) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + 2\lambda \int_{t_n}^{t_{n+1}} \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \, ds \\
+ 2(\theta_{t_n} - \overline{\theta}_{t_n}^\lambda) \int_{t_n}^{t_{n+1}} (h(\theta_s) - h(\theta_{t_n})) \, ds),
\]
Using \(|a, b| \leq \epsilon\|a\|^2 + (4\epsilon)^{-1}\|b\|^2\) then yields
\[
\left\| \theta_{t_{n+1}} - \overline{\theta}_{t_{n+1}}^\lambda \right\|^2 \leq (1 - 2\tilde{a}\lambda - \epsilon) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + (2\lambda + (2\epsilon)^{-1}) \int_{t_n}^{t_{n+1}} \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \, ds.
\]
Let \((\mathcal{F}_t)_{t \geq 0}\) denote the natural filtration of the Brownian motion \((B_t)_{t \geq 0}\). By \((B1)\) and Lemma \[5.3\] we have that
\[
E \left( \left\| \theta_{t_{n+1}} - \overline{\theta}_{t_{n+1}}^\lambda \right\|^2 \mid \mathcal{F}_{t_n} \right) \\
\leq (1 - 2\tilde{a}\lambda - \epsilon) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + (2\lambda + (2\epsilon)^{-1}) \int_{t_n}^{t_{n+1}} E \left( \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \right) \, ds \\
\leq (1 - 2\tilde{a}\lambda - \epsilon) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + (2\lambda + (2\epsilon)^{-1}) L_1^2 \int_{t_n}^{t_{n+1}} E \left( \left\| h(\theta_s) - h(\theta_{t_n}) \right\|^2 \right) \, ds \\
\leq (1 - 2\tilde{a}\lambda - \epsilon) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + (2\lambda + (2\epsilon)^{-1}) L_1^2 \int_{t_n}^{t_{n+1}} (2ds + \frac{3}{2\epsilon} L_2^2 \left\| \theta_{t_n} - x^* \right\|^2 + \frac{1}{\epsilon} dL_3^2 s^3) \, ds \\
\leq (1 - 2\tilde{a}\lambda - \epsilon) \left\| \theta_{t_n} - \overline{\theta}_{t_n}^\lambda \right\|^2 + (2\lambda + (2\epsilon)^{-1}) L_1^2 (d\lambda^2 + \frac{1}{2} \lambda^3 L_2^2 \left\| \theta_{t_n} - x^* \right\|^2 + \frac{1}{12} \lambda^4 dL_1^2) .
\]
Taking \(\epsilon = \tilde{a}/2\) and using induction one obtains
\[
E \left( \left\| \theta_{t_{n+1}} - \overline{\theta}_{t_{n+1}}^\lambda \right\|^2 \right) \leq (1 - \tilde{a})^{n+1} \int_{\mathbb{R}^d} \| y - x \|^2 \pi(dy) + \sum_{k=0}^{n} (2\lambda + \tilde{a}^{-1}) L_1^2 (d\lambda^2 + \frac{1}{12} \lambda^4 dL_1^2)(1 - \tilde{a})^k \\
+ \sum_{k=0}^{n} (2\lambda + \tilde{a}^{-1}) L_1^2 \frac{1}{2} \lambda^3 L_2^2 \overline{\theta}_k (1 - \tilde{a})^{n-k},
\]
where \(\overline{\theta}_k = e^{-\tilde{a} t_k} E(\|\theta_0 - x^*\|^2) + d/a(1 - e^{-\tilde{a} t_k}) \leq d/a, \) due to Lemma \[5.2\] (ii). Hence
\[
E \left( \left\| \theta_{t_{n+1}} - \overline{\theta}_{t_{n+1}}^\lambda \right\|^2 \right) \leq (1 - \tilde{a})^{n+1} (\| x - x^* \|^2 + d/a) + \lambda L_1^2 \tilde{a}^{-1}(2\lambda + \tilde{a}^{-1})(d + \frac{1}{12} \lambda^2 dL_1^2) \\
+ \frac{1}{2} \lambda^2 L_1^2 (2\lambda + \tilde{a}^{-1}) d/(a\tilde{a}) . \tag{16}\]

8
By Lemma 6.3 below, for all \( x \in \mathbb{R}^d \), \((\delta_x R^n_\lambda)_{n \geq 0}\) converges in Wasserstein distance to \( \pi_\lambda \) as \( n \to \infty \), we have that
\[
W_2(\pi, \pi_\lambda) = \lim_{n \to \infty} W_2(\pi, \delta_x R^n_\lambda) \leq c\sqrt{\lambda},
\]
where \( c = (L_1^2\lambda^{-1}(2\lambda + \ddot{a}^{-1})(d + \frac{1}{12}\lambda^2L_1^2d + \frac{1}{2}L_1^2\lambda d/a))^{1/2} \).

Note that for the Langevin SDE (11), the Euler and Milstein schemes coincide, which implies that the optimal rate of convergence is 1 instead of 1/2. The bound provided in Theorem 3.6 can thus be improved under an additional assumption, see Subsection 5.3.

6 Ergodic properties of the recursive scheme (2)

For a fixed step size \( \lambda \in (0, 1) \), consider the Markov kernel \( R_\lambda \) given for all \( A \in \mathcal{B}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \) by
\[
R_\lambda(\theta, A) = \int_A (4\pi\lambda)^{-d/2}\exp\left(\frac{-1}{4\lambda}(\|y - \theta + \lambda h(\theta)\|^2)\right)dy.
\]

The discrete-time Langevin recursion (2) is a time-homogeneous Markov chain, and for any \( n \geq 1 \), and for any bounded (or non-negative) Borel function \( f : \mathbb{R}^d \to \mathbb{R} \),
\[
E[f(\tilde{\theta}^n_\lambda)] = R_\lambda f(\tilde{\theta}^n_{n-1}) = \int_{\mathbb{R}^d} f(y)R_\lambda(\tilde{\theta}^n_{n-1}, dy).
\]

We say that a function \( V : \mathbb{R}^d \to [1, \infty) \) satisfies a Foster-Lyapunov drift condition for \( R_\lambda \) if there exist constants \( \lambda > 0 \), \( \alpha \in [0, 1) \), \( c > 0 \) such that for all \( \lambda \in (0, \lambda] \),
\[
R_\lambda V \leq \alpha^\lambda V + \lambda c.
\]

The following lemma shows that (18) holds for a suitable polynomial Lyapunov function.

**Lemma 6.1.** Under Assumptions 3.1, 3.2 and 3.3, for each integer \( p \geq 1 \), the function \( V(x) := \|x\|^{2p} \) satisfies the Foster-Lyapunov drift condition for the Markov kernel \( R_\lambda \).

**Proof.** This is almost identical to the proof of Lemma 7.1 below, hence omitted.

As a first consequence of Lemma 6.1, we have

**Lemma 6.2.** Under Assumptions 3.1, 3.2 and 3.3,
\[
\sup_n EV(\tilde{\theta}^n_\lambda) < \infty.
\]

**Proof.** Using the fact that the function \( V \) satisfies the Foster-Lyapunov drift condition for \( R_\lambda \), see Lemma 6.1 we compute
\[
EV(\tilde{\theta}^n_\lambda) = E_E[V(\tilde{\theta}^n_\lambda)|\tilde{\theta}^n_{n-1}] = E[R_\lambda V(\tilde{\theta}^n_{n-1})] \leq \alpha^\lambda E[V(\tilde{\theta}^n_{n-1})] + \lambda c \leq \lambda c \leq \lambda c...
\]
\[
\leq \alpha^n V(\tilde{\theta}^n_0) + (a^{(n-1)} + \ldots + \alpha + 1)\lambda c \leq \alpha^n V(\tilde{\theta}^n_0) + (1 - a^n)(1 - \alpha)^{-1}\lambda c.
\]

Noting that \( \sup_{0 < \lambda < 1} \frac{\lambda}{1 - \alpha} < \infty \), we have \( \sup_{0 < \lambda < 1} \sup_n EV(\tilde{\theta}^n_\lambda) < \infty \).

In the following, contraction property of the Markov chain \( (\tilde{\theta}^n_{n})_{n \in \mathbb{N}} \) is derived. Together with Lemma 6.1 \( R_\lambda \) admits a unique stationary distribution \( \pi_\lambda \), which may differ from \( \pi \). It is also proved that the law of the Markov chain with kernel \( R_\lambda \) converges to its stationary distribution \( \pi_\lambda \) exponentially fast in the Wasserstein-2 distance.

**Lemma 6.3.** Let Assumption 3.3 hold ((B2) is thereby implied). Fix \( \lambda \in (0, 2/(a + L_1)) \). Then
(i) For all $x \in \mathbb{R}^d$, $n \geq 1$,
\[
\int_{\mathbb{R}^d} \|y - x^*\|^2 R_n^a(y, dx) \leq (1 - 2\bar{a})^n \|x - x^*\|^2 + (d/\bar{a})(1 - (1 - 2\bar{a})^n).
\]

(ii) There exists a unique stationary distribution $\pi_n$ on the Borel sets of $\mathbb{R}^d$ such that
\[
\int_{\mathbb{R}^d} \|x - x^*\|^2 \pi_n(dx) \leq d/\bar{a}.
\]

(iii) For all $x \in \mathbb{R}^d$, $n \geq 1$,
\[
W_2(\delta_x R_n^a, \pi_n) \leq e^{-\bar{a}n}\sqrt{2}\|x - x^*\|^2 + d/\bar{a}^{1/2}.
\]

Proof. See Proposition 2, 3 of \cite{[9]}.

\section{Analysis for the scheme (4)}

The following inequalities, derived from Assumptions 3.2 and 3.3, are often used:
\[
\|H(\theta, x)\| \leq L_1\|\theta\| + L_2\|x\| + \|H(0, 0)\|, \quad \langle \theta, H(\theta, x) \rangle \geq a\|\theta\|^2 + \langle \theta, H(0, x) \rangle.
\]

The process in (2) is Markovian while the one in (4) is not. Nonetheless we can prove the following.

\textbf{Lemma 7.1.} Let Assumptions 3.2, 3.3 and 3.4 hold. Let $V(x) = \|x\|^2$ for some integer $p \geq (a+1)/a$. The process $\theta^\lambda$ satisfies, for $\rho \in (0, 1)$, $\lambda < \min \{1/(2a), a^{1/2p}/(8pL_1^2)\}$
\[
EV(\theta^\lambda_n) \leq \rho EV(\theta^\lambda_{n-1}) + C, \quad n \geq 1,
\]
where $C = C_0 + 2^{2p-3}(pL_1)^{2p}EV(X_0) + \|H(0, 0)\|^2 + 2^{2p-1}(dpL_1(2p - 1))^p$, and $C_0$ is given explicitly in the proof. As a result, $\sup_n \bar{E}(\theta^\lambda_n) < \infty$.

Proof. Define $\hat{B}_t^\lambda := B_t/\sqrt{\lambda}$ for $t \geq 0$, $\lambda > 0$. Notice that $(\hat{B}_t^\lambda)_{t \geq 0}$ is a Brownian motion, and $\bar{F}_t^\lambda := \bar{F}_t^\lambda$, $t \geq 0$ is the natural filtration of $(\hat{B}_t^\lambda)_{t \geq 0}$. For each $\lambda > 0$, and for each $x = (x_0, x_1, \ldots) \in (\mathbb{R}^m)^{\mathbb{N}}$, consider the process, for all $t \geq 0$
\[
dY_t^\lambda(x) = -\lambda H(Y_t^\lambda, x_{t+1}) dt + \sqrt{2\lambda}d\hat{B}_t^\lambda.
\]
with $Y_0^\lambda(x) = \theta_0$. Let $\Theta_t^\lambda := Y_t^\lambda(X)$, $t \geq 0$, where $X = (X_t)_{t \in \mathbb{N}}$ is a random element in $(\mathbb{R}^m)^{\mathbb{N}}$. One notes that $\text{Law}(\Theta_t^\lambda) = \text{Law}(\theta_t^\lambda)$, then $EV(\theta_t^\lambda) = EV(Y_t^\lambda(X))$. The shorthand notation $Y_t = Y_t^\lambda(X)$ is used in the proof. Consider the Lyapunov function $V(\theta) = \|\theta\|^{2p}$ for all $\theta \in \mathbb{R}^d$, $p \geq 1$. To calculate $EV(\theta_t^\lambda) = EV(Y_t^\lambda)$, we first show that $EV(Y_t^\lambda) < \rho EV(\theta_0) + C$ for $\rho \in (0, 1)$ and for some constant $C > 0$, then by induction, one can obtain $\sup_n EV(Y_n^\lambda) < \infty$. Consider $t \in [0, 1)$, applying Itô’s formula to $V(Y_t)$ yields, almost surely
\[
dV(Y_t) = \lambda(\Delta V(Y_t) - \langle H(\theta_0, X_1), \nabla V(Y_t) \rangle) dt + \sqrt{2\lambda}(\nabla V(Y_t), d\hat{B}_t^\lambda).
\]
Then, since the stochastic integral is a martingale (see Appendix [5.4] for a detailed proof), one obtains,
\[
EV(Y_t) = EV(\theta_0) + \lambda \int_0^t E(\Delta V(Y_s) - \langle H(\theta_0, X_1), \nabla V(Y_s) \rangle) ds.
\]
Differentiating on both sides yields
\[
\frac{d}{dt} EV(Y_t) = \lambda E(\Delta V(Y_t) - \langle H(Y_t, X_1), \nabla V(Y_t) \rangle) + \lambda E(\langle H(Y_t, X_1), \nabla V(Y_t) \rangle - \langle H(\theta_0, X_1), \nabla V(Y_t) \rangle),
\]
(21)
where the second inequality holds by Young’s inequality with $2p$. Denote by $\| \cdot \|_A(Y_i)$

Substituting (23) into (21) yields

Then, one calculates, for a set $C$

where the first inequality holds due to Assumption 3.3, while the second inequality holds by Assumption 3.1. Denote by $M' = \max\{8\|H(0, 0)\|/a, \sqrt{8d + 16(p - 1)/a}, (8L_2/a)^2, (8L_2E\|X_0\|^{4p - 1}/a)^{1/2}\}$, one observes that, on the set $\{\|Y_i\| \geq M'\}$ the following inequalities hold

and this implies by taking into consideration equation (22)

where $\bar{\eta} = ap$. As for the set $\{\|Y_i\| < M'\}$, similarly, one obtains

where $C_{\bar{\eta}} = 2p\|H(0, 0)\|(M')^{2p-2} + 2pL_2(M')^{2p-\frac{3}{2}} + 2pL_2E\|X_0\|^{4p-1} + (2pd + 4p(p - 1))(M')^{2p-2}$. Then, combining the two cases yields

Substituting (23) into (21) yields

where the second inequality holds by Young’s inequality with $2p$ and $\frac{2p}{2p-1}$, whereas the last inequality holds due to Assumption 3.2 and note that $EV(X_0) < \infty$ by Assumption 3.1. By using $1 - e^{-x} \leq x$ for all $x \geq 0$ and Assumption 3.2 one obtains

where $C = C_{\bar{\eta}} + 2^{8p-3}(pL_1^{2p}E\|X_0\|^{2p}) + 2^{5p-1}(dp^2L_2^2(2p - 1))^p$, and this implies by continuity

Let $p > (a + 1)/a$, then by Taylor expansion,

$$EV(Y_1) \leq \left(1 - a\lambda + \frac{a^2}{2}\lambda^2 + 2^{6p-2}(pL_1^2)^{2p}\lambda^{2p+1}\right)EV(\theta_0) + C.$$
Set $\lambda^2a^2/2 < a\lambda/4$ and $\lambda^2\rho^{p-2}(pL^2_1)^{2p}\lambda^2p^{p+1} < a\lambda/4$, one obtains the restrictions for $\lambda$:

$$\lambda < \min\left\{1/(2a), a^{1/2p}/(8pL^2_1)\right\},$$

which yields

$$EV(Y_1) \leq \left(1 - \frac{a}{2}\right) EV(\theta_0) + C.$$ 

Denote by $\rho = 1 - a\lambda/2$, and one notes that with the restriction (24), $\rho \in (0,1)$ is satisfied. Finally, by induction, one obtains, for each $n \in N$

$$EV(Y_n) \leq \rho^nEV(\theta_0) + \frac{C}{1-\rho} < \infty,$$

which implies that $\sup_n EV(Y_n) < \infty$.

Uniform $L^2$ bounds for the process in (4) are obtained in [23] under dissipativity condition on $\nabla U$ and the $L^2$ error of the stochastic gradient, i.e. $E\|H(\theta,X_n) - h(\theta)\|^2$, see their Assumptions (A.3), (A.4). In that paper a large size mini-batch could be used to reduce the variance of the estimator, which requires more computational costs. We could also incorporate mini-batches in our algorithm but this is not pursued here. For stability, the variance of the estimator has to be controlled, see [20].

8 Appendix

8.1 Some results for conditional L-mixing processes

The following maximal inequality is pivotal for our arguments. Define

$$\mathcal{H}_n := G_n \cup \sigma(\xi_j, \ j \in \mathbb{N}), \quad \mathcal{H}_n^+ := G^+_n, \ n \in \mathbb{N}.$$

**Theorem 8.1.** Fix $n \in \mathbb{N}$. Let $(W_k)_{k \in \mathbb{N}}$ be a conditionally L-mixing process with respect to $(\mathcal{H}_k, \mathcal{H}_k^+), k \in \mathbb{N}$, satisfying $E[W_k|\mathcal{H}_n] = 0$ a.s. for all $k \geq n$. Let $m > n$ and let $b_k, n < k \leq m$ be deterministic numbers. Then we have, for each $r > 2$,

$$E^{1/r}\left[\sup_{n < k \leq m} \left| \sum_{j=n+1}^{k} b_j W_j \right|^r \mathbb{I}(\mathcal{H}_n) \right] \leq C_r \left( \sum_{s=n+1}^{m} b_s^2 \right)^{1/2} \sqrt{M^r_n(W)\Gamma^r_n(W)},$$

almost surely, where $C_r$ is a deterministic constant depending only on $r$ but independent of $n,m$ and $d$.

**Proof.** In Theorem 2.6 of [3] this is proved in the case where $\mathcal{H}_k = \sigma(\xi_j, \ j \leq k), k \in \mathbb{N}$ where $(\xi_j)_{j \in \mathbb{Z}}$ is an i.i.d. sequence in a Polish space. With trivial modifications this can be extended to the case where $\mathcal{H}_k = \sigma(Z, \xi_j, \ j \leq k)$ where $Z$ is a (Polish space-valued) random variable that is independent of $(\xi_j)_{j \in \mathbb{Z}}$. In the present setting we can thus choose $Z := (\xi_j)_{j \in \mathbb{N}}$.

**Lemma 8.2.** Let $X_n, n \in \mathbb{N}$ be a conditionally L-mixing process. Let Assumption 3.2 hold true. Then, for each $j \in \mathbb{N}$, the random field $H(\theta, X_n), n \in \mathbb{N}, \theta \in B(j)$ is uniformly conditionally L-mixing.

**Proof.** Let $\theta \in B(j)$. Noting that $r > 1$, the first inequality of (19) and Minkowski’s inequality imply for $k \geq n$,

$$E^{1/r}[\|H(\theta, X_k)\|^{r}|\mathcal{H}_n] \leq L_1 j + L_2 E^{1/r}[X_k\|\mathcal{H}_n] + \|H(0, 0)\|$$

and hence

$$M^r_n(H(\theta, X), B(j)) \leq L \left\{M^r_n(X) + j + L^{-1}\|H(0, 0)\| \right\},$$

where $L = \max\{L_1, L_2\}$. We also have, using Lemma 8.3 that

$$E^{1/r}[\|H(\theta, X_k) - E[H(\theta, X_k)|\mathcal{H}_n \cup \mathcal{H}_n^+]\|^{r}|\mathcal{H}_n] \leq 2E^{1/r}[\|H(\theta, X_k) - H(\theta, E[X_k|\mathcal{H}_n \cup \mathcal{H}_n^+])\|^{r}|\mathcal{H}_n]$$

$$\leq 2L_2 E^{1/r}[\|X_k - E[X_k|\mathcal{H}_n \cup \mathcal{H}_n^+]\|^{r}|\mathcal{H}_n]$$

which implies $\Gamma^r_n(H(\theta, X), B(j)) \leq 2L_2 \Gamma^r_n(X)$. 

\[\square\]
Lemma 8.3. Let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be sigma-algebras. Let $X, Y$ be random variables in $L^p$ such that $Y$ is measurable with respect to $\mathcal{H} \vee \mathcal{G}$. Then for any $p \geq 1$,

$$E^{1/p} \left[ |X - E[X|\mathcal{H} \vee \mathcal{G}]|^p |\mathcal{G} \right] \leq 2E^{1/p} \left[ |X - Y|^p |\mathcal{G} \right].$$

Proof. See Lemma 6.1 of [5]. □

8.2 Proof of Theorem 3.5

Clearly, since $(X_n)_{n \in \mathbb{N}}$ is conditionally $L$-mixing with respect to $(\mathcal{G}_n, \mathcal{G}_n^+)_{n \in \mathbb{N}}$, it remains conditionally $L$-mixing with respect to $(\mathcal{H}_n, \mathcal{H}_n^+)_{n \in \mathbb{N}}$, too.

For each $\theta \in \mathbb{R}^d$, $0 \leq i \leq j$, we recursively define

$$z^\lambda(j, i, \theta) := \theta, \quad z^\lambda(j + 1, i, \theta) := z^\lambda(j, i, \theta) - \lambda h(z(j, i, \theta)) + \sqrt{2\lambda}\xi_{j+1}.$$ 

Let $T := [1/\lambda]$. We then set, for each $n \in \mathbb{N}$ and for each $nT \leq k < (n + 1)T$, $\bar{z}_k^\lambda := z^\lambda(k, nT, \bar{\theta}_k^\lambda)$. Note that $\bar{\theta}_k^\lambda$ is then defined for all $k \in \mathbb{N}$ and $\bar{\theta}_k^\lambda = z^\lambda(k, 0, \theta_0)$.

Lemma 8.4. There is $C^b > 0$ such that

$$\sup_{\lambda < 1} \sup_{n \in \mathbb{N}} \left( \|H(\theta_n^\lambda, X_{n+1})\|_2 + \|h(\bar{z}_n^\lambda)\|_2 \right) \leq C^b.$$

Proof. The first inequality of (19) implies

$$E \left[ \|H(\theta_n^\lambda, X_{n+1})\|^2 \right] \leq 2L_2^2E\|\theta_n^\lambda\|^2 + 2E\|H(0, X_n)\|^2 \leq 2L_2^2E\|\theta_n^\lambda\|^2 + 4\|H(0, 0)\|^2 + 4L_2^2E\|X_n\|^2.$$ 

Combining this inequality with Lemma 7.1 shows that $\sup_{\lambda < 1} \sup_{n} \|H(\theta_n^\lambda, X_{n+1})\|_2$ is finite. A similar argument can be applied to $h(\bar{z}_n^\lambda)$ since $h$ and $H$ are functions with the same linear upper bound. □

Lemma 8.5. Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of random variables such that for some $p > 1$

$$M = \sup_{i \in \mathbb{N}} E\|X_i\|^p < \infty.$$

Then for $0 < r < p$,

$$E^{p/2r} \left( \sup_{1 \leq i \leq j} \|X_i\|^r \right) \leq j^{r/p}M^{r/p}.$$ 

Proof. One has

$$E^{p/2r} \left( \sup_{1 \leq i \leq j} \|X_i\|^r \right) \leq E \left( \sup_{1 \leq i \leq j} \|X_i\|^p \right) \leq E \left( \sum_{i=1}^j \|X_i\|^p \right) \leq jM,$$

by Jensen’s inequality. □

Lemma 8.6. Let $k > n$. There exists a version $h_{k,nT}: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ of $E[H(\theta, X_t)|\mathcal{H}_{nT}]$, $\theta \in \mathbb{R}^d$ which is jointly measurable.

Proof. For a fixed $\theta \in \mathbb{R}^d$, the following conditional expectation

$$E[H(\theta, X_k)|\mathcal{H}_{nT}], \quad \theta \in \mathbb{R}^d.$$ 

is a $\mathcal{H}_{nT}$-measurable random variable. However, we actually need a function

$$h_{k,nT}: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$$ 

that is a.s. continuous in its second variable and, for all $\theta \in \mathbb{R}^d$, $h_{k,nT}$ is a version of $E[H(\theta, X_k)|\mathcal{F}_{nT}]$. (26)
It is enough to construct \( h_{t,nT}(\theta), \theta \in \mathcal{B}(N) \) for each \( N \in \mathbb{N} \). Consider \( \mathcal{B}(N) := C(\mathcal{B}(N); \mathbb{R}^d) \), the usual Banach space of continuous, \( \mathbb{R}^d \)-valued functions defined on \( \mathcal{B}(N) \), equipped with the maximum norm. The function

\[
\omega \in \Omega \rightarrow G_N(\omega) := (H(\theta, X_{[t]}(\omega))_{\theta \in \mathcal{B}(N)}, \omega \in \Omega,
\]

is a \( \mathcal{B}(N) \)-valued random variable and

\[
\sup_{\theta \in \mathcal{B}(N)} |H(\theta, X_{[t]})| \leq K[N + |X_{[t]}|] + |H(0,0)|,
\]

which clearly has finite expectation as the process \( X_n, n \in \mathbb{N} \) is conditionally \( L \)-mixing. Proposition V.2.5 of [22] implies the existence of a \( \mathcal{B}(N) \)-valued random variable \( \mathcal{G}_N \) such that, for each \( b \) in the dual space \( \mathcal{B}'(N) \) of \( \mathcal{B}(N) \),

\[
E[b(G_N)|\mathcal{H}_{nT}] = b(\mathcal{G}_N).
\]

This implies, in particular, that for all \( \theta \in \mathcal{B}(N) \),

\[
E[H(\theta, X_{[t]})|\mathcal{H}_{nT}] = \mathcal{G}_N(\theta).
\]

We may thus set \( h_{t,nT} := \mathcal{G}_N(\omega) \). Since \( (\omega, \theta) \rightarrow \mathcal{G}_N(\omega, \theta) \) is measurable in its first variable and continuous in the second, it is, in particular, jointly measurable, see e.g. Lemma 4.50 of [4]. \( \square \)

**Proof of Theorem 3.7** Fix \( n \in \mathbb{N} \) and let \( nT \leq k < (n+1)T \) be arbitrary. By the triangle inequality, we decompose the estimation into two parts

\[
|\theta_k^\lambda - \bar{\theta}_k^\lambda| \leq |\theta_k^\lambda - \bar{\theta}_k^\lambda| + |\bar{\theta}_k^\lambda - \bar{\theta}_k^\lambda|.
\]

Let \( h_{k,nT} \) be the jointly measurable version as in Lemma 8.6. We estimate

\[
|\theta_k^\lambda - \bar{\theta}_k^\lambda| \leq \lambda \sum_{i=nT+1}^k (H(\theta_i^\lambda, X_i) - h(h_i^\lambda)) \leq \lambda \sum_{i=nT+1}^k (H(\theta_i^\lambda, X_i) - H(\lambda, X_i)) + \lambda \sum_{i=nT+1}^k h_i,nT(\bar{\theta}_i^\lambda) - h(h_i^\lambda)) \leq \lambda L_1 \sum_{i=nT+1}^k |\theta_i^\lambda - \bar{\theta}_i^\lambda| + \lambda \max_{nT+1 \leq m < (n+1)T} \left\| \sum_{i=nT+1}^m (H(\theta_i^\lambda, X_i) - h_i,nT(\bar{\theta}_i^\lambda)) \right\| + \lambda \sum_{i=nT+1}^\infty |h_i,nT(\bar{\theta}_i^\lambda) - h(h_i^\lambda)|,
\]

by Assumption 3.2. A discrete-time version of Gronwall’s lemma and taking squares lead to

\[
|\theta_k^\lambda - \bar{\theta}_k^\lambda|^2 \leq 2\lambda^2 e^{2L_1T} \lambda \left[ \max_{nT+1 \leq m < (n+1)T} \left\| \sum_{i=nT+1}^m (H(\theta_i^\lambda, X_i) - h_i,nT(\bar{\theta}_i^\lambda)) \right\| \right]^2 + \left( \sum_{i=nT+1}^\infty |h_i,nT(\bar{\theta}_i^\lambda) - h(h_i^\lambda)| \right)^2,
\]

noting also \((x + y)^2 \leq 2(x^2 + y^2), x, y \in \mathbb{R}\). Let us define the \( \mathcal{H}_{nT} \)-measurable random variable

\[
N := \max_{nT+1 \leq k < (n+1)T} |\bar{\theta}_k^\lambda|.
\]

Now, recalling the definition of \( T \) and taking \( \mathcal{H}_{nT} \)-conditional expectations, we can write

\[
E \left[ |\theta_k^\lambda - \bar{\theta}_k^\lambda|^2 | \mathcal{H}_{nT} \right] \leq 2\lambda^2 e^{2L_1} \sum_{j=1}^\infty \mathbb{I}_{(j-1 \leq N < j)} E \left[ \max_{nT+1 \leq m < (n+1)T} \left\| \sum_{i=nT+1}^m (H(\theta_i^\lambda, X_i) - h_i,nT(\bar{\theta}_i^\lambda)) \right\| | \mathcal{H}_{nT} \right] + 2\lambda^2 e^{2L_1} E \left[ \bar{\theta}_k^\lambda^2 | \mathcal{H}_{nT} \right],
\]

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where
\[ \Xi_n := \left( \sum_{i=nT+1}^{\infty} \| h_{i,nT}(\xi^i_k) - h(\xi^i_k) \| \right)^2 |H_nT| . \]

Using the $H_{nT}$-measurability of $\xi^k$, $nT \leq k < (n+1)T$, Theorem 8.1 and taking expectations, we can continue our estimations as
\[ E\|\theta_k^\lambda - \bar{\xi}_k^\lambda\|^2 \leq 2C_2\lambda^2 e^{2L_1}\sum_{j=1}^{\infty} E\{ \mathbb{1}_{(j-1\leq N<j)} T \Gamma^n_{2}(Z^\lambda(j), B(j)) M^n_{2}(Z^\lambda(j), B(j)) \} + 2\lambda^2 e^{2L_1} E [\Xi_n^2] , \]

where $C_2$ is a constant. The second moment of $\Xi_n^2$ is finite, see Lemma 8.7. We remark that, for each $j \in \mathbb{N}$, the process defined by
\[ Z^\lambda_k(j) := (H(\xi^\lambda_k, X_k) - h_{k,nT}(\xi^k_k)) \mathbb{1}_{\{ |\xi^k_k| \leq j \}}, \quad nT \leq k < (n+1)T, \quad n \in \mathbb{N} \tag{27} \]

satisfies
\[ \Gamma_{pT}^{nT}(Z^\lambda(j), B(j)) \leq 2L_2 \Gamma_{pT}^{nT}(X), \quad M_{pT}^{nT}(Z^\lambda(j), B(j)) \leq 2L[M_{pT}^{nT}(X) + j + L^{-1}|H(0,0)|], \tag{28} \]

by Lemma 6.3 and Remark 6.4 of [5] and by Lemma 8.2 above.

Let $p' > 3$ be an arbitrary integer and set $p := 2p'$. By the Hölder inequality, the trivial $\{ j-1 \leq N < j \} \subset \{ j \leq N + 1 \}$, the Markov inequality, we estimate
\[
\sum_{j=1}^{\infty} E\{ \mathbb{1}_{(j-1\leq N<j)} \Gamma^n_{2}(Z^\lambda(j), B(j)) M^n_{2}(Z^\lambda(j), B(j)) \} \\
\leq \sum_{j=1}^{\infty} P_{1/2}(N + 1 \geq j) E_{1/2}[\Gamma^n_{2}(Z^\lambda(j), B(j)) M^n_{2}(Z^\lambda(j), B(j))]^2 ] \\
\leq \sum_{j=1}^{\infty} \sqrt{\frac{E(N + 1)^6}{j^6}4L_2L E_{1/4}[\Gamma^n_{2}(X))]^4 M^n_{2}(X) + j + L^{-1}|H(0,0)|]^3] \\
\leq C' \sum_{j=1}^{\infty} jT^{3/p} j^{3} \leq C'T^{3/p} ,
\]

for suitable $\tilde{C}, \tilde{C}' > 0$, using (28), Lemma 8.3 and the fact that $X$ was assumed to be conditionally $L$-mixing. We conclude that
\[ E^{1/2}\|\theta_k^\lambda - \bar{\xi}_k^\lambda\|^2 \leq C^5|\lambda\sqrt{T}T^{3/(2p)} + \lambda| \leq C^* \lambda^{1/2} - \frac{1}{\sqrt{p}} , \]

with some $C^*, C^* > 0$, for all $k \in \mathbb{N}$. From these estimations, the constant $C^*$ is of order $e^{2L_1}L_2L$ and independent from $d$.

Now we turn to estimating $\|\bar{\xi}_k^\lambda - \bar{\theta}_k^\lambda\|$ for $nT \leq k \leq (n+1)T$. We compute
\[
\|\bar{\xi}_k^\lambda - \bar{\theta}_k^\lambda\| = \sum_{i=1}^{n} \| z^\lambda(k, iT, \theta^\lambda_{iT}) - z^\lambda(k, (i-1)T, \theta^\lambda_{(i-1)T}) \| \\
= \sum_{i=1}^{n} \| z^\lambda(k, iT, \theta^\lambda_{iT}) - z^\lambda(k, iT, z^\lambda(iT, (i-1)T, \theta^\lambda_{(i-1)T})) \|. 
\]

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By Lemma 6.3, we estimate
\[
\|z^\lambda(k, iT, \theta_{iT}^\lambda) - z^\lambda(k, iT, z^\lambda(iT, (i-1)T, \theta_{(i-1)T}^\lambda))\|_2
\]
\[
\leq (1 - 2\alpha \lambda)\|z^\lambda(k-1, iT, \theta_{iT}^\lambda) - z^\lambda(k-1, iT, z^\lambda(iT, (i-1)T, \theta_{(i-1)T}^\lambda))\|_2
\]
\[
\leq (1 - 2\alpha \lambda)^k\|\theta_{iT}^\lambda - z^\lambda(iT, (i-1)T, \theta_{(i-1)T}^\lambda)\|_2
\]
\[
\leq (1 - 2\alpha \lambda)^k\|\theta_{iT}^\lambda - H(\theta_{iT}^\lambda, X_{iT}) - \sum_{i=1}^{\infty}\lambda H(\theta_{iT}^\lambda, X_{iT}) + h(\sum_{i=1}^{\infty}h(\theta_{iT}^\lambda, X_{iT}))\|_2
\]
\[
\leq (1 - 2\alpha \lambda)^k\|\theta_{iT}^\lambda - \sum_{i=1}^{\infty}\lambda H(\theta_{iT}^\lambda, X_{iT}) + h(\sum_{i=1}^{\infty}h(\theta_{iT}^\lambda, X_{iT}))\|_2
\]

Using Lemma 8.4, the estimation continues as follows
\[
\|z^\lambda_n - \bar{\theta}_k\|_2 \leq \sum_{i=1}^{n} e^{-2\alpha \lambda(k-iT)} \|\theta_{iT}^\lambda - \sum_{i=1}^{\infty}\lambda H(\theta_{iT}^\lambda, X_{iT}) + h(\sum_{i=1}^{\infty}h(\theta_{iT}^\lambda, X_{iT}))\|_2
\]
\[
\leq \frac{C^\uparrow}{1 - e^{-2\alpha}} \left[ 1 + C^\uparrow \right] \lambda^\frac{\lambda^\uparrow}{\lambda^\uparrow}
\]
for some $C^\uparrow > 0$. The proof is complete.

\begin{lemma}
There exist random variables $\Xi_n$, $n \in \mathbb{N}$ such that, for all $\theta \in \mathbb{R}^d$,
\[
\sum_{k=nT+1}^{\infty} \|h_{k,nT}(\theta) - h(\theta)\| \leq \Xi_n,
\]
and $\sup_{n \in \mathbb{N}} E[\Xi_n^2] < \infty$.
\end{lemma}

\begin{proof}
Notice that, since $E[X_k|\mathcal{H}_{nT}^+]$ is independent of $\mathcal{H}_{nT}$,
\[
E[H(\theta, E[X_k|\mathcal{F}_{nT}^+])|\mathcal{F}_{nT}] = E[H(\theta, E[X_k|\mathcal{F}_{nT}^+])], \quad \forall k \geq nT + 1.
\]
This implies that
\[
\|h_{k,nT}(\theta) - h(\theta)\|
\]
\[
\leq \left[ \|E[H(\theta, X_k)|\mathcal{H}_{nT}] - E[H(\theta, E[X_k|\mathcal{H}_{nT}])|\mathcal{H}_{nT}]\| + \|E[H(\theta, E[X_k|\mathcal{H}_{nT}^+]) - E[H(\theta, X_k)]\| + \|E[H(\theta, X_k|\mathcal{H}_{nT}^+)}|\mathcal{H}_{nT}] + E[H(\theta, E[X_k|\mathcal{H}_{nT}^+])|\mathcal{H}_{nT}]\|
\]
\[
\leq L_2 E[|X_k - E[X_k|\mathcal{H}_{nT}^+]|]|\mathcal{H}_{nT}| + L_2 E[|X_k - E[X_k|\mathcal{H}_{nT}^+]|]|\mathcal{H}_{nT}|
\]
\[
\leq L_2 \|\gamma_1^{nT}(X, k - nT) + E\gamma_1^{nT}(X, k - nT)\|.
\]
Hence
\[
\sum_{k=nT+1}^{\infty} \|h_{k,nT}(\theta) - h(\theta)\| \leq L_2 \|\Gamma_1^{nT}(X) + E\Gamma_1^{nT}(X)\|.
\]
Since $X$ is conditionally $L$-mixing, $\sup_{n \in \mathbb{N}} E[(\Gamma_1^{nT}(X))^2]$ is finite. This implies the statement.
\end{proof}

\section{An improved convergence rate}

\begin{assumption}
$U$ is twice continuously differentiable. There exists $L_3$ such that for all $\theta_1, \theta_2 \in \mathbb{R}^d$
\[
\|\nabla^2 U(\theta_1) - \nabla^2 U(\theta_2)\| \leq L_3 \|\theta_1 - \theta_2\|,
\]
where the matrix norm is defined as the Hilbert-Schmidt norm.
\end{assumption}

\begin{lemma}
Let Assumption 8.8 hold. For any $\theta_1, \theta_2 \in \mathbb{R}^d$,
\[
\|h(\theta_1) - h(\theta_2) - (\nabla^2 U(\theta_2))^T (\theta_1 - \theta_2)\| \leq L_3 \|\theta_1 - \theta_2\|^2.
\]
\end{lemma}
Proof. For any \( \theta_1, \theta_2 \in \mathbb{R}^d \), there exists \( t \in [0, 1] \) such that
\[
\begin{align*}
\|h(\theta_1) - h(\theta_2) - (\nabla^2 U(\theta_2)) (\theta_1 - \theta_2)\| \\
= \| (\nabla^2 U(t\theta_1 + (1-t)\theta_2)) (\theta_1 - \theta_2) - (\nabla^2 U(\theta_2)) (\theta_1 - \theta_2)\| \\
\leq L_3 \|\theta_1 - \theta_2\|^2,
\end{align*}
\]
by the mean value theorem and Assumption 8.8.

Lemma 8.10. Let Assumptions 5.2 and 5.3 hold, \((B1), (B2)\) are thereby implied.

(i) For all \( t \geq 0 \) and \( x \in \mathbb{R}^d \),
\[
\int_{\mathbb{R}^d} \|y - x\|^4 P_t(x, \mathrm{d}y) \leq e^{-4\alpha t}\|x - x^*\|^4 + (3d^2 / \alpha^2) (1 - e^{-4\alpha t}) + (6d / \alpha)e^{-2\alpha t} (1 - e^{-2\alpha t}) (\|x - x^*\|^2 - d / \alpha).
\]
(ii) The stationary distribution \( \pi \) satisfies
\[
\int_{\mathbb{R}^d} \|x - x^*\|^4 \pi(\mathrm{d}x) \leq 3d^2 / \alpha^2.
\]

Proof. (i) Denote by \( x \) the starting point of the process \((\theta_t)_{t \geq 0}\). By applying Itô’s formula to \( \|\theta_t - x^*\|^4 \), one obtains
\[
E(\|\theta_t - x^*\|^4 | \theta_0 = x) = \|x - x^*\|^4 + (4d + 8)E \left( \int_0^t \|\theta_s - x^*\|^2 \mathrm{d}s \middle| \theta_0 = x \right) - 4E \left( \int_0^t \langle h(\theta_s), (\theta_s - x^*)\rangle | \theta_0 = x \right),
\]
which by using \( h(x^*) = 0 \), \((B2)\) and Lemma 5.2(i) implies
\[
\frac{d}{dt} E(\|\theta_t - x^*\|^4 | \theta_0 = x) = -4E \left( \langle h(\theta_t) - h(x^*), (\theta_t - x^*)\rangle | \theta_0 = x \right) + (4d + 8)E \left( \|\theta_t - x^*\|^2 | \theta_0 = x \right)
\leq -4\alpha E(\|\theta_t - x^*\|^4 | \theta_0 = x) + 12d(e^{-2\alpha t}\|x - x^*\|^2 + d / \alpha(1 - e^{-2\alpha t}))
\]
By using Grönwall’s lemma we have
\[
E(\|\theta_t - x^*\|^4 | \theta_0 = x) \leq e^{-4\alpha t}\|x - x^*\|^4 + (3d^2 / \alpha^2) (1 - e^{-4\alpha t}) + (6d / \alpha)e^{-2\alpha t} (1 - e^{-2\alpha t}) (\|x - x^*\|^2 - d / \alpha).
\]
(ii) The proof follows the same lines as the proof of Lemma 5.2(ii).

Theorem 8.11. Let Assumptions 5.2 and 5.3 hold, \((B1), (B2)\) are thereby implied, and Assumption 8.8 hold. Let \( \lambda \in (0, 1 / (a + L_1)) \), then
\[
W_2(\pi, \pi_\lambda) \leq \tilde{c} \lambda,
\]
where \( \tilde{c} = (dL_1^2 \left( 2 / \alpha + L_1^2 \lambda^2 / (6\alpha) + 64dL_2^2 / (\alpha^2L_1^2) + \lambda L_1^2 / \alpha^2 + 24L_2^2L_1^2d / (\alpha^2L_1^2) + L_1^2 / (\alpha^2L_1^2) \right)^{1/2} \).

Proof. Let \( x \in \mathbb{R}^d \), \( n \geq 1 \) and \( \zeta_0 = \pi \otimes \delta_x \). \((\theta_0, \bar{\theta}_0)\) is distributed according to \( \zeta_0 = \pi \otimes \delta_x \). Define \( e_{in} = \theta_{in} - \bar{\theta}_{in} \). By (15), we have
\[
\|e_{in+1}\|^2 = \|e_{in}\|^2 + \int_{t_n}^{t_{n+1}} \left( h(\theta_s) - h(\bar{\theta}_{in}) \right) \mathrm{d}s - 2\lambda \left( e_{in}, h(\theta_s) - h(\bar{\theta}_{in}) \right) - 2 \int_{t_n}^{t_{n+1}} \langle e_{in}, h(\theta_s) - h(\theta_{in}) \rangle \mathrm{d}s,
\]
which yields, due to Young’s inequality, \((B2)\) and Cauchy-Schwarz inequality
\[
\|e_{in+1}\|^2 = \|e_{in}\|^2 + 2\lambda^2 \|h(\theta_{in}) - h(\bar{\theta}_{in})\|^2 + 2\lambda \int_{t_n}^{t_{n+1}} \|h(\theta_s) - h(\theta_{in})\|^2 \mathrm{d}s
\]
\[
- 2\lambda \|e_{in}\|^2 - \frac{2\lambda}{\alpha + L_1} \|h(\theta_{in}) - h(\bar{\theta}_{in})\|^2 - 2 \int_{t_n}^{t_{n+1}} \langle e_{in}, h(\theta_s) - h(\theta_{in}) \rangle \mathrm{d}s.
\]
Since \( \lambda < 1/(a + L_1) \), by using \( |\langle a, b \rangle| \leq \varepsilon \|a\|^2 + (4\varepsilon)^{-1} \|b\|^2 \), one obtains

\[
\|e_{t_n+1}\|^2 \leq (1 - 2\tilde{\alpha}\lambda)\|e_{t_n}\|^2 + 2\lambda \int_{t_n}^{t_{n+1}} \|h(\theta_s) - h(\theta_{t_n})\|^2 \, ds \\
- 2\int_{t_n}^{t_{n+1}} \left\langle e_{t_n}, h(\theta_s) - h(\theta_{t_n}) - (\nabla h(\theta_{t_n}))^T (\theta_s - \theta_{t_n}) \right\rangle \, ds \\
- 2\int_{t_n}^{t_{n+1}} \left\langle e_{t_n}, (\nabla h(\theta_{t_n}))^T (\theta_s - \theta_{t_n}) \right\rangle \, ds.
\]

Note that for any \( \theta \in \mathbb{R}^d \), (B1) implies \( \|\nabla h(\theta)y\| \leq L_1 \|y\| \), for any \( y \in \mathbb{R}^d \), then using Young’s inequality and Lemma 8.9 yield

\[
E\left[\|e_{t_n+1}\|^2 | \mathcal{F}_{t_n}\right] \leq (1 - 2(\tilde{\alpha} - \varepsilon)\lambda)\|e_{t_n}\|^2 + 2\lambda \int_{t_n}^{t_{n+1}} E\left[\|h(\theta_s) - h(\theta_{t_n})\|^2 | \mathcal{F}_{t_n}\right] \, ds \\
+ (\varepsilon)^{-1} \int_{t_n}^{t_{n+1}} E\left[\left\|h(\theta_s) - h(\theta_{t_n}) - (\nabla h(\theta_{t_n}))^T (\theta_s - \theta_{t_n})\right\|^2 | \mathcal{F}_{t_n}\right] \, ds \\
+ (\varepsilon)^{-1} \int_{t_n}^{t_{n+1}} E\left[\left\|\nabla h(\theta_{t_n})\right\|^2 \left(\int_{t_n}^s -h(\theta_r) \, dr \right) \right\|^2 | \mathcal{F}_{t_n}\right] \, ds \\
\leq (1 - 2(\tilde{\alpha} - \varepsilon)\lambda)\|e_{t_n}\|^2 + 2\lambda L_3^2 \int_{t_n}^{t_{n+1}} E\left[\|\theta_s - \theta_{t_n}\|^2 | \mathcal{F}_{t_n}\right] \, ds \\
+ (\varepsilon)^{-1} L_3^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^s E\left[\|\theta_r - \theta_{t_n}\|^4 | \mathcal{F}_{t_n}\right] \, dr \, ds \\
+ (\varepsilon)^{-1} \lambda L_3^4 \int_{t_n}^{t_{n+1}} \int_{t_n}^s E\left[\|h(\theta_r)\|^4 | \mathcal{F}_{t_n}\right] \, dr \, ds.
\]

By using (B1) and Young’s inequality, one obtains

\[
E\left[\|\theta_s - \theta_{t_n}\|^4 | \mathcal{F}_{t_n}\right] = E\left[\left\|\int_{t_n}^s h(\theta_r) \, dr + \sqrt{2}\int_{t_n}^s dw_r \right\|^4 | \mathcal{F}_{t_n}\right] \leq 8\lambda^3 \int_{t_n}^s E\left[\|h(\theta_r)\|^4 | \mathcal{F}_{t_n}\right] \, dr + 96d^2(s - t_n)^2.
\]

Then, by Lemma 5.3, Lemma 8.10 \( \nabla h(x^*) = 0 \) and (B1), we have

\[
E\left[\|e_{t_n+1}\|^2 | \mathcal{F}_{t_n}\right] \leq (1 - 2(\tilde{\alpha} - \varepsilon)\lambda)\|e_{t_n}\|^2 + 2\lambda^3 d L_1^2 + d L_1^4 \lambda^5 / 6 + \lambda^3 L_1^2 \|	heta_{t_n} - x^*\|^2 \\
+ 8\lambda^3 (\varepsilon)^{-1} L_3^2 L_4^4 \int_{t_n}^{t_{n+1}} \int_{t_n}^s E\left[\|\theta_r - x^*\|^4 | \mathcal{F}_{t_n}\right] \, dr \, ds + 32d^2 \lambda^3 (\varepsilon)^{-1} L_3^2 \\
+ (\varepsilon)^{-1} \lambda L_3^4 \int_{t_n}^{t_{n+1}} \int_{t_n}^s E\left[\|\theta_r - x^*\|^2 | \mathcal{F}_{t_n}\right] \, dr \, ds.
\]

Finally, taking \( \varepsilon = \tilde{\alpha}/2 \), and by induction and Lemma 8.10, one obtains

\[
E\|e_{t_n+1}\|^2 \leq (1 - \tilde{\alpha}\lambda)^{n+1} E\|e_{t_n}\|^2 + \lambda^2 d L_1^2 \left(2/\tilde{\alpha} + L_1^2 \lambda^2 / (6\tilde{\alpha}) + 64d L_3^2 / (\tilde{\alpha}^2 L_1^2)\right) \\
+ \lambda^3 L_1^4 d / \tilde{\alpha}^2 + \left(24\lambda^3 L_3^2 L_1^4 d / (\tilde{\alpha}^2) + \lambda^2 L_1^4 d / (\tilde{\alpha}^2)\right).
\]

As \( n \to \infty \), \( \lim_{n \to \infty} W_2(\delta_x R^n_\lambda, \pi) = W_2(\pi_\lambda, \pi) \), which implies

\[
W_2(\pi_\lambda, \pi) \leq c \lambda,
\]

where \( c = (d L_1^2 (2/\tilde{\alpha} + L_1^2 \lambda^2 / (6\tilde{\alpha}) + 64d L_3^2 / (\tilde{\alpha}^2 L_1^2) + \lambda L_1^2 / \tilde{\alpha}^2 + 24\lambda^2 L_3^2 L_1^4 d / (\tilde{\alpha}^2) + L_1^4 / (\tilde{\alpha}^2))^{1/2} \).
8.4 An auxiliary proof for Lemma 7.1

To show that the stochastic integral \( \int_0^t \sqrt{2X} \langle \nabla V(Y_s), d\tilde{B}_s \rangle \) is a martingale, it is enough to show, for any \( t > 0, p \geq 1 \)

\[
2\lambda E \int_0^t \| \nabla V(Y_s) \|^2 \, ds \leq 2\lambda E \int_0^t (\| Y_s \|^2 + 1)^{2p} \, ds < \infty.
\]

Consider the Lyapunov function \( \bar{V}_p(\theta) = (\| \theta \|^2 + 1)^{2p} \) for all \( \theta \in \mathbb{R}^d, p \geq 1 \). Define the stopping time \( \tau_n = \inf\{ t > 0 : \| Y_t \|^2 > n \} \). In this proof, we write \( Y_t = Y_t^X(X) \). For \( t \in [0,1) \), applying Itô’s formula to \( \bar{V}_p(Y_t^X) \) yields

\[
E\bar{V}_p(Y_{t\wedge \tau_n}) = E\bar{V}_p(\theta_0) + E \int_0^t 1_{\{s \leq \tau_n\}} \lambda (\bar{V}_p(Y_s) - \langle H(Y_s, X_{[s]+1}), \nabla \bar{V}_p(Y_s) \rangle) \, ds
+ E \int_0^t 1_{\{s \leq \tau_n\}} \lambda (H(Y_s, X_{[s]+1}) - H(Y_{[s]}, X_{[s]+1}), \nabla \bar{V}_p(Y_s)) \, ds.
\]

(29)

The second term above can be further estimated as

\[
E \left[ 1_{\{s \leq \tau_n\}} (\bar{V}_p(Y_s) - \langle H(Y_s, X_{[s]+1}), \nabla \bar{V}_p(Y_s) \rangle) \right]
= E \left[ 1_{\{s \leq \tau_n\}} \left((4pd + 8p(2p - 1))(\| Y_s \|^2 + 1)^{2p-1} - 4p(\| Y_s \|^2 + 1)^{2p-1} \langle Y_s, H(Y_s, X_{[s]+1}) \rangle)\right) \right]
\leq C E \left[ 1_{\{s \leq \tau_n\}} \bar{V}_p(Y_s) \right] + C.
\]

(30)

Substituting (30) into (29) yields

\[
E\bar{V}_p(Y_{t\wedge \tau_n}) \leq E\bar{V}_p(\theta_0) + \lambda C \int_0^t E(\bar{V}_p(Y_{s\wedge \tau_n}) + 1) \, ds
+ E \int_0^t 1_{\{s \leq \tau_n\}} \| H(Y_s, X_{[s]+1}) - H(Y_{[s]}, X_{[s]+1}) \|^2 \, ds + \lambda \frac{4p}{4p-1} \int_0^t E\bar{V}_p(Y_{s\wedge \tau_n}) \, ds
\leq C_1 \int_0^t E\bar{V}_p(Y_{s\wedge \tau_n}) \, ds + C_2 \int_0^t E\bar{V}_p(Y_{s\wedge \tau_n}) \, ds + C_3,
\]

(31)

where the first inequality holds by using Young’s inequality with \( 4p \) and \( \frac{4p}{4p-1} \), and the second inequality holds due to Assumption 3.2. Note that \( C_1, C_2, C_3 \) are independent of \( t \wedge \tau_n \), but \( C_3 \) depends on \( \theta_0 \) and \( E\| X_0 \|^p \), for some \( p \geq 1 \); in addition, for all \( p \geq 1 \), \( E\| X_0 \|^p < \infty \) by Assumption 3.1. One observes \( \sup_{0 \leq s < t} E\bar{V}_p(Y_{s\wedge \tau_n}) \) is dominated by the RHS of (31), then,

\[
\sup_{0 \leq s < t} E\bar{V}_p(Y_{s\wedge \tau_n}) < C_4 \int_0^t \sup_{0 \leq s < t} E\bar{V}_p(Y_{s\wedge \tau_n}) \, ds + C_3 < \infty.
\]

The application of Gronwall’s lemma yields

\[
\sup_{0 \leq s < t} E\bar{V}_p(Y_{s\wedge \tau_n}) \leq C_3 e^{C_4 t},
\]

which implies by Fatou’s lemma

\[
\sup_{0 \leq s < t} E\bar{V}_p(Y_t) \leq C_3 e^{C_4 t}.
\]

References


