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FORMALITY AND SPLITTING OF REAL NON-ABELIAN MIXED HODGE STRUCTURES

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ABSTRACT. We define and construct mixed Hodge structures on real schematic homotopy types of complex projective varieties, giving mixed Hodge structures on their homotopy groups and pro-algebraic fundamental groups. We also show that these split on tensoring with the ring \( \mathbb{R}[x] \) equipped with the Hodge filtration given by powers of \((x - i)\), giving new results even for simply connected varieties. Thus the mixed Hodge structures can be recovered from cohomology groups of local systems, together with the monodromy action at the Archimedean place. As the basepoint varies, these all become real variations of mixed Hodge structure.

INTRODUCTION

The main aims of this paper are to construct mixed Hodge structures on the real schematic homotopy types of complex varieties, and to investigate how far these can be recovered from the structures on cohomology groups of local systems.

In [Mor], Morgan established the existence of natural mixed Hodge structures on the minimal model of the rational homotopy type of a smooth variety \( X \), and used this to define natural mixed Hodge structures on the rational homotopy groups \( \pi_\ast(X \otimes \mathbb{Q}) \) of \( X \). This construction was extended to singular varieties by Hain in [Hai2].

When \( X \) is also projective, [DGMS] showed that its rational homotopy type is formal; in particular, this means that the rational homotopy groups can be recovered from the cohomology ring \( H^\ast(X, \mathbb{Q}) \). However, in [CCM], examples were given to show that the mixed Hodge structure on homotopy groups could not be recovered from that on integral cohomology. We will describe how formality interacts with the mixed Hodge structure, showing the extent to which the mixed Hodge structure on \( \pi_\ast(X \otimes \mathbb{R}, x_0) \) can be recovered from the pure Hodge structure on \( H^\ast(X, \mathbb{R}) \).

This problem was suggested to the author by Carlos Simpson, who asked what happens when we vary the formality quasi-isomorphism. [DGMS] proved formality by using the \( d \bar{d} \) Lemma (giving real quasi-isomorphisms), while most subsequent work has used the \( \partial \bar{\partial} \) Lemma (giving Hodge-filtered quasi-isomorphisms). The answer (Corollary 2.12) is that, if we define the ring \( S := \mathbb{R}[x] \) to be pure of weight 0, with the Hodge filtration on \( S \otimes \mathbb{R} \mathbb{C} \) given by powers of \((x - i)\), then there is an \( S \)-linear isomorphism

\[
\pi_\ast(X \otimes \mathbb{R}, x_0) \otimes \mathbb{R} S \cong \pi_\ast(H^\ast(X, \mathbb{R}) \otimes \mathbb{R} S,
\]

preserving the Hodge and weight filtrations, where the homotopy groups \( \pi_\ast(H^\ast(X, \mathbb{R})) \) are given the Hodge structure coming from the Hodge structure on the cohomology ring \( H^\ast(X, \mathbb{R}) \), regarded as a real homotopy type.

This is proved by replacing \( d^c \) with \( d^c + xd \) in the proof of [DGMS], so \( x \in S \) is the parameter for varying formality quasi-isomorphisms. In several respects, \( S \otimes \mathbb{R} \mathbb{C} \) behaves like Fontaine’s ring \( B_{st} \) of semi-stable periods, and the MHS can be recovered from a pro-nilpotent operator on the real homotopy type \( H^\ast(X, \mathbb{R}) \), which we regard as monodromy at the Archimedean place. The isomorphism above says that the MHS on

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Section 2 contains most of the results related to real homotopy types. Corollary 2.12 constructs a non-abelian mixed Hodge structure on the real homotopy type. Moreover, there is an $S$-equivariant morphism $\text{row}_1: \text{SL}_2 \rightarrow C^*$ corresponding to projection of the first row; all of the structures split on pulling back along $\text{row}_1$, and these pullbacks can be recovered from cohomology of local systems. This is because the principle of two types (or the $dd^c$-lemma) holds for any pair $ud + ve^c, xd + yd^c$ of operators, provided $(u, v) \in \text{GL}_2$. The pullback $\text{row}_1$ corresponds to tensoring with the algebra $S$ described above. Proposition 2.18 shows how this pullback to $\text{SL}_2$ can be regarded as an analogue of
the limit mixed Hodge structure, while Proposition 2.13, Corollary 2.21 and Proposition 2.14 show how it is closely related to real Deligne cohomology, Consani’s Archimedean cohomology and Deninger’s $\Gamma$-factor of $X$ at the Archimedean place.

Section 3 is mostly a review of the relative Malcev homotopy types introduced in [Pri3], generalising both schematic and real homotopy types, with some new results in §3.3 on homotopy types over general bases (rather than just fields). Major new results are Theorem 3.10 and Corollary 3.11, which show how relative homotopy types arise naturally in the study of fibrations. Theorem 3.28 adapts the main comparison result of [Pri3] to the case of fixed basepoints.

In Section 4, the constructions of Section 1 are then extended to homotopy types. The main result is Theorem 4.20, showing how non-abelian algebraic mixed Hodge and twistor structures on relative Malcev homotopy types give rise to such structures on homotopy groups, while Proposition 5.6 shows that these are compatible with Morgan’s mixed Hodge structures on rational homotopy types and groups.

In the next two sections, we establish the existence of algebraic mixed Hodge structures on various relative Malcev homotopy types of compact Kähler manifolds, giving more information than rational homotopy types when $X$ is not nilpotent (Corollaries 5.16 and 6.12). The starting point is the Hodge structure defined on the reductive complex pro-algebraic fundamental group $\varpi_1(X, x_0)_{\text{red}}$ in [Sim3], in the form of a discrete $\mathbb{C}^*$-action. We only make use of the induced action of $U_1 \subset \mathbb{C}^*$, since this preserves the real form $\varpi_1(X, x_0)_{\text{red}}$, respects the harmonic metric, and has has the important property that the map
\[ \pi_1(X, x_0) \times U_1 \to \varpi_1(X, x_0)_{\text{red}} \]
is real analytic. We regard this as a kind of pure weight 0 Hodge structure on $\varpi_1(X, x_0)_{\text{red}}$, since a pure weight 0 Hodge structure is the same as an algebraic $U_1$-action. We extend this to a mixed Hodge structure on the schematic (or relative Malcev) homotopy type (Theorem 5.14 and Proposition 6.3).

In some contexts, the unitary action is incompatible with the homotopy type. In these cases, we instead only have mixed twistor structures (as defined in [Sim2]) on the homotopy type (Theorem 6.1) and homotopy groups (Corollary 6.2).

Section 7 shows how representations of $\varpi_1(X, x_0)$ in the category of mixed Hodge structures correspond to variations of mixed Hodge structure (VMHS) on $X$ (Theorem 7.6). This implies (Corollary 7.7) that the relative Malcev homotopy groups become VMHS as the basepoint varies. Taking the case of $\pi_1$, this proves [Ara] Conjecture 5.5 (see Remarks 5.18 and 7.9 for details).

Section 8 is dedicated to describing the mixed Hodge structure on homotopy types in terms of a pro-nilpotent derivation on the split Hodge structure over $\text{SL}_2$. It provides an explicit description of this derivation in terms of standard operators on the complex of $\mathcal{C}^\infty$ forms on $X$, and in particular shows that the real Hodge structure on $\pi_3(X) \otimes \mathbb{R}$ is split whenever $X$ is simply connected (Examples 8.15.2).

In Section 9, we extend the results of Sections 5 and 6 to simplicial compact Kähler manifolds, and hence to singular proper complex varieties.

I would like to thank Carlos Simpson for drawing my attention to the questions addressed in this paper, and for much useful discussion. I would also like to thank Jack Morava for suggesting that non-abelian mixed Hodge structures should be related to Archimedean $\Gamma$-factors.
1. Non-abelian structures

1.1. Hodge filtrations. In this section, we will define algebraic Hodge filtrations on real affine schemes. This construction is essentially that of [Sim1] §5, with the difference that we are working over \( \mathbb{R} \) rather than \( \mathbb{C} \).

**Definition 1.1.** Define \( C \) to be the real affine scheme \( \prod_{\mathbb{C}/\mathbb{R}} \mathbb{A}^1 \) obtained from \( \mathbb{A}^1_\mathbb{C} \) by restriction of scalars, so for any real algebra \( A \), \( C(A) = A_{\mathbb{C}}(\mathbb{A} \otimes_{\mathbb{R}} \mathbb{C}) \cong A \otimes_{\mathbb{R}} \mathbb{C} \). Choosing \( i \in \mathbb{C} \) gives an isomorphism \( C \cong \mathbb{A}^2_\mathbb{R} \), and we let \( C^* \) be the quasi-affine scheme \( C - \{0\} \).

Define \( S \) to be the real algebraic group \( \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \) obtained as in [Del1] 2.1.2 from \( \mathbb{G}_m, \mathbb{C} \) by restriction of scalars. Note that there is a canonical inclusion \( \mathbb{G}_m \to S \), and that \( S \)
acts on $C$ and $C^*$ by inverse multiplication, i.e.
$$S \times C \to C$$
$$(\lambda, w) \mapsto (\lambda^{-1}w).$$

**Remark 1.2.** A more standard $S$-action is given by the inclusion $S \hookrightarrow \mathbb{A}^2 \cong C$. However, we wish $C$ to be of weight $-1$ rather than $+1$.

Fix an isomorphism $C \cong \mathbb{A}^2$, with co-ordinates $u, v$ on $C$ so that the isomorphism $C(\mathbb{R}) \cong \mathbb{C}$ is given by $(u, v) \mapsto u + iv$. Thus the algebra $O(C)$ associated to $C$ is the polynomial ring $C = \mathbb{R}[u, v]$. $S$ is isomorphic to the scheme $\mathbb{A}^2_{\mathbb{R}} - \{(u, v) : u^2 + v^2 = 0\}$.

**Definition 1.3.** Given an affine scheme $X$ over $\mathbb{R}$, we define an algebraic Hodge filtration $X_\mathcal{F}$ on $X$ to consist of the following data:
1. an $S$-equivariant affine morphism $X_\mathcal{F} \to C^*$,
2. an isomorphism $X \cong X_{\mathcal{F}, 1} := X_\mathcal{F} \times_{C^*, 1} \text{Spec} \mathbb{R}$.

**Definition 1.4.** A real splitting of the Hodge filtration $X_\mathcal{F}$ consists of an $S$-action on $X$, and an $S$-equivariant isomorphism
$$X \times C^* \cong X_\mathcal{F}$$
over $C^*$.

**Remark 1.5.** Note that giving $X_\mathcal{F}$ as above is equivalent to giving the affine morphism
$$[X_\mathcal{F}/S] \to [C^*/S]$$
of stacks. This fits in with the idea in [KPS] that if $\mathcal{O}_B$ is an $\infty$-stack parametrising some $\infty$-groupoid of objects, then the groupoid of non-abelian filtrations of this object is $\text{Hom}([A^1/\mathcal{G}_m], \mathcal{O}_B)$.

Now, we may regard a quasi-coherent sheaf $\mathcal{F}$ on a stack $\mathcal{X}$ as equivalent to the affine cogroup $\text{Spec}(\mathcal{O}_X \oplus \mathcal{F})$ over $\mathcal{X}$. This gives us a notion of an algebraic Hodge filtration on a real vector space. We now show how this is equivalent to the standard definition.

**Lemma 1.6.** There is an equivalence of categories between flat quasi-coherent $\mathcal{G}_m$-equivariant sheaves on $\mathbb{A}^1$, and exhaustive filtered vector spaces, where $\mathcal{G}_m$ acts on $\mathbb{A}^1$ via the standard embedding $\mathcal{G}_m \hookrightarrow \mathbb{A}^1$.

**Proof.** Let $t$ be the co-ordinate on $\mathbb{A}^1$, and $M$ global sections of a $\mathcal{G}_m$-equivariant sheaf on $\mathbb{A}^1$. Since $M$ is flat, $0 \to M \xrightarrow{t} M \xrightarrow{} M \otimes_{k[t], 0} k \to 0$ is exact, so $t$ is an injective endomorphism. The $\mathcal{G}_m$-action is equivalent to giving a decomposition $M = \bigoplus M_n$, and we have $t : M_n \to M_{n+1}$. Thus the images of $\{M_n\}_{n \in \mathbb{Z}}$ give a filtration on $M \otimes_{k[t], 1} k$.

Conversely, set $M$ to be the Rees module $\xi(V, F) := \bigoplus F_n V$, with $\mathcal{G}_m$-action given by setting $F_n V$ to be weight $n$, and the $k[t]$-module structure determined by letting $t$ be the inclusion $F_n V \hookrightarrow F_{n+1} V$. If $I$ is a $k[t]$-ideal, then $I = (f)$, since $k[t]$ is a principal ideal domain. The map $M \otimes I \to M$ is thus isomorphic to $f : M \to M$. Writing $f = \sum a_n t^n$, we see that it is injective on $M = \bigoplus M_n$. Thus $M \otimes I \to M$ is injective, so $M$ is flat by [Mat] Theorem 7.7. □

**Remark 1.7.** We might also ask what happens if we relax the condition that the filtration be flat, since non-flat structures might sometimes arise as quotients.

An arbitrary algebraic filtration on a real vector space $V$ is a system $W_r$ of complex vector spaces with (not necessarily injective) linear maps $s : W_r \to W_{r+1}$, such that $\lim_{r \to \infty} W_r \cong V$.

**Corollary 1.8.** The category of flat algebraic Hodge filtrations on real vector spaces is equivalent to the category of pairs $(V, F)$, where $V$ is a real vector space and $F$ an exhaustive decreasing filtration on $V \otimes_{\mathbb{R}} \mathbb{C}$. A real splitting of the Hodge filtration is equivalent to giving a real Hodge structure on $V$ (i.e. an $S$-action).
Proof. The flat algebraic Hodge filtration on $V$ gives an $S$-module $\xi(V, F)$ on $C^*$, with $\xi(V, F)|_1 = V$. Observe that $C^* \otimes \mathbb{C} \cong \mathbb{A}^2_\mathbb{C} - \{0\}$, and $S \otimes \mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$, compatible with the usual actions, the isomorphisms given by $(u, v) \mapsto (u + iv, u - iv)$. Writing $\mathbb{A}^2_\mathbb{C} - \{0\} = (\mathbb{A}^1 \times \mathbb{G}_m) \cup (\mathbb{G}_m \times \mathbb{A}^1)$, we see that giving $\xi(V, F) \otimes \mathbb{C}$ amounts to giving two filtrations $(F, F')$ on $V \otimes _\mathbb{R} \mathbb{C}$, which is the fibre over $(1, 1)$ in the new co-ordinates. The real structure determines behaviour under complex conjugation, with $F' = \overline{F}$. If we set $M \subset \xi(V \otimes C; F, \overline{F})$ to be the real elements, then $\xi(V, F) = J^{-1}M$. \hfill \Box

Remark 1.9. Although flat quasi-coherent sheaves on $[C^*/S]$ also correspond to flat quasi-coherent sheaves on $[C/S]$, we do not follow [Kap] in working over the latter, since many natural non-flat objects arise on $[C/S]$ whose behaviour over $0 \in C$ is pathological. However, our approach has the disadvantage that we cannot simply describe the bigraded vector space $\text{gr}_F \text{gr}_F V$, which would otherwise be given by pulling back along $[0/S] \rightarrow [C/S]$.

The motivating example comes from the embedding $\mathcal{H}^* \rightarrow \mathbb{A}^*$ of real harmonic forms into the real de Rham algebra of a compact Kähler manifold. This gives a quasi-isomorphism of the associated complexes on $[C^*/S]$, since the maps $F^p(\mathcal{H}^* \otimes \mathbb{C}) \rightarrow F^p(\mathbb{A}^* \otimes \mathbb{C})$ are quasi-isomorphisms. However, the associated map on $[C/S]$ is not a quasi-isomorphism, as this would force the derived pullbacks to $0 \in C$ to be quasi-isomorphic, implying that the maps $\mathcal{H}^{pq} \rightarrow \mathbb{A}^{pq}$ be isomorphisms.

Remark 1.10. We might also ask what happens if we relax the condition that the Hodge filtration be flat.

An arbitrary algebraic Hodge filtration on a real vector space $V$ is a system $F^p$ of complex vector spaces with (not necessarily injective) linear maps $s : F^p \rightarrow F^{p-1}$, such that $\lim_{p \rightarrow -\infty} F^p \cong V \otimes \mathbb{C}$.

Definition 1.11. For $C^*$ as in §1.1, fix an isomorphism $C \cong \mathbb{A}^2$, with co-ordinates $(u, v)$, so that the isomorphism $C(\mathbb{R}) \cong \mathbb{C}$ is given by $(u, v) \mapsto u + iv$. Let $\widetilde{C}^* \rightarrow C^*$ be the étale covering of $C^*$ given by cutting out the divisor $\{u - iv = 0\}$ from $C^* \otimes \mathbb{R}$.\hfill \Box

Lemma 1.12. There is an equivalence of categories between flat $S$-equivariant quasi-coherent sheaves on $\widetilde{C}^*$, and exhaustive filtrations on complex vector spaces.

Proof. First, observe that there is an isomorphism $\widetilde{C}^* \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{G}_m, \mathbb{C}$, given by $(u, v) \mapsto (u + iv, u - iv)$. As in Corollary 1.8, $S_C \cong \mathbb{G}_m, \mathbb{C} \times \mathbb{G}_m, \mathbb{C}$ under the same isomorphism. Thus $S$-equivariant quasi-coherent sheaves on $\widetilde{C}^*$ are equivalent to $\mathbb{G}_m, \mathbb{C} \times 1$-equivariant quasi-coherent sheaves on the scheme $\mathbb{A}^1_{\mathbb{C}} \subset \widetilde{C}^*0$ given by $u - iv = 1$. Now apply Lemma 1.6. \hfill \Box

1.1.1. $\text{SL}_2$.

Definition 1.13. Define maps $\text{row}_1, \text{row}_2 : \text{GL}_2 \rightarrow \mathbb{A}^2$ by projecting onto the first and second rows, respectively. If we make the identification $C = \mathbb{A}^2$ of Definition 1.1, then these are equivariant with respect to the right $S$-action $\text{GL}_2 \times S \rightarrow \text{GL}_2$, given by $(A, \lambda) \mapsto A \left( \begin{smallmatrix} \mathbb{R}_2 & \mathbb{A}^1 \mathbb{R} \\ \mathbb{A}^1 \mathbb{R} & \mathbb{R}_2 \end{smallmatrix} \right)^{-1}$.

Definition 1.14. Define an $S$-action on $\text{SL}_2$ by

$$(\lambda, A) \mapsto \left( \begin{smallmatrix} 0 & 0 \\ \mathbb{A} \lambda & 0 \end{smallmatrix} \right) A \left( \begin{smallmatrix} \mathbb{R}_2 & \mathbb{A}^1 \mathbb{R} \\ \mathbb{A}^1 \mathbb{R} & \mathbb{R}_2 \end{smallmatrix} \right)^{-1} = \left( \begin{smallmatrix} \mathbb{A} \lambda & 0 \\ 0 & 1 \end{smallmatrix} \right) A \left( \begin{smallmatrix} \mathbb{R}_2 & \mathbb{A}^1 \mathbb{R} \\ \mathbb{A}^1 \mathbb{R} & \mathbb{R}_2 \end{smallmatrix} \right).$$

Let $\text{row}_1 : \text{SL}_2 \rightarrow C^*$ be the $S$-equivariant map given by projection onto the first row.

Remark 1.15. Observe that, as an $S$-equivariant scheme over $C^*$, we may decompose $\text{GL}_2$ as $\text{GL}_2 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & \mathbb{G}_m \end{smallmatrix} \right) \times \text{SL}_2$, where the $S$-action on $\mathbb{G}_m$ has $\lambda$ acting as multiplication by $(\lambda \lambda)^{-1}$.

We may also write $C^* = [\text{SL}_2/\mathbb{G}_a]$, where $\mathbb{G}_a$ acts on $\text{SL}_2$ as left multiplication by $\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, where the $S$-action on $\mathbb{G}_a$ has $\lambda$ acting as multiplication by $\lambda \lambda$. 

Lemma 1.16. The morphism \( \text{row}_1 : \text{SL}_2 \to C^* \) is weakly final in the category of \( S \)-equivariant affine schemes over \( C^* \).

Proof. We need to show that for any affine scheme \( U \) equipped with an \( S \)-equivariant morphism \( f : U \to C^* \), there exists a (not necessarily unique) \( S \)-equivariant morphism \( g : U \to \text{SL}_2 \) such that \( f = \text{row}_1 \circ g \).

If \( U = \text{Spec} \, A \), then \( A \) is a \( O(C) = \mathbb{R}[u,v] \)-algebra, with the ideal \( (u,v)_A = A \), so there exist \( a, b \in A \) with \( ua - vb = 1 \). Thus the map factors through \( \text{row}_1 : \text{SL}_2 \to C^* \).

Complexifying, and writing \( w = u + iv, \bar{w} = u - iv \) gives an expression \( \alpha w + \beta \bar{w} = 1 \). Now splitting \( \alpha, \beta \) into types, we have \( \alpha^{10} w + \beta^{01} \bar{w} = 1 \). Similarly, \( \frac{1}{2} (\alpha^{10} + \beta^{01}) w + \frac{1}{2} (\beta^{01} + \alpha^{10}) \bar{w} = 1 \), on conjugating and averaging. Write this as \( \alpha' w + \beta' \bar{w} = 1 \). Finally, note that \( y := \alpha' + \beta', -x := ia' - i\beta' \) are both real, giving \( uy - vx = 1 \), with \( x, y \) having the appropriate \( S \)-action to regard \( A \) as an \( O(\text{SL}_2) \)-algebra when \( \text{SL}_2 \) has co-ordinates \((u,v)\).

\[ \square \]

Remark 1.17. Observe that for our action of \( \mathbb{G}_m \subset S \) (corresponding to left multiplication by diagonal matrices) on \( \text{SL}_2 \), the stack \( [\text{SL}_2/\mathbb{G}_m] \) is just the affine scheme \( \mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1) \).

Here, \( \Delta \) is the diagonal embedding, and the projections to \( \mathbb{P}^1 \) correspond to the maps \( \text{row}_1, \text{row}_2 : [\text{SL}_2/\mathbb{G}_m] \to [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] \) (noting that for \( \text{row}_2 \) this means taking the inverse of our usual \( \mathbb{G}_m \)-action on \( C^* \)). Lemma 1.16 can then be reformulated to say that \( \mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1) \) is weakly final in the category of \( U_1 \)-equivariant affine schemes over \( \mathbb{P}^1 \).

Lemma 1.18. The affine scheme \( \text{SL}_2 \xrightarrow{\text{row}_1} C^* \) is a flat algebraic Hodge filtration, corresponding to the algebra

\[ S := \mathbb{R}[x], \]

with filtration \( F^p(S \otimes \mathbb{C}) = (x - i)^p \mathbb{C}[x] \).

Proof. Since \( \text{row}_1 \) is flat and equivariant for the inverse right \( S \)-action, we know by Corollary 1.8 that we have a filtration on \( S \otimes \mathbb{C} \), for \( \text{Spec} \, S = \text{SL}_2 \times_{\text{row}_1, C^*, 1} \text{Spec} \, \mathbb{R} \). \( \text{Spec} \, S \) consists of invertible matrices \(( \frac{1}{x} \frac{0}{1} ) \), giving \( S \) the ring structure claimed.

To describe the filtration, we use Lemma 1.12, considering the pullback of \( \text{row}_1 \) along \( \mathbb{C}^* \to C^* \). The scheme \( \text{SL}_2 := \text{SL}_2 \times_{\text{row}_1, C} \mathbb{C}^* \) is isomorphic to \( C^* \times \mathbb{A}^1 \), with projection onto \( A_{\mathbb{C}}^1 \) given by \( (x,y) \mapsto x - iy \). This isomorphism is moreover \( S_{\mathbb{C}} \)-equivariant over \( C^* \), when we set the co-coordinates of \( \mathbb{A}^1 \) to be of type \((1,0)\).

The filtration \( F \) on \( S \otimes \mathbb{C} \) then just comes from the decomposition on \( \mathbb{C}[x - iy] \) associated to the action of \( \mathbb{G}_{m, \mathbb{C}} \times \{1\} \subset S_{\mathbb{C}} \), giving

\[ F^p \mathbb{C}[x - iy] = \bigoplus_{p' \geq p} (x - iy)^{-p'} \mathbb{C}. \]

The filtration on \( S \otimes \mathbb{C} \) is given by evaluating this at \( y = 1 \), giving \( F^p(S \otimes \mathbb{C}) = (x - i)^p \mathbb{C}[x] \), as required.

For an explicit inverse construction, the complex Rees module \( \bigoplus_{pq} u^p v^q F^q F^p S \) associated to \( S \) is the \( \mathbb{C}[w, \bar{w}] \)-subalgebra of \( (S \otimes \mathbb{C})[w, w^{-1}, \bar{w}, \bar{w}^{-1}] \) generated by \( z := w^{-1}(x - i) \) and \( z := \bar{w}^{-1}(x + i) \). These satisfy the sole relation \( wz - \bar{w}z = -2i \), so \( (\frac{x}{y} \frac{v}{u}) \in \text{SL}_2 \), where \( z = \xi + i\eta, \bar{z} = \xi - i\eta \).

\[ \square \]

Remark 1.19. We may now reinterpret Lemma 1.16 in terms of Hodge filtrations. An \( S \)-equivariant affine scheme, flat over \( C^* \), is equivalent to a real algebra \( A \), equipped with an exhaustive decreasing filtration \( F \) on \( A \otimes_{\mathbb{R}} \mathbb{C} \), such that \( \text{gr}_F \text{gr}_F (A \otimes_{\mathbb{R}} \mathbb{C}) = 0 \). This last condition is equivalent to saying that \( 1 \in F^1 + F^1 \), or even that there exists \( \alpha \in F^1(A \otimes_{\mathbb{R}} \mathbb{C}) \) with \( \Re \alpha = 1 \). We then define a homomorphism \( f : S \to A \) by setting \( f(x) = 3\alpha \), noting that \( f(1 + ix) = \alpha \in F^1(A \otimes_{\mathbb{R}} \mathbb{C}) \), so \( f \) respects the Hodge filtration.
preserving the quasi-MHS structure, and inducing the identity on taking $W$.

Since $W \otimes V$ is Hausdorff, satisfying the opposedness condition.

We proceed by induction on the weight filtration.

Proof. for $i$ bounded below if $\otimes V$ is Hausdorff, satisfying the opposedness condition.

Example 1.23. regarded as an $O_j$ torsor for the group.

Definition 1.20. The $G_a$-action on $SL_2$ of Remark 1.15 gives rise to an action of the associated Lie algebra $g_a \cong \mathbb{R}$ on $O(SL_2)$. Explicitly, define the standard generator $N \in g_a$ to act as the derivation with $N x = u, Ny = v, Nu = Nv = 0$, for co-ordinates $(u \ u \ v \ v)$ on $SL_2$.

This is equivalent to the $O(SL_2/C)$-linear isomorphism $\Omega(SL_2/C) \to O(SL_2)$ given by $dx \mapsto u, dy \mapsto v$. This is not $S$-equivariant, but has type $(-1, -1)$, so we write $\Omega(SL_2/C) \cong O(SL_2)(-1)$.

The DG algebra $O(SL_2) \to O(SL_2)(-1)\epsilon$, for $\epsilon$ of degree 1, is an algebra over $O(C) = \mathbb{R}[u, v]$, so we may consider the DG algebra $j^{-1}O(SL_2) \to j^{-1}O(SL_2)(-1)\epsilon$ on $C^*$, for $j : C^* \to C$. This is an acyclic resolution of the structure sheaf $\mathcal{O}_C$, so

\[ Rj_*\mathcal{O}_C \cong j_*j^{-1}O(SL_2) \to j^{-1}O(SL_2)(-1)\epsilon = (O(SL_2) \to O(SL_2)(-1)\epsilon), \]

regarded as an $O(C)$-algebra. This construction is moreover $S$-equivariant.

Definition 1.21. From now on, we will denote the DG algebra $O(SL_2) \to O(SL_2)(-1)\epsilon$ by $RO(C^*)$, thereby making a canonical choice of representative in the equivalence class $Rj_*\mathcal{O}_C$.

Definition 1.22. Define a (real) quasi-MHS to be a real vector space $V$, equipped with an exhaustive increasing filtration $W$ on $V$, and an exhaustive decreasing filtration $F$ on $V \otimes \mathbb{C}$.

We adopt the convention that a (real) MHS is a finite-dimensional quasi-MHS on which $W$ is Hausdorff, satisfying the opposedness condition

\[ \text{gr}^W_n \text{gr}^F_j (V \otimes \mathbb{C}) = 0 \]

for $i + j \neq n$.

Define a (real) ind-MHS to be a filtered direct limit of MHS. Say that an ind-MHS is bounded below if $W_n V = 0$ for $N \ll 0$.

Example 1.23. The ring $\mathcal{S}$ of Lemma 1.18 can be given the structure of a quasi-MHS with the weight filtration $W_0 \mathcal{S} = \mathcal{S}, W_{-1} \mathcal{S} = 0$, but is not an ind-MHS.

Definition 1.24. Given a quasi-MHS $V$, define the decreasing filtration $\gamma^* \otimes V = V \cap F^p(V \otimes \mathbb{C})$.

Proposition 1.25. Every (finite-dimensional abelian) MHS $V$ admits an $\mathcal{S}$-splitting, i.e. an $\mathcal{S}$-linear isomorphism

\[ V \otimes \mathcal{S} \cong (\text{gr}^W V) \otimes \mathcal{S}, \]

of quasi-MHS, inducing the identity on the grading associated to $W$. The set of such splittings is a torsor for the group $i + W_{-1} \gamma^0 \text{End}((\text{gr}^W V) \otimes \mathcal{S})$.

Proof. We proceed by induction on the weight filtration. $\mathcal{S}$-linear extensions $0 \to W_{n-1} V \otimes \mathcal{S} \to W_n V \otimes \mathcal{S} \to \text{gr}^W_n V \otimes \mathcal{S} \to 0$ of quasi-MHS are parametrised by

\[ \text{Ext}^1_{\mathbb{A}^1 \times SL_2} (\text{gr}^W V \otimes O(\mathbb{A}^1) \otimes O(SL_2), \xi(W_{n-1} V, W, F, F) \otimes \text{End}(O(C))) O(SL_2) \times \mathcal{S}, \]

since $\mathbb{A}^1 \times S$ is (linearly) reductive. Now, $\text{gr}^W V \otimes O(\mathbb{A}^1) \otimes O(SL_2)$-module, so its higher Exts are all 0, and all $\mathcal{S}$-linear quasi-MHS extensions of $\text{gr}^W V \otimes \mathcal{S}$ by $W_{n-1} V \otimes \mathcal{S}$ are isomorphic, so $W_n V \otimes \mathcal{S} \cong W_{n-1} V \otimes \mathcal{S} \otimes \text{gr}^W V \otimes \mathcal{S}$.

Finally, observe that any two splittings differ by a unique automorphism of $(\text{gr}^W V) \otimes \mathcal{S}$, preserving the quasi-MHS structure, and inducing the identity on taking $\text{gr}^W$. This group is just $i + W_{-1} \gamma^0 \text{End}((\text{gr}^W V) \otimes \mathcal{S})$, as required. \qed
1.1.2. **Cohomology of Hodge filtrations.** Given a complex $\mathcal{F}^*$ of algebraic Hodge filtrations, we now show how to calculate hypercohomology $\mathbb{H}^*(\mathbb{C}^*/\mathbb{S}, \mathcal{F}^*)$, and compare this with Beilinson’s weak Hodge cohomology.

Considering the étale pushout $C^* = \mathcal{C}^* \cup_{\mathbb{S} C} S$ of affine schemes, $\mathbf{R}\Gamma(C^*, \mathcal{F}^*)$ is the cone of the morphism

$$\mathbf{R}\Gamma(\mathcal{C}^*, \mathcal{F}^*) \oplus \mathbf{R}\Gamma(S, \mathcal{F}^*) \to \mathbf{R}\Gamma(S_{\mathbb{S} C}, \mathcal{F}^*).$$

If $\mathcal{F}^*$ is a flat complex, it corresponds under Corollary 1.8 to a complex $V^*$ of real vector spaces, equipped with an exhaustive filtration $F$ of $V^*_C := V^* \otimes \mathbb{C}$. The expression above then becomes

$$(\bigoplus_{n \in \mathbb{Z}} F^n(V^*_C) w^{-n}) [\bar{w}, w^{-1}] \oplus V^*_R [u, v, (u^2 + v^2)^{-1}] \to V^*_C [w, w^{-1}, \bar{w}, \bar{w}^{-1}],$$

for co-ordinates $u, v$ on $C^*$ as in Corollary 1.8, and $w = u + iv, \bar{w} = u - iv$.

Since $S$ is a reductive group, taking $S$-invariants is an exact functor, so $\mathbf{R}\Gamma([C^*/\mathbb{S}, \mathcal{F}^*])$ is the cone of the morphism

$$\mathbf{R}\Gamma(\mathcal{C}^*, \mathcal{F}^*)^S \oplus \mathbf{R}\Gamma(S, \mathcal{F}^*)^S \to \mathbf{R}\Gamma(S_{\mathbb{S} C}, \mathcal{F}^*)^S$$

which is just

$$F^0(V^*_C) \oplus V^*_R \to V^*_C,$$

which is just the functor $\mathbf{R}\Gamma_{\mathbb{S} C}$ from [Bei].

Therefore

$$\mathbf{R}\Gamma([C^*/\mathbb{S}, \mathcal{F}^*]) \simeq \mathbf{R}\Gamma_{\mathbb{S} C}(U^*, V^*).$$

**Remark 1.26.** For $\mathbb{S}$ as in Lemma 1.18, and a complex $V^*$ of $\mathbb{S}$-modules, with compatible filtration $F$ on $V^* \otimes \mathbb{C}$, let $\mathcal{F}^*$ be the associated bundle on $[C^*/\mathbb{S}]$. By Lemma 1.18, this is a row1$1_{\mathcal{S} \mathbb{S} L_2/S}$-module, so $\mathcal{F}^* = \text{row}_{1_{\mathcal{S}}} \mathcal{E}^*$, for some quasi-coherent complex $\mathcal{E}^*$ on $[\mathbb{S} \mathbb{S} L_2/S]$, and

$$\mathbf{R}\Gamma([C^*/\mathbb{S}, \mathcal{F}^*]) = \mathbf{R}\Gamma([C^*/\mathbb{S}], \text{row}_{1_{\mathcal{S}}} \mathcal{E}^*)$$

$$\simeq \mathbf{R}\Gamma([\mathbb{S} \mathbb{S} L_2/S], \mathcal{E}^*)$$

$$\simeq \Gamma([\mathbb{S} \mathbb{S} L_2/S], \mathcal{E}^*)$$

$$\simeq \Gamma([C^*/\mathbb{S}, \mathcal{F}^*]),$$

since $\mathbb{S} \mathbb{S} L_2$ and $\text{row}_{1_{\mathcal{S}}}$ are both affine.

In other words,

$$\mathbf{R}\Gamma_{\mathbb{S} C}(V^*) \simeq \gamma_0 V^*,$$

for $\gamma$ as in Definition 1.24, which is equivalent to saying that $V \oplus F^0(V \otimes \mathbb{C}) \to V \otimes \mathbb{C}$ is necessarily surjective for all $\mathbb{S}$-modules $V$.

1.2. **Twistor filtrations.**

**Definition 1.27.** Given an affine scheme $X$ over $\mathbb{K}$, we define an algebraic (real) twistor filtration $X_T$ on $X$ to consist of the following data:

1. a $\mathbb{G}_m$-equivariant affine morphism $T : X_T \to C^*$,
2. an isomorphism $X \cong X_{T, 1} := X_T \times_{C^*, 1} \text{Spec } \mathbb{R}$.

**Definition 1.28.** A real splitting of the twistor filtration $X_T$ consists of a $\mathbb{G}_m$-action on $X$, and an $\mathbb{G}_m$-equivariant isomorphism

$$X \times C^* \cong X_T$$

over $C^*$. 
Definition 1.29. Adapting [Sim2] §1 from complex to real structures, say that a twistor structure on a real vector space $V$ consists of a vector bundle $\mathcal{E}$ on $\mathbb{P}^1_\mathbb{R}$, with an isomorphism $V \cong \mathcal{E}_1$, the fibre of $\mathcal{E}$ over $1 \in \mathbb{P}^1$.

Proposition 1.30. The category of finite flat algebraic twistor filtrations on real vector spaces is equivalent to the category of twistor structures.

Proof. The flat algebraic twistor filtration is a flat $\mathbb{G}_m$-equivariant quasi-coherent sheaf $M$ on $C^*$, with $M|_1 = V$. Taking the quotient by the right $\mathbb{G}_m$-action, $M$ corresponds to a flat quasi-coherent sheaf $M_{\mathbb{G}_m}$ on $[C^*/\mathbb{G}_m]$. Now, $[C^*/\mathbb{G}_m] \cong [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] = \mathbb{P}^1$, so Lemma 1.6 implies that $M_{\mathbb{G}_m}$ corresponds to a flat quasi-coherent sheaf $\mathcal{E}$ on $\mathbb{P}^1$. Note that $\mathcal{E}_1 = (M|_{\mathbb{G}_m})_{\mathbb{G}_m} \cong M_1 \cong V$, as required. \hfill \Box

Definition 1.31. Define the real algebraic group $U_1$ to be the circle group, whose $A$-valued points are given by $\{(a,b) \in A^2 : a^2 + b^2 = 1\}$. Note that $U_1 \hookrightarrow S$, and that $S/\mathbb{G}_m \cong U_1$. This latter $S$-action gives $U_1$ a split Hodge filtration.

Lemma 1.32. There is an equivalence of categories between algebraic twistor filtrations $X_T$ on $X$, and extensions $\tilde{X}$ of $X$ over $U_1$, $\{X = \tilde{X}_T\}$ equipped with algebraic Hodge filtrations $\tilde{X}_T$, compatible with the standard Hodge filtration on $U_1$.

Proof. Given an algebraic Hodge filtration $\tilde{X}_T$ over $U_1 \times C^*$, take

$$X_T := \tilde{X}_T \times_{U_1, \text{Spec } \mathbb{R}} \text{Spec } \mathbb{R},$$

and observe that this satisfies the axioms of an algebraic twistor filtration. Conversely, given an algebraic twistor filtration $X_T$ (over $C^*$), set

$$\tilde{X}_T = (X_T \times U_1)/(-1,-1),$$

with projection $\pi(x,t) = (pr(x)t^{-1}, t^2) \in C^* \times U_1$. \hfill \Box

Corollary 1.33. A flat algebraic twistor filtration on a real vector space $V$ is equivalent to the data of a flat $O(U_1)$-module $\check{V}^{U_1}$ with $\check{V}^{U_1} \otimes_{O(U_1)} \mathbb{R} = V$, together with an exhaustive decreasing filtration $F$ on $(\check{V}^{U_1}) \otimes \mathbb{C}$, with the morphism $O(U_1) \otimes \mathbb{R} \check{V}^{U_1} \to \check{V}^{U_1}$ respecting the filtrations (for the standard Hodge filtration on $O(U_1) \otimes \mathbb{C}$). In particular, the filtration is given by $F^p(\check{V}^{U_1} \otimes \mathbb{C}) = (a + ib)^p F^0(\check{V}^{U_1} \otimes \mathbb{C})$.

Definition 1.34. Given a flat algebraic twistor filtration on a real vector space $V$ as above, define $\text{gr}_F \check{V}^{U_1}$ to be the real part of $\text{gr}_F \check{V}^{U_1}(\check{V}^{U_1} \otimes \mathbb{C})$. Note that this is an $O(U_1)$-module, and define $\text{gr}_T V := (\text{gr}_F \check{V}^{U_1}) \otimes_{O(U_1)} \mathbb{R}$.

These results have the following trivial converse.

Lemma 1.35. An algebraic Hodge filtration $X_T \to C^*$ on $X$ is equivalent to an algebraic twistor filtration $T : X_T \to C^*$ on $X$, together with a $U_1$-action on $X_T$ with respect to which $T$ is equivariant, and for which $-1 \in U_1$ acts as $-1 \in \mathbb{G}_m$.

Proof. The subgroups $U_1$ and $\mathbb{G}_m$ of $S$ satisfy $(\mathbb{G}_m \times U_1)/(-1,-1) \cong S$. \hfill \Box

1.3. Mixed Hodge structures. We now define algebraic mixed Hodge structures on real affine schemes.

Definition 1.36. Given an affine scheme $X$ over $\mathbb{R}$, we define an algebraic mixed Hodge structure $X_{\text{MHS}}$ on $X$ to consist of the following data:

1. an $\mathbb{G}_m \times S$-equivariant affine morphism $X_{\text{MHS}} \to \mathbb{A}^1 \times C^*$,
2. a real affine scheme $\text{gr}_X X_{\text{MHS}}$ equipped with an $S$-action,
3. an isomorphism $X \cong X_{\text{MHS}} \times_{(\mathbb{A}^1 \times C^*)_{(1,1)}} \text{Spec } \mathbb{R}$,
Definition 1.37. Given an algebraic mixed Hodge structure \( X_{MHS} \) on \( X \), define \( \text{gr}^W X_{MHS} := X_{MHS} \times_{\mathbb{A}^1,0} \text{Spec} \mathbb{R} \), noting that this is isomorphic to \( \text{gr} X_{MHS} \times C^* \). We also define \( X_{\mathcal{F}} := X_{MHS} \times_{\mathbb{A}^1,1} \text{Spec} \mathbb{R} \), noting that this is a Hodge filtration on \( X \).

Definition 1.38. A real splitting of the mixed Hodge structure \( X_{MHS} \) is a \( \mathbb{G}_m \times S \)-equivariant isomorphism

\[
\mathbb{A}^1 \times \text{gr} X_{MHS} \times C^* \cong X_{MHS},
\]

giving the opposedness isomorphism on pulling back along \( \{0\} \to \mathbb{A}^1 \).

Remarks 1.39. (1) Note that giving \( X_{MHS} \) as above is equivalent to giving the affine morphisms \([X_{MHS}/\mathbb{G}_m \times S] \to [\mathbb{A}^1/\mathbb{G}_m] \times [C^*/S]\) and \( \text{gr} X_{MHS} \to BS \) of stacks, satisfying an opposedness condition.

(2) To compare this with the non-abelian mixed Hodge structures postulated in [KPS], note that pulling back along the morphism \( \tilde{C}^* \to C^* \) gives an object over \([\mathbb{A}^1/\mathbb{G}_m] \times [C^*/S] \cong [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m]_C \); this is essentially the stack \( X_{dR} \) of [KPS]. The stack \( X_{B,R} \) of [KPS] corresponds to pulling back along \( 1 : \text{Spec} \mathbb{R} \to C^* \). Thus our algebraic mixed Hodge structures give rise to pre-non-abelian mixed Hodge structures (pre-NAMHS) in the sense of [KPS]. Our treatment of the opposedness condition is also similar to the linearisation condition for a pre-NAMHS, by introducing additional data corresponding to the associated graded object.

As for Hodge filtrations, this gives us a notion of an algebraic mixed Hodge structure on a real vector space. We now show how this is equivalent to the standard definition.

Proposition 1.40. The category of flat \( \mathbb{G}_m \times S \)-equivariant quasi-coherent sheaves \( M \) on \( \mathbb{A}^1 \times C^* \) is equivalent to the category of quasi-MHS.

Under this equivalence, bounded below ind-MHS \((V, W, F)\) correspond to flat algebraic mixed Hodge structures \( M \) on \( V \) whose weights with respect to the \( \mathbb{G}_m \times 1 \)-action are bounded below.

A real splitting of the Hodge filtration is equivalent to giving a (real) Hodge structure on \( V \) (i.e. an \( S \)-action).

Proof. Adapting Corollary 1.8, we see that a flat \( \mathbb{G}_m \times S \)-equivariant module \( M \) on \( \mathbb{A}^1 \times C^* \) corresponds to giving exhaustive filtrations \( W \) on \( V = M|_{(1,1)} \) and \( F \) on \( V \otimes \mathbb{C} \), i.e. a quasi-MHS on \( V \). Write \( \xi(V, MHS) \) for the \( \mathbb{G}_m \times S \)-equivariant quasi-coherent sheaf on \( \mathbb{A}^1 \times C^* \) associated to a quasi-MHS \((V, W, F)\).

A flat algebraic mixed Hodge structure is a flat \( \mathbb{G}_m \times S \)-equivariant module \( M \) on \( \mathbb{A}^1 \times C^* \), with \( M|_{(1,1)} = V \), together with a \( \mathbb{G}_m \times S \)-equivariant splitting of the algebraic Hodge filtration \( M|_{(0)} \times C^* \). Under the equivalence above, this gives a quasi-MHS \((V, W, F)\), with \( W \) bounded below, satisfying the split opposedness condition

\[
(\text{gr}_n^W V) \otimes \mathbb{C} = \bigoplus_{p+q=n} F^p(\text{gr}_n^W V \otimes \mathbb{C}) \cap F^q(\text{gr}_n^W V \otimes \mathbb{C}).
\]

When the weights of \( M \) are bounded below, we need to express this as a filtered direct limit of MHS. Since \( W \) is exhaustive, it will suffice to prove that each \( W_r V \) is an ind-MHS. Now \( W_N V = 0 \) for some \( N \), so split opposedness means that \( W_{N+1} V \) is a direct sum of pure Hodge structures (i.e. an \( S \)-representation), hence an ind-MHS. Assume inductively that \( W_{r-1} V \) is an ind-MHS, and consider the exact sequence

\[
0 \to W_{r-1} V \to W_r V \to \text{gr}_r^W V \to 0.
\]
of quasi-MHS. Again, split opposedness shows that \( \text{gr}_r^W V \) is an ind-MHS, so we may express it as \( \text{gr}_r^W V = \lim_{\alpha} U_\alpha \), with each \( U_\alpha \) a MHS. Thus \( W_r V = \lim_{\alpha} W_r V \times_{\text{gr}_r^W V} U_\alpha \), so we may assume that \( \text{gr}_r^W V \) is finite-dimensional (replacing \( W_r \) with \( W_r V \times_{\text{gr}_r^W V} U_\alpha \)).

Then quasi-MHS extensions of \( \text{gr}_r^W V \) by \( W_{r-1} V \) are parametrised by

\[
\text{Ext}^1_{A^1 \times C^*}(\xi(\text{gr}_r^W V, \text{MHS}), \xi(W_{r-1} V, \text{MHS}))^{G_m \times S}.
\]

Express \( W_{r-1} V \) as a filtered direct limit \( \lim_{\beta} T_\beta \) of MHS, and note that

\[
\text{Ext}^1_{A^1 \times C^*}(\xi(\text{gr}_r^W V, \text{MHS}), \xi(W_{r-1} V, \text{MHS}))^{G_m \times S} = \lim_{\beta} \text{Ext}^1_{A^1 \times C^*}(\xi(\text{gr}_r^W V, \text{MHS}), \xi(T_\beta, \text{MHS}))^{G_m \times S},
\]

since \( \xi(\text{gr}_r^W V, \text{MHS}) \) is finite and locally free. Thus the extension \( W_r V \to \text{gr}_r^W V \) is a pushout of an extension

\[
0 \to T_\beta \to E \to \text{gr}_r^W V \to 0
\]

for some \( \beta \), so \( W_r V \) can be expressed as the ind-MHS \( W_r V = \lim_{\beta', \beta} E \oplus T_\beta T_{\beta'} \).

Conversely, any MHS \( V \) satisfies the split opposedness condition by [Del1] Proposition 1.2.5, so the same holds for any ind-MHS. Thus every ind-MHS corresponds to a flat algebraic MHS under the equivalence above.

Finally, note that the split opposedness condition determines the data of any real splitting.

\[ \square \]

**Remark 1.41.** Note that the proof of [Del1] Proposition 1.2.5 does not adapt to infinite filtrations. For instance, the quasi-MHS \( S \) of Example 1.23 satisfies the opposedness condition, but does not give an ind-MHS. Geometrically, this is because the fibre over \( \{0\} \in C \) is empty. Algebraically, it is because the Hodge filtration on the ring \( S = \text{gr}_0^W S \) is not split, but \( \text{gr}_F \text{gr}_F(S \otimes \mathbb{C}) = 0 \), which is a pure Hodge structure of weight 0.

### 1.3.1. Cohomology of MHS

Given a complex \( \mathcal{F}^* \) of algebraic MHS, we now show how to calculate hypercohomology \( H^\ast([C^*/S] \times [A^1/G_m], \mathcal{F}^*) \), and compare this with Beilinson’s absolute Hodge cohomology. By Proposition 1.40, \( \mathcal{F}^* \) gives rise to a complex \( V^\ast \) of quasi-MHS.

Since \( A^1 \) is affine and \( G_m \) reductive, \( R\text{pr}_\ast = \text{pr}_\ast \) for the projection \( \text{pr} : [C^*/S] \times [A^1/G_m] \to [C/S] \). Thus

\[
R\Gamma([C^*/S] \times [A^1/G_m], \mathcal{F}^*) \simeq R\Gamma([C^*/S], \text{pr}_\ast \mathcal{F}^*),
\]

and \( \text{pr}_\ast \mathcal{F}^* \) just corresponds under Corollary 1.8 to the complex \( W_0 V^\ast \) with filtration \( F \) on \( W_0 V^\ast \).

Hence §1.1.2 implies that \( R\Gamma([C^*/S] \times [A^1/G_m], \mathcal{F}^*) \) is just the cone of

\[
W_0 F^0(V^\ast) \oplus W_0 V^\ast \to W_0 V^\ast,
\]

which is just the absolute Hodge functor \( R\Gamma_H \) from [Bei].

Therefore

\[
R\Gamma([C^*/S] \times [A^1/G_m], \mathcal{F}^*) \simeq R\Gamma_H(V^\ast),
\]

Likewise, if \( \mathcal{E}^\ast \) is another such complex, coming from a complex \( (U^\ast, W, F) \), then

\[
R\text{Hom}_{[C^*/S] \times [A^1/G_m]}(\mathcal{E}^\ast, \mathcal{F}^*) \simeq R\text{Hom}_H(U^\ast, V^\ast).
\]

### 1.4. Mixed twistor structures.

**Definition 1.42.** Given an affine scheme \( X \) over \( \mathbb{R} \), we define an algebraic mixed twistor structure \( X_{\text{MTS}} \) on \( X \) to consist of the following data:

1. an \( G_m \times G_m \)-equivariant affine morphism \( X_{\text{MTS}} \to A^1 \times C^* \),
2. a real affine scheme \( W_{\text{MTS}} \) equipped with a \( G_m \)-action,
3. an isomorphism \( X \cong X_{\text{MTS}} \times_{(A^1 \times C^*) \times \text{Spec} \mathbb{R}} \).
(4) a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant isomorphism $\text{gr}X_{\text{MTS}} \times C^* \cong X_{\text{MTS}} \times_{\mathbb{A}^1, 0} \text{Spec} \mathbb{R}$. This is called the opposedness isomorphism.

**Definition 1.43.** Given an algebraic mixed twistor structure $X_{\text{MTS}}$ on $X$, define $\text{gr}^W X_{\text{MTS}} := X_{\text{MTS}} \times_{\mathbb{A}^1, 0} \text{Spec} \mathbb{R}$, noting that this is isomorphic to $\text{gr}X_{\text{MTS}} \times C^*$. We also define $X_{T} := X_{\text{MTS}} \times_{\mathbb{A}^1, 1} \text{Spec} \mathbb{R}$, noting that this is a twistor filtration on $X$.

**Definition 1.44.** A real splitting of the mixed twistor structure $X_{\text{MTS}}$ is a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant isomorphism

$$\mathbb{A}^1 \times \text{gr}X_{\text{MTS}} \times C^* \cong X_{\text{MTS}},$$

giving the opposedness isomorphism on pulling back along $\{0\} \to \mathbb{A}^1$.

**Remark 1.45.** Note that giving $X_{\text{MTS}}$ as above is equivalent to giving the affine morphism $[X_{\text{MTS}}/\mathbb{G}_m \times \mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m] \times [C^*/\mathbb{G}_m]$ of stacks, satisfying a split opposedness condition.

**Definition 1.46.** Adapting [Sim2] §1 from complex to real structures, say that a (real) mixed twistor structure (real MTS) on a real vector space $V$ consists of a finite locally free sheaf $\mathcal{E}$ on $\mathbb{P}^1_{\mathbb{R}}$, equipped with an exhaustive Hausdorff increasing filtration by locally free subsheaves $W_i \mathcal{E}$, such that for all $i$ the graded bundle $\text{gr}_i^W \mathcal{E}$ is semistable of slope $i$ (i.e. a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(i)$). We also require an isomorphism $V \cong \mathcal{E}_1$, the fibre of $\mathcal{E}$ over $1 \in \mathbb{P}^1$.

Define a quasi-MTS on $V$ to be a flat quasi-coherent sheaf $\mathcal{E}$ on $\mathbb{P}^1_{\mathbb{R}}$, equipped with an exhaustive increasing filtration by quasi-coherent subsheaves $W_i \mathcal{E}$, together with an isomorphism $V \cong \mathcal{E}_1$. Define an ind-MTS to be a filtered direct limit of real MTS, and say that an ind-MTS $\mathcal{E}$ on $V$ is bounded below if $W_N \mathcal{E} = 0$ for $N \ll 0$.

Applying Corollary 1.33 gives the following result.

**Lemma 1.47.** A flat algebraic mixed twistor structure on a real vector space $V$ is equivalent to giving an $O(U_1)$-module $V'$, equipped with a mixed Hodge structure (compatible with the weight 0 real Hodge structure on $O(U_1)$), together with an isomorphism $V' \otimes_{O(U_1)} \mathbb{R} \cong V$.

**Proposition 1.48.** The category of flat $\mathbb{G}_m \times \mathbb{G}_m$-equivariant quasi-coherent sheaves on $\mathbb{A}^1 \times C^*$ is equivalent to the category of quasi-MTS.

Under this equivalence, bounded below ind-MTS on $V$ correspond to flat algebraic mixed twistor structures $\xi(V, \text{MTS})$ on $V$ whose weights with respect to the $\mathbb{G}_m \times 1$-action are bounded below.

**Proof.** The first statement follows by combining Lemma 1.30 with Lemma 1.6.

Now, given a flat algebraic mixed twistor structure $\xi(V, \text{MTS})$ on $V$ whose weights with respect to the $\mathbb{G}_m \times 1$-action are bounded below, the proof of Proposition 1.40 adapts (replacing $S$ with $\mathbb{G}_m$) to show that $\xi(V, \text{MTS})$ is a filtered direct limit of finite flat algebraic mixed twistor structures. It therefore suffices to show that finite flat algebraic mixed twistor structures correspond to MTS.

A finite flat algebraic mixed twistor structure is a finite locally free $\mathbb{G}_m \times \mathbb{G}_m$-equivariant module $M$ on $\mathbb{A}^1 \times C^*$, with $M]\mid_{\{1, 1\}} = V$, together with a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant splitting of the algebraic twistor filtration $M]\mid_{\{0\}} \times C^*$. Taking the quotient by the right $\mathbb{G}_m$-action, $M$ corresponds to a finite locally free $\mathbb{G}_m$-equivariant module $M_{\mathbb{G}_m}$ on $\mathbb{A}^1 \times [C^*/\mathbb{G}_m]$. Note that $[C^*/\mathbb{G}_m] \cong [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] = \mathbb{P}^1$, so Lemma 1.6 implies that $M_{\mathbb{G}_m}$ corresponds to a finite locally free module on $\mathcal{E}$ on $\mathbb{P}^1$, equipped with a finite filtration $W$.

Now, $\text{gr}^W X_{\text{MTS}}$ corresponds to a $\mathbb{G}_m$-representation $V$, or equivalently a graded vector space $V = \bigoplus V^n$. If $\pi$ denotes the projection $\pi : C^* \to \mathbb{P}^1$, then the opposedness isomorphism is equivalent to a $\mathbb{G}_m$-equivariant isomorphism

$$\text{gr}^W \mathcal{E} \cong V \otimes_{\mathbb{G}_m} (\pi_* \mathcal{E}_{C^*}) = \bigoplus_n V^n \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{P}^1}(n),$$
so $\text{gr}_n^W \mathscr{E} \cong V^n \otimes_{\mathbb{R}} \theta_{\mathbb{R}^1}(n)$, as required.

Remark 1.49. Note that every MHS $(V, W, F)$ has an underlying MTS $\mathscr{E}$ on $V$, given by forming the $S$-equivariant Rees module $\xi(V, F)$ on $C^*$ as in Corollary 1.8, and setting $\mathscr{E}$ to be the quotient $\xi(V, F)_{G_m}$ by the action of $G_m \subset S$. Beware that if $\mathscr{E}$ is the MTS underlying $V$, then $\mathscr{E}(-2n)$ is the MTS underlying the MHS $V(n)$.

2. S-splittings for real homotopy types

Fix a compact Kähler manifold $X$.

In [Mor], Theorem 9.1, a mixed Hodge structure was given on the rational homotopy groups of a smooth complex variety $X$. Here, we study the consequences of formality quasi-isomorphisms for this Hodge filtration when $X$ is a connected compact Kähler manifold.

Let $A^\bullet(X)$ be the differential graded algebra of real $C^\infty$ forms on $X$. As in [DGMS], this is the real (nilpotent) homotopy type of $X$. If we write $J$ for the complex structure on $A^\bullet(X)$, then there is a differential $d^c := J^{-1}dJ$ on the underlying graded algebra $A^\bullet(X)$. Note that $dd^c + d^c d = 0$.

2.1. The mixed Hodge structure.

Definition 2.1. Define the DGA $\tilde{A}^\bullet(X)$ on $C$ by

$$\tilde{A}^\bullet(X) = (A^\bullet(X) \otimes_{\mathbb{R}} O(C), ud + vd^c),$$

for co-ordinates $u, v$ as in §1.1. We denote the differential by $\tilde{d} := ud + vd^c$. Note that $\tilde{d}$ is indeed flat:

$$\tilde{d}^2 = u^2d^2 + uv(dd^c + d^cd) + v^2(d^c)^2 = 0.$$

Definition 2.2. There is an action of $S$ on $A^\bullet(X)$, which we will denote by $a \mapsto \lambda \circ a$, for $\lambda \in \mathbb{C}^* = S(\mathbb{R})$. For $a \in (A^\bullet(X) \otimes \mathbb{C})^{pq}$, it is given by

$$\lambda \circ a := \lambda^p \bar{\lambda}^q a.$$

Lemma 2.3. There is a natural algebraic $S$-action on $\tilde{A}^\bullet(X)$ over $C$.

Proof. For $\lambda \in S(\mathbb{R}) = \mathbb{C}^*$, this action is given on $A^\bullet(X)$ by $a \mapsto \lambda \circ a$, extending to $\tilde{A}^\bullet(X)$ by tensoring with the action on $C$ from Definition 1.1. We need to verify that this action respects the differential $\tilde{d}$.

Taking the co-ordinates $(u, v)$ on $C$, we will consider the co-ordinates $w = u + iv, \bar{w} = u - iv$ on $C_{\mathbb{C}}$. Now, we may decompose $d$ and $d^c$ into types (over $\mathbb{C}$) as $d = \partial + \bar{\partial}$ and $d^c = i\partial - i\bar{\partial}$. Thus $\tilde{d} = w\partial + \bar{w}\bar{\partial}$, so

$$\tilde{d} : (A^\bullet(X) \otimes \mathbb{C})^{p,q} \to w(A^\bullet(X) \otimes \mathbb{C})^{p+1,q} \oplus \bar{w}(A^\bullet(X) \otimes \mathbb{C})^{p,q+1},$$

which is equivariant under the $S$-action given, with $\lambda$ acting as multiplication by $\lambda^p \bar{\lambda}^q$ on both sides.

As in [Mor], there is a natural quasi-MHS on $A^\bullet(X)$. The weight filtration is given by the good truncation $W_i A^\bullet(X) = \tau^{\leq i} A^\bullet(X)$, and Hodge filtration on $A^\bullet(X) \otimes_{\mathbb{R}} \mathbb{C}$ is $F^p (A^\bullet(X) \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{i \geq p} A^{p,q}(X, \mathbb{C})$.

Lemma 2.4. The $S$-equivariant $C^\infty$-bundle $j^* \tilde{A}^\bullet(X)$ corresponds under Corollary 1.8 to the Hodge filtration on $A^\bullet(X, \mathbb{C})$.

Proof. We just need to verify that $\tilde{A}^\bullet(X) \otimes \mathbb{C}$ is isomorphic to the Rees algebra $\xi(A^\bullet(X), F, \bar{F})$ (for $F$ the Hodge filtration), with the same complex conjugation.

Now,

$$\xi(A^\bullet(X), F, \bar{F}) = \bigoplus_{pq} F^p \cap \bar{F}^q,$$
with $\lambda \in S(\mathbb{R}) \cong \mathbb{C}^*$ acting as $\lambda^p \xi^q$ on $F^p \cap F^q$, and inclusion $F^p \to F^{p-1}$ corresponding to multiplication by $w = u + iv$. We therefore define an $O(C)$-linear map $f : \hat{A}^*(X) \to \xi(A^*(X), F, \tilde{F})$ by mapping $(A(X) \otimes \mathbb{C})^p$ to $F^p \cap \tilde{F}^q$. It only remains to check that this respects the differentials.

For $a \in (A(X) \otimes \mathbb{C})^p$, $f(d\tilde{a}) = f(w\tilde{a} + \tilde{w}\tilde{a}) = w(\tilde{a}) + \tilde{w}(\tilde{a}) \in w(F^p+1 \cap \tilde{F}^q) + \tilde{w}(F^p \cap \tilde{F}^{q+1})$.

But $w(F^p+1 \cap \tilde{F}^q) = F^p \cap \tilde{F}^q = \tilde{w}(F^p \cap \tilde{F}^{q+1})$, so this is just $\partial a + \tilde{\partial} a = da$ in $F^p \cap \tilde{F}^q$, which is just $df(a)$, as required.

Combining this with the weight filtration means that the bundle $\xi(A^*(X), \text{MHS})$ on $[A^1/\mathbb{G}_m] \times [C^*/S]$ associated to the quasi-MHS $A^*(X)$ is just the Rees algebra $\xi(j^*\hat{A}^*(X), W)$, regarded as a $\mathbb{G}_m \times S$-equivariant $A^1 \times C^*$-bundle.

### 2.2. The family of formality quasi-isomorphisms.

**Lemma 2.5.** Given a graded module $V^*$ over a ring $B$, equipped with operators $d, d^c$ of degree 1 such that $[d, d^c] = d^2 = (d^c)^2 = 0$, then for $(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}) \in \text{GL}_2(B)$,

$$\ker d \cap \ker d^c = \ker(ud + vd^c) \cap \ker(xd + yd^c),$$

$$\text{Im}(ud + vd^c) + \text{Im}(xd + yd^c) = \text{Im}d + \text{Im}d^c,$$

$$\text{Im}(ud + vd^c)(xd + yd^c) = \text{Im}dd^c.$$

**Proof.** Observe that if we take any matrix, the corresponding inequalities (with $\leq$ replacing $=$) all hold. For invertible matrices, we may express $d, d^c$ in terms of $(ud + vd^c), (xd + yd^c)$ to give the reverse inequalities.

**Proposition 2.6.** If the pair $(d, d^c)$ of Lemma 2.5 satisfies the principle of two types, then so does $(ud + vd^c), (xd + yd^c)$.

**Proof.** The principle of two types states that

$$\ker d \cap \ker d^c \cap (\text{Im} d + \text{Im} d^c) = \text{Im} dd^c.$$

**Corollary 2.7.** On the graded algebra

$$A^*_\mathbb{R}(X) \otimes O(\text{SL}_2),$$

for $X$ compact Kähler, the operators $(ud + xd^c), (xd + yd^c)$ satisfy the principle of two types, where

$$O(\text{SL}_2) = \mathbb{Z}[u, v, x, y]/(uy - vx - 1)$$

is the ring associated to the affine group scheme $\text{SL}_2$.

**Definition 2.8.** We therefore set $d^c := xd + yd^c$.

The principle of two types now gives us a family of quasi-isomorphisms:

**Corollary 2.9.** We have the following $S$-equivariant quasi-isomorphisms of DGAs over $\text{SL}_2$, with notation from Definition 1.14:

$$\text{row}^1_j^* \hat{A}^*(X) \xrightarrow{i} \ker(d^c) \xrightarrow{p} \text{row}^2_2 H^*(j^*\hat{A}^*(X)) \cong H^*(A^*(X)) \otimes_{\mathbb{R}} O(\text{SL}_2),$$

where $\ker(d^c)$ means $\ker(d^c) \cap \text{row}^1_j^* \hat{A}^*(X)$, with differential $\tilde{d}$.

**Proof.** The principle of two types implies that $i$ is a quasi-isomorphism, and that we may define $p$ as projection onto $H^*_d(A^*(X) \otimes O(\text{SL}_2))$, on which the differential $\tilde{d}$ is 0. The final isomorphism now follows from the description $H^*(A^*(X)) \cong \frac{\ker d^c}{\text{Im} dd^c}$, which clearly maps to $H^*(j^*\hat{A}^*(X))$, the principle of two types showing it to be an isomorphism. \qed
Since the weight filtration is just defined in terms of good truncation, this also implies that
\[ \xi(\text{row} j^*\tilde{A}^*(X), W) \simeq \xi(H^*(X, \mathbb{R}), W) \otimes \mathcal{O}_{\mathbb{A}^1, \mathbb{SL}_2} \]
as \mathbb{G}_m \times S\text{-equivariant dg algebras over } \mathbb{A}^1 \times \mathbb{SL}_2.

**Corollary 2.10.** For \( S \) as in Example 1.23, we have the following \( W \)-filtered quasi-isomorphisms of DGAs
\[ A^*(X) \otimes S \xleftarrow{i'} \ker(\delta^c + xd) \xrightarrow{p'} H^*(X, \mathbb{R}) \otimes S, \]
where \( \ker(\delta^c + xd) := \ker(\delta^c + xd) \cap (A^*(X) \otimes S) \), with differential \( d \). Moreover, on tensoring with \( \mathbb{C} \), these become \( (W, F) \)-bifiltered quasi-isomorphisms.

**Proof.** Under the equivalence of Lemma 1.18, \( \text{Spec } S \) corresponds to \( \left( \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} \right) \subset \mathbb{SL}_2 \), equipped with a Hodge filtration. Then Corollary 2.9 is equivalent to the statement that \( i' \) and \( p' \) are quasi-isomorphisms preserving the filtrations, rather than filtered quasi-isomorphisms.

That these are also \( W \)-filtered quasi-isomorphisms is immediate, since \( W \) is defined as good truncation. \( \square \)

**Remarks 2.11.** Note that we cannot deduce Corollary 2.10 directly from Lemma 2.5 for the pair \( d, \delta^c + xd \), since that would only establish that \( i' \), \( p' \) are quasi-isomorphisms preserving the filtrations, rather than filtered quasi-isomorphisms.

Setting \( x = 0 \) recovers the real formality quasi-isomorphism of [DGMS], while \( x = i \) gives the complex filtered quasi-isomorphism used in [Mor].

Since \( A^*(X, \mathbb{R}) \otimes S \) is not a mixed Hodge structure in the classical sense (as \( F \) is not bounded on \( S \)), we cannot now apply the theory of mixed Hodge structures on real homotopy types from [Mor] to infer consequences for Hodge structures on homotopy groups. In \S\S 4 and 5, we will develop theory allowing us to deduce the following result on the interaction between formality and the mixed Hodge structure.

**Corollary 2.12.** For \( x_0 \in X \), \( S \) as in Example 1.23, and for all \( n \), there are \( S \)-linear isomorphisms
\[ \pi_*(X \otimes \mathbb{R}, x_0) \otimes \mathbb{R} S \cong \pi_*(H^*(X, \mathbb{R})) \otimes \mathbb{R} S, \]
of inverse systems of quasi-MHS, compatible with Whitehead brackets and Hurewicz maps. The associated graded map from the weight filtration is just the pullback of the standard isomorphism \( \text{gr}_W \pi_*(X \otimes \mathbb{R}, x_0) \cong \pi_*(H^*(X, \mathbb{R})) \) (coming from the opposedness isomorphism).

Here, \( \pi_*(H^*(X, \mathbb{R})) \) are the real homotopy groups \( H_{*-1}\tilde{G}(H^*(X, \mathbb{R})) \) (see Definition 3.22) associated to the formal homotopy type \( H^*(X, \mathbb{R}) \), with a real Hodge structure coming from the Hodge structure on \( H^*(X, \mathbb{R}) \).

**Proof.** Corollary 5.4 will show how \( j^*\tilde{A}^* \) determines an ind-MHS on \( \pi_*(X \otimes \mathbb{R}) \), and we will see in Proposition 5.6 that this is the same as the Hodge structure of [Mor]. The \( S \)-splitting of ind-MHS is then proved as Corollary 5.5. \( \square \)

In \S7, we will see how these MHS become variations of mixed Hodge structure as the basepoint \( x_0 \in X \) varies, while \S8 will show how to recover the MHS explicitly from the formality quasi-isomorphisms.

### 2.3. Real Deligne cohomology

Now, consider the derived direct image of \( j^*\tilde{A}^*(X) \) under the morphism \( q : [C^*/S] \to [\mathbb{A}^1/\mathbb{G}_m] \) given by \( u, v \mapsto u^2 + v^2 \). This is equivalent to \( (R_j^*j^*\tilde{A}^*(X))^{U_1} \) for \( j : C^* \to C \), since \( U_1 \) is reductive, \( \mathbb{G}_m = S/U_1 \) and \( \mathbb{A}^1 = C^*/U_1 \).
Proposition 2.13. There are canonical isomorphisms
\[(\mathbb{R}^m j_* j^* \tilde{A}^\bullet(X))^U_1 \simeq (\bigoplus_{a < 0} H^m(X, \mathbb{R})) \oplus (\bigoplus_{a \geq 0} (2\pi i)^{-a} H^m_D(X, \mathbb{R}(a))),\]
where \(a\) is the weight under the action of \(S/U_1 \cong \mathbb{G}_m\), and \(H^m_D(X, \mathbb{R}(a))\) is real Deligne cohomology.

Proof. The isomorphism \(\mathbb{G}_m = S/U_1\) allows us to regard \(O(\mathbb{G}_m)\) as an \(S\)-representation, and
\[\left(\mathbb{R} q_* j^* \tilde{A}^\bullet(X)\right)^U_1 \simeq \mathbb{R} \Gamma([C^\bullet/S], j^* \tilde{A}^\bullet(X) \otimes O(\mathbb{G}_m)).\]
Now, \(O(\mathbb{G}_m) = \mathbb{R}[s, s^{-1}]\), with \(s\) of type \((-1, -1)\), so \(O(\mathbb{G}_m) \cong \bigoplus_a (2\pi i)^{-a} \mathbb{R}(a)\), giving (by \(\S\).1.2.1)
\[\left(\mathbb{R} q_* j^* \tilde{A}^\bullet(X)\right)^U_1 \simeq \bigoplus_a (2\pi i)^{-a} \mathbb{R} \Gamma_{\text{tw}}(A^\bullet(X)(a)),\]
which is just real Deligne cohomology by [Bei].

We may also compare these cohomology groups with the groups considered in [Den1] and [Den2] for defining \(\Gamma\)-factors of smooth projective varieties at Archimedean places.

Proposition 2.14. The torsion-free quotient of the \(\mathbb{G}_m\)-equivariant \(A^1\)-module \(\left(\mathbb{R}^m j_* j^* \tilde{A}^\bullet(X)\right)^U_1\) is the Rees module of \(H^m(X, \mathbb{R})\) with respect to the filtration \(\gamma\).

Proof. The results of \(\S\).1.2 give a long exact sequence
\[
\ldots \to \left(\mathbb{R}^m j_* j^* \tilde{A}^\bullet(X)\right)^U_1 \to \bigoplus_{a \in \mathbb{Z}} (F^a H^m(X, \mathbb{C}) \oplus H^m(X, \mathbb{R})) \to \bigoplus_{a \in \mathbb{Z}} H^m(X, \mathbb{C}) \to \ldots,
\]
and hence
\[0 \to \bigoplus_{a \in \mathbb{Z}} F^a H^{m-1}(X, \mathbb{C}) \to \left(\mathbb{R}^m j_* j^* \tilde{A}^\bullet(X)\right)^U_1 \to \bigoplus_{a \in \mathbb{Z}} \gamma^a H^m(X, \mathbb{R}) \to 0.
\]
Since multiplication by the standard co-ordinate of \(A^1\) corresponds to the embedding \(F^{n+1} \hookrightarrow F^n\), the left-hand module is torsion, giving the required result. □

Remark 2.15. In [Den1] and [Con], \(\Gamma\)-factors of real varieties were also considered. If we let \(\sigma\) denote the de Rham conjugation of the associated complex variety, then we may replace \(S\) throughout this paper by \(S \rtimes \langle \sigma \rangle\), with \(\sigma\) acting on \(S(\mathbb{R})\) by \(\lambda \mapsto \bar{\lambda}\), and on \(\text{SL}_2\) by \((u v \ x y) \mapsto (a b \ x y)\) (i.e. conjugation by \((1 0 \ 0 1)\), noting that \(\sigma(d^c) = -d^c\). In that case, the cohomology group considered in [Den1] is the torsion-free quotient of \(\left(\mathbb{R}^m j_* j^* \tilde{A}^\bullet(X)\right)^U_1 \rtimes \langle \sigma \rangle\).

Lemma 2.16. There is a canonical \(S\)-equivariant quasi-isomorphism
\[R j_* j^* \tilde{A}^\bullet(X) \simeq H^\bullet(X, \mathbb{R}) \otimes_\mathbb{R} RO(C^\bullet)\]
of \(C\)-modules, where \(H^\bullet(X, \mathbb{R})\) is equipped with its standard \(S\)-action (the real Hodge structure), and \(RO(C^\bullet)\) is from Definition 1.21.

Proof. The natural inclusion \(H^\bullet \otimes O(C) \to \tilde{A}^\bullet\) of real harmonic forms gives rise to a morphism
\[H^\bullet \otimes \theta(C^\bullet) \to j^* \tilde{A}^\bullet\]
of \(S\)-equivariant cochain complexes over \(C^\bullet\), which is a quasi-isomorphism by Lemma 2.5, and hence
\[H^\bullet \otimes RO(C^\bullet) \simeq R j_* j^* \tilde{A}^\bullet,\]
as required. □
Corollary 2.17. As an $S$-representation, the summand of $H^n(C^*, j^*\tilde{A}^*) \otimes \mathbb{C}$ of type $(p, q)$ is given by
\[ \bigoplus_{p' \geq p, q' \geq q} \mathcal{H}^{p'q'} \oplus \bigoplus_{p' < p, q' < q} \mathcal{H}^{p'q'}. \]
In particular, this describes Deligne cohomology by taking invariants under complex conjugation when $p = q$.

Proof. This follows from Lemma 2.16, since $H^*(C, \mathcal{O}_{C^*}) \cong \bigoplus_n H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. \hfill \Box

2.4. Analogies with limit Hodge structures. If $\Delta$ is the open unit disc, and $f : X \to \Delta$ a proper surjective morphism of complex Kähler manifolds, smooth over the punctured disc $\Delta^*$, then Steenbrink ([Ste]) defined a limit mixed Hodge structure at 0. Take the universal covering space $\tilde{\Delta}^*$ of $\Delta^*$, and let $\tilde{X}^*$ := $X \times_{\Delta} \tilde{\Delta}^*$. Then the limit Hodge structure is defined as a Hodge structure on
\[ \lim_{t \to 0} H^*(X_t) := H^*(\tilde{X}^*) \]
[Ste] (2.19) gives an exact sequence
\[ \ldots \to H^i(X^*) \to H^i(\tilde{X}^*) \xrightarrow{N} H^i(\tilde{X}^*)(-1) \to \ldots, \]
where $N$ is the monodromy operator associated to the deck transformation of $\tilde{\Delta}^*$.

Since we are working with quasi-coherent cohomology, connected affine schemes replace contractible topological spaces, and Lemma 1.16 implies that we may then regard $\text{SL}_2$ as the universal cover of $C^*$, with deck transformations $\mathbb{G}_a$. We then substitute $C$ for $\Delta$, $C^*$ for $\Delta^*$ and $\text{SL}_2$ for $\tilde{\Delta}^*$. We also replace $\mathcal{O}_{X^*}$ with $j^*\tilde{A}^*(X)$, so $\mathcal{O}_{\tilde{X}^*}$ becomes $\text{row}^*j^*\tilde{A}^*(X)$.

This suggests that we should think of $\text{row}^*j^*\tilde{A}^*(X)$ (with its natural $S$-action) as the limit mixed Hodge structure at the Archimedean special fibre.

The derivation $N$ of Definition 1.20 then acts as the monodromy transformation. Since $N$ is of type $(-1, -1)$ with respect to the $S$-action, the weight decomposition given by the action of $\mathbb{G}_m \subset S$ splits the monodromy-weight filtration. The following result allows us to regard $\text{row}^*j^*\tilde{A}^*(X)$ as the limit Hodge structure at the special fibre corresponding to the Archimedean place.

Proposition 2.18. $Rj_*j^*\tilde{A}^*(X)$ is naturally isomorphic to the cone complex of the diagram $\text{row}^*j^*\tilde{A}^*(X) \xrightarrow{N} \text{row}^*j^*\tilde{A}^*(X)(-1)$, where $N$ is the locally nilpotent derivation given by differentiating the $\mathbb{G}_a$-action on $\text{SL}_2$.

Proof. This follows from the description of $RO(C^*)$ in §1.1.1. \hfill \Box

2.5. Archimedean cohomology. As in §2.4, the $S$-action gives a real (split) Hodge structure on the cohomology groups $H^q(\text{row}^*j^*\tilde{A}^*(X))$. In order to avoid confusion with the weight filtration on $j^*\tilde{A}^*(X)$, we will denote the associated weight filtration by $M$.

Corollary 2.19. There are canonical isomorphisms
\[ \text{gr}^{M}_{q+p} H^q(\text{row}^*j^*\tilde{A}^*(X)) \cong H^q(X, \mathbb{R}) \otimes \text{gr}^{M}_{q} O(\text{SL}_2) \]

Proof. This is an immediate consequence of the splitting in Corollary 2.9. \hfill \Box

Lemma 2.20. $\ker N \cap H^q(\text{row}^*j^*\tilde{A}^*(X)) \cong H^q(X, \mathbb{R}) \otimes \mathbb{R}[u, v]$, and $\text{coker} N \cap H^q(\text{row}^*j^*\tilde{A}^*(X)(-1)) \cong H^q(X, \mathbb{R}) \otimes \mathbb{R}[x, y][-1]$, for $N$ as in Definition 1.20.

Proof. This is a direct consequence of Corollary 2.19, since $\mathbb{R}[u, v] = \ker N|_{O(\text{SL}_2)}$ and the map $\mathbb{R}[x, y] \to \text{coker} N|_{O(\text{SL}_2)}$ is an isomorphism. \hfill \Box
Corollary 2.21. The $U_1$-invariant subspace $H^q(\text{row}_1^*j^*\hat{A}^*(X))^{U_1}$ is canonically isomorphic as an $N$-representation to the Archimedean cohomology group $H^q(X^*)$ defined in [Con].

Proof. First observe that $N$ acts on $O(\text{SL}_2)$ as the derivation $(0 \ 0)
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acting on the left, and that differentiating the action of $G_m \subset S$ on $O(\text{SL}_2)$ gives the derivation $(-1 \ 0)
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, also acting on the left. Therefore decomposition by the weights of the $G_m$-action gives a splitting of the filtration $M$ associated to the locally nilpotent operator $N$.

By Proposition 2.13 and [Con] Proposition 4.1, we know that Deligne cohomology arises as the cone of $N : H^* \to H^*(-1)$, for both cohomology theories $H^*$.

It now follows from Corollary 2.19 and Lemma 2.20 that the graded $N$-module $H^q(\text{row}_1^*j^*\hat{A}^*(X))^{U_1}$ shares all the properties of [Con] Corollary 4.4, Proposition 4.8 and Corollary 4.10, which combined are sufficient to determine the graded $N$-module $H^q(X^*)$ up to isomorphism. \hfill $\square$

2.5.1. Archimedean periods. We can construct the ring $\mathcal{S}$ without choosing co-ordinates as follows. Given any $\mathbb{R}$-algebra $A$, let $UA$ be the underlying $\mathbb{R}$-module. Then

$$\mathcal{S} = \mathbb{R}[UC] \otimes_{\mathbb{R}[U]} \mathbb{R},$$

For an explicit comparison, write $UC = \mathbb{R}[x_1 \oplus \mathbb{R}x_i]$, so the right-hand side is $\mathbb{R}[x_1, x_i] \otimes_{\mathbb{R}[x]} \mathbb{R} = \mathbb{R}[x_i]$.

The filtration $F$ is then given by powers of the augmentation ideal of the canonical map $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$, since the ideal is $(x_i - i)$. The derivation $N$ (from Definition 1.20) is differentiation $\mathcal{S} \to \Omega(\mathcal{S}/\mathbb{R})$, and $\Omega(\mathcal{S}/\mathbb{R}) = \mathcal{S} \otimes_{\mathbb{R}} (\mathbb{C}/\mathbb{R})$. There is also an action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathcal{S}$, determined by the action on $UC$, which corresponds to the generator $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acting $\mathbb{C}$-linearly as $\sigma(x) = -x$.

For $K = \mathbb{R}, \mathbb{C}$, we therefore define $B(K) := \mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$, with Frobenius $\phi$ acting as complex conjugation, and the Hodge filtration, $\text{Gal}(\mathbb{C}/K)$-action and $N$ defined as above. However, beware that $B(K)$ differs from the ring $B_{st}$ from [Den1].

We think of $B(K)$ as analogous to the ring $B_{st}$ of semi-stable periods (see e.g. [II]) used in crystalline cohomology. For a $p$-adic field $K$, recall that $B_{st}(K)$ is a $\mathbb{Q}_p$-algebra equipped with a $\text{Gal}(\bar{K}/K)$-action, a Frobenius-linear automorphism $\phi$, a decreasing filtration $F^iB_{st}(K)$, and a nilpotent derivation

$$N : B_{st}(K) \to B_{st}(K)(-1).$$

Thus we think of $X \otimes \mathbb{R}$ as being of semi-stable reduction at $\infty$, with nilpotent monodromy operator $N$ on the Archimedean fibre $X \otimes \mathcal{S}$. The comparison with $B_{st}$ is further justified by comparison with [Pri2], where the crystalline comparison from [Ols] is used to show that for a variety of good reduction, the $p$-adic étale homotopy type $(X_{\text{et}} \otimes \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B^\phi_{\text{cris}}$ is formal as Galois representation in homotopy types, and that formality preserves $N$ (since good reduction means that $N$ acts trivially on $(X_{\text{et}} \otimes \mathbb{Q}_p)$, while $B_{st}^\phi \cap \ker N = B_{\text{cris}}^\phi$).

In our case, $B(K)^\phi = \mathcal{S}$, so $(X \otimes \mathbb{R}) \otimes_{\mathbb{R}} B(K)^\phi$ is formal as a $\text{Gal}(\bar{K}/K)$-representation in non-abelian MHS. However, formality does not preserve $N$ since we only have semi-stable, not good, reduction at $\infty$.

In keeping with the philosophy of Arakelov theory, there should be a norm $\langle -, - \rangle$ on $B(K)$ to compensate for the finiteness of $\text{Gal}(\mathbb{C}/K)$. In order to ensure that $d^\ast = -[A, d^\ast]$, we define a semilinear involution $*$ on $O(\text{SL}_2) \otimes \mathbb{C}$ by $u^* = y, v^* = -x$. This corresponds to the involution $A \mapsto (A^t)^{-1}$ on $\text{SL}_2(\mathbb{C})$, so the most natural metric on $O(\text{SL}_2)$ comes via Haar measure on $SU_2(\mathbb{C})$ (the unit quaternions). However, the ring homomorphism $O(\text{SL}_2) \to B(K)$ (corresponding to $(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}) \leq \text{SL}_2$) is not then bounded for any possible norm on $B(K)$, suggesting that we should think of $\text{SL}_2$ as being more fundamental than $\mathcal{S}$.
Remark 2.22. If we wanted to work with $k$-MHS for a subfield $k \subset \mathbb{R}$, we could replace $S$ with the ring

$$S_k = k[U_k \mathbb{C}] \otimes_{k[U_k \mathbb{R}]} k,$$

where $U_k B$ is the $k$-module underlying a $k$-algebra $B$. The results of §§1.1.2, 1.3.1 then carry over, including Remark 1.26.

We can use this to find the analogue of Definitions 1.3 and 1.36 for $k$-MHS. First, note that $S$-equivariant $\mathrm{SL}_2$-modules are quasi-coherent sheaves on $[\mathrm{SL}_2/S]$, and that $[\mathrm{SL}_2/S] = [\mathrm{SSym}_2/\mathbb{G}_m]$, where $\mathrm{SSym}_2 \subset \mathrm{SL}_2$ consist of symmetric matrices, and the identification $\mathrm{SL}_2/\mathbb{U}_1 = \mathrm{SSym}_2$ is given by $A \mapsto AA^t$ (noting that $\mathbb{U}_1$ acts on $\mathrm{SL}_2$ as right multiplication by $\mathbb{O}_2$). The action of $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathrm{SSym}_2$ is conjugation by $(\frac{1}{2} \ 0 \ -\frac{1}{2})$, while the involution $*$ is given by $B \mapsto B^{-1}$.

Note that $\mathrm{SSym}_2 = \text{Spec} \xi(S, \gamma)$, for $\xi(S, \gamma)$ the Rees algebra with respect to the filtration $\gamma$.

Now, algebraic Hodge filtrations on real complexes correspond to $S$-equivariant $\mathbf{RO}(C^*)$-complexes (for $\mathbf{RO}(C^*)$ as in Definition 1.21). The identifications above and Remark 1.26 ensure that these are equivalent to $\mathbb{G}_m$-equivariant $\mathbf{RO}(C^*)^{U_1}$-complexes, where $\mathbf{RO}(C^*)^{U_1}$ is the cone of $\xi(S, \gamma) \xrightarrow{N} \xi(\Omega(S/\mathbb{R}), \gamma)$.

Therefore we could define algebraic Hodge filtrations on $k$-complexes to be $\mathbb{G}_m$-equivariant $\xi(\Omega^*(S_k/k), \gamma)$-complexes, where $\gamma^p V = V \cap F^p(V \otimes_k \mathbb{C})$.

To complete the analogy with étale and crystalline homotopy types, there should be a graded homotopy type $\xi(X_{st}, \gamma)$ over the generalised ring (in the sense of [Har]) $\xi(B, \gamma)$ of norm 1 Galois-invariant elements in the Rees algebra $\xi(B, \gamma) = O(\mathrm{SSym}_2)$, equipped with a monodromy operator $N$ and complex conjugation $\phi$.

The generalised tensor product $\xi(X_{st}, \gamma) \otimes_{\xi(B, \gamma)} \xi(B, \gamma)$ should then be equivalent to $\xi((X \otimes \mathbb{R}) \otimes_{\mathbb{R}} B, \gamma)$, and then we could recover the rational homotopy type as the subalgebra

$$(X \otimes \mathbb{R}) = \xi(X_{st} \otimes B, \gamma)^{\mathbb{G}_m, \phi=1,N=0} = F^0(X_{st} \otimes B)^{\phi=1,N=0}.$$

By comparison with étale cohomology, the existence of a Hodge filtration on $X \otimes \mathbb{R}$ seems anomalous, but it survives this process because (unlike the crystalline case) $F^0 B = B$.

3. Relative Malcev homotopy types

Given a reductive pro-algebraic group $R$, a topological space $X$, and a Zariski-dense morphism $\rho : \pi_1(X, x) \to R(k)$, [Pri3] introduced the Malcev homotopy type $X^\rho,\text{Mal}$ of $X$ relative to $\rho$. If $R = 1$ and $k = \mathbb{Q}$ (resp. $k = \mathbb{R}$), then this is just the rational (resp. real) homotopy type of $X$. If $R$ is the reductive pro-algebraic fundamental group of $X$, then $X^\rho,\text{Mal}$ is the schematic homotopy type of $X$.

Readers uninterested in non-nilpotent topological spaces or homotopy fibres can skip straight to §3.3, restricting to the case $R = 1$.

3.1. Review of pro-algebraic homotopy types. Here we give a summary of the results from [Pri3] which will be needed in this paper. Fix a field $k$ of characteristic zero.

Definition 3.1. Given a pro-algebraic group $G$ (i.e. an affine group scheme over $k$), define the reductive quotient $G^\text{red}$ of $G$ by

$$G^\text{red} = G/R_u(G),$$

where $R_u(G)$ is the pro-unipotent radical of $G$. Observe that $G^\text{red}$ is then a reductive pro-algebraic group, and that representations of $G^\text{red}$ correspond to semisimple representations of $G$.
Proposition 3.2. For any pro-algebraic group $G$, there is a Levi decomposition $G = G^{\text{red}} \ltimes \Ru(G)$, unique up to conjugation by $\Ru(G)$.

Proof. See [HM]. \hfill \square

3.1.1. The pointed pro-algebraic homotopy type of a topological space. We now recall the results from [Pri3] §1.

Definition 3.3. Let $\mathcal{S}_0$ be the category of reduced simplicial sets, i.e. simplicial sets with one vertex, and $s\text{Gp}$ the category of simplicial groups. Let $\text{Top}_0$ denote the category of pointed connected compactly generated Hausdorff topological spaces.

Note that there is a functor from $\text{Top}_0$ to $\mathcal{S}_0$ which sends $(X,x)$ to the simplicial set $\text{Sing}(X,x)_n := \{ f \in \text{Hom}_{\text{Top}}(|\Delta^n|, X) : f(v) = x \ \forall v \in \Delta^n_0 \}$.

this is a right Quillen equivalence, the corresponding left equivalence being geometric realisation. For the rest of this section, we will therefore restrict our attention to reduced simplicial sets.

As in [GJ] Ch.V, there is a classifying space functor $\bar{W} : s\text{Gp} \rightarrow \mathcal{S}_0$. This has a left adjoint $G : \mathcal{S}_0 \rightarrow s\text{Gp}$, Kan’s loop group functor ([Kan]), and these give a Quillen equivalence of model categories. In particular, $\pi_i(G(X)) = \pi_{i+1}(X)$. This allows us to study simplicial groups instead of pointed topological spaces.

Definition 3.4. Given a simplicial object $G_\bullet$ in the category of pro-algebraic groups, define $\pi_0(G_\bullet)$ to be the coequaliser

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\partial_1} & G_0 \\
\downarrow \partial_0 & & \downarrow \pi_0(G)
\end{array}
\]

in the category of pro-algebraic groups.

Definition 3.5. Define a pro-algebraic simplicial group to consist of a simplicial complex $G_\bullet$ of pro-algebraic groups, such that the maps $G_n \rightarrow \pi_0(G)$ are pro-unipotent extensions of pro-algebraic groups, i.e. $\ker(G_n \rightarrow \pi_0(G))$ is pro-unipotent. We denote the category of pro-algebraic simplicial groups by $s\text{AGp}$.

There is a forgetful functor $(k) : s\text{AGp} \rightarrow s\text{Gp}$, given by sending $G_\bullet$ to $G_\bullet(k)$. This functor clearly commutes with all limits, so has a left adjoint $G_\bullet \mapsto (G_\bullet)^{\text{alg}}$. We can describe $(G_\bullet)^{\text{alg}}$ explicitly. First let $(\pi_0(G))^{\text{alg}}$ be the pro-algebraic completion of the abstract group $\pi_0(G)$, then let $(G^{\text{alg}})_n$ be the relative Malcev completion (in the sense of [Hai4]) of the morphism

$G_n \rightarrow (\pi_0(G))^{\text{alg}}$.

In other words, $G_n \rightarrow (G^{\text{alg}})_n \xrightarrow{f} (\pi_0(G))^{\text{alg}}$ is the universal diagram with $f$ a pro-unipotent extension.

Proposition 3.6. The functors $(k)$ and $^{\text{alg}}$ give rise to a pair of adjoint functors

$\text{Ho}(s\text{Gp}) \xrightarrow{\bot} \text{Ho}(s\text{AGp})$

on the homotopy categories, with $(G^{\text{alg}})(X) = G(X)^{\text{alg}}$, for any $X \in \mathcal{S}_0$.

Proof. As in [Pri3] Proposition 1.36, it suffices to observe that $(k)$ preserves fibrations and trivial fibrations. \hfill \square
Definition 3.7. Given a reduced simplicial set (or equivalently a pointed, connected topological space) $(X, x)$, define the pro-algebraic homotopy type $(X, x)^\text{alg}$ of $(X, x)$ over $k$ to be the object

$$G(X, x)^\text{alg}$$

in $\text{Ho}(\text{sAGp})$. Define the pro-algebraic fundamental group by $\varpi_1(X, x) := \pi_0(G(X, x)^\text{alg})$. Note that $\pi_0(G^\text{alg})$ is the pro-algebraic completion of the group $\pi_0(G)$.

We then define the higher pro-algebraic homotopy groups $\varpi_n(X, x)$ by

$$\varpi_n(X, x) := \pi_{n-1}(G(X, x)^\text{alg}).$$

3.1.2. Pointed relative Malcev homotopy types.

Definition 3.8. Assume we have an abstract group $G$, a reductive pro-algebraic group $R$, and a representation $\rho : G \to R(k)$ which is Zariski-dense on morphisms. Define the Malcev completion $(G, \rho)^\text{Mal}$ (or $G^\rho, \text{Mal}$, or $G^{R, \text{Mal}}$) of $G$ relative to $\rho$ to be the universal diagram

$$G \to (G, \rho)^\text{Mal} \xrightarrow{\rho} R,$$

with $\rho$ a pro-unipotent extension, and the composition equal to $\rho$.

Note that finite-dimensional representations of $(G, \rho)^\text{Mal}$ correspond to $G$-representations which are Artinian extensions of $R$-representations.

Definition 3.9. Given a Zariski-dense morphism $\rho : \pi_1(X, x) \to R(k)$, let the Malcev completion $G(X, x)^{R, \text{Mal}}$ (or $G(X, x)^{R, \text{Mal}}$) of $(X, x)$ relative to $\rho$ be the pro-algebraic simplicial group $(G(X, x), \rho)^\text{Mal}$. Observe that the Malcev completion of $(X, x)$ relative to $(\pi_1(X, x))^{\text{red}}$ is just $G(X, x)^\text{alg}$. Let $\varpi_1(X^{R, \text{Mal}}, x) = \pi_0(G(X, x), \rho)^\text{Mal}$ and $\varpi_n(X^{R, \text{Mal}}, x) = \pi_{n-1}(G(X, x), \rho)^\text{Mal}$. Observe that $\varpi_1(X^{R, \text{Mal}}, x)$ is just the relative Malcev completion $\varpi_1(X, x)^{R, \text{Mal}}$ of $\rho : \pi_1(X, x) \to R(k)$.

Note that for any cosimplicial $G(X, x)^{R, \text{Mal}}$-representation (i.e. $O_G(X, x)^{R, \text{Mal}}$-comodule, and in particular any $\varpi_1(X^{R, \text{Mal}}, x)$-representation) $V$, the canonical map $\mathbb{H}^*(G(X, x)^{R, \text{Mal}}, V) \to \mathbb{H}^*(X, V)$ is an isomorphism.

Theorem 3.10. Take a fibration $f : (X, x) \to (Y, y)$ (of pointed connected topological spaces) with connected fibres, and set $F := f^{-1}(y)$. Take a Zariski-dense representation $\rho : \pi_1(X, x) \to R(k)$ to a reductive pro-algebraic group $R$, let $K$ be the closure of $\rho(\pi_1(F, x))$, and $T := R/K$. If the monodromy action of $\pi_1(Y, y)$ on $H^*(F, V)$ factors through $\varpi_1(Y, y)^T, \text{Mal}$ for all $K$-representations $V$, then $G(F, x)^{K, \text{Mal}}$ is the homotopy fibre of $G(X, x)^{R, \text{Mal}} \to G(Y, y)^{T, \text{Mal}}$.

In particular, there is a long exact sequence

$$\cdots \to \varpi_n(F, x)^{K, \text{Mal}} \to \varpi_n(X, x)^{R, \text{Mal}} \to \varpi_n(Y, y)^{T, \text{Mal}} \to \varpi_{n-1}(F, x)^{K, \text{Mal}} \to \cdots \to \varpi_1(F, x)^{K, \text{Mal}} \to \varpi_1(X, x)^{R, \text{Mal}} \to \varpi_1(Y, y)^{T, \text{Mal}} \to 1.$$

Proof. First observe that $\rho(\pi_1(F, x))$ is normal in $\pi_1(X, x)$, so $K$ is normal in $R$, and $T$ is therefore a reductive pro-algebraic group, so $(Y, y)^{T, \text{Mal}}$ is well-defined. Next, observe that since $K$ is normal in $R$, $R_n(K)$ is also normal in $R$, and is therefore $1$, ensuring that $K$ is reductive, so $(F, x)^{K, \text{Mal}}$ is also well-defined.

Consider the complex $O(R) \otimes_{O(T)} O(G(Y)^T, \text{Mal})$ of $(X, x)^{R, \text{Mal}}$-representations, regarded as a complex of sheaves on $X$. The Leray spectral sequence for $f$ with coefficients in this complex is is

$$E_2^{a, b} = \mathbb{H}^a(Y, \mathbb{H}^b(F, O(R)) \otimes_{O(T)} O(G(Y)^T, \text{Mal}))) \implies \mathbb{H}^{a+b}(X, O(R) \otimes_{O(T)} O(G(Y)^T, \text{Mal}))).$$
Regarding $O(R)$ as a $K$-representation, $H^*(F,O(R))$ is a $\varpi_1(Y,y)^{\text{Mal}}$-representation by hypothesis. Hence $H^*(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal}})$ is a cosimplicial $G(Y)^{\text{Mal}}$-representation, so
\[ \mathbb{H}^i(Y, H^j(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal}})) \cong \mathbb{H}^i(G(Y)^{\text{Mal}}, H^j(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal}})). \]

Now, $H^*(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal}})$ is a fibrant cosimplicial $G(Y)^{\text{Mal}}$-representation, so
\[
\mathbb{H}^i(G(Y)^{\text{Mal}}, H^j(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal}})) \\
\cong H^i(T, H^j(G(Y)^{\text{Mal}}, H^j(F,O(R)) \otimes_{O(T)} O(G(Y)^{\text{Mal})))) \\
= \begin{cases} 
H^j(F,O(R)) \otimes_{O(T)} k = H^j(F,O(K)) & i = 0, \\
0 & i \neq 0,
\end{cases}
\]

so
\[ \mathbb{H}^j(X,O(R) \otimes_{O(T)} O(G(Y)^{\text{Mal}})) \cong H^j(F,O(K)). \]

Now, let $F$ be the homotopy fibre of $G(X,x)^{R,\text{Mal}} \to G(Y,y)^{T,\text{Mal}}$, noting that there is a natural map $G(F,x)^{K,\text{Mal}} \to F$. We have
\[ \mathbb{H}^j(X,O(R) \otimes_{O(T)} O(G(Y)^{T,\text{Mal}})) = \mathbb{H}^j(G(X)^{R,\text{Mal}}, O(R) \otimes_{O(T)} O(G(Y)^{T,\text{Mal}})), \]
and a Leray-Serre spectral sequence
\[ \mathbb{H}^i(G(Y)^{T,\text{Mal}}, H^j(F,O(R)) \otimes_{O(T)} O(G(Y)^{T,\text{Mal}})) \Rightarrow \mathbb{H}^{i+j}(G(X)^{R,\text{Mal}}, O(R) \otimes_{O(T)} O(G(Y)^{T,\text{Mal}})). \]

The reasoning above adapts to show that this spectral sequence also collapses, yielding
\[ H^j(F,O(K)) = \mathbb{H}^j(X,O(R) \otimes_{O(T)} O(G(Y)^{T,\text{Mal}})). \]

We have therefore shown that the map $G(F,x)^{K,\text{Mal}} \to F$ gives an isomorphism
\[ H^*(F,O(K)) \to H^*(G(F,x)^{K,\text{Mal}}, O(K)), \]
and hence isomorphisms $H^*(F,V) \to H^*(G(F,x)^{K,\text{Mal}}, V)$ for all $K$-representations $V$. Since this is a morphism of simplicial pro-unipotent extensions of $K$, [Pri3] Corollary 1.55 implies that $G(F,x)^{K,\text{Mal}} \to F$ is a weak equivalence. \hfill $\square$

A special case of Theorem 3.10 has appeared in [KPT1] Proposition 4.20, when $F$ is simply connected and of finite type, and $T = \varpi_1(Y,y)^{\text{red}}$.

**Corollary 3.11.** Given a fibration $f : (X,x) \to (Y,y)$ with connected fibres, assume that the fibre $F := f^{-1}(y)$ has finite-dimensional cohomology groups $H^i(F,k)$ and let $R$ be the reductive quotient of the Zariski closure of the homomorphism $\pi_1(Y,y) \to \prod_i \text{GL}(H^i(F,k))$. Then the Malcev homotopy type $(F \otimes k,x)$ is the homotopy fibre of
\[ G(X,x)^{R,\text{Mal}} \to G(Y,y)^{R,\text{Mal}}. \]

**Proof.** This is just Theorem 3.10, with $R = T$ and $K = 1$. \hfill $\square$

**Remark 3.12.** Beware that even when $Y$ is a $K(\pi,1)$, the relative completion $Y^{R,\text{Mal}}$ need not be so. For instance, [Hai3] and [Hai1] are concerned with studying the exact sequence $1 \to T_g \to \Gamma_g \to \text{Sp}_g(\mathbb{Z}) \to 1$, where $\Gamma_g$ is the mapping class group and $T_g$ the Torelli group. Taking $R = \text{Sp}_g/\mathbb{Q}$, we get $H^1(\text{Sp}_g(\mathbb{Z}),O(R)) = 0$, but $H^2(\text{Sp}_g(\mathbb{Z}),O(R)) \cong \mathbb{Q}$. Therefore $\varpi_1(B\text{Sp}_g(\mathbb{Z}))^{R,\text{Mal}} = R$, but the Hurewicz theorem gives $\varpi_2(B\text{Sp}_g(\mathbb{Z}))^{R,\text{Mal}} = \mathbb{Q}$. Thus the long exact sequence for homotopy has
\[ \mathbb{Q} \to T_g \otimes \mathbb{Q} \to \Gamma_g^{R,\text{Mal}} \to \text{Sp}_g(\mathbb{Z})^{R,\text{Mal}} \to 1. \]

This is consistent with [Hai3] Proposition 7.1 and [Hai1] Theorem 3.4, which show that $T_g \otimes \mathbb{Q} \to \Gamma_g^{R,\text{Mal}}$ is a central extension by $\mathbb{Q}$. \hfill $\square$
Definition 3.13. Define a group $\Gamma$ to be good with respect to a Zariski-dense representation $\rho : \Gamma \to R(k)$ to a reductive pro-algebraic group if the homotopy groups $\pi_n(B\Gamma)^{R,\text{Mal}}$ are 0 for all $n \geq 2$.

By [Pri3] Examples 3.20, the fundamental group of a compact Riemann surface is algebraically good with respect to all representations, as are finite groups, free groups and finitely generated nilpotent groups.

Lemma 3.14. A group $\Gamma$ is good relative to $\rho : \Gamma \to R(k)$ if and only if the map

$$H^n(\Gamma^{\rho,\text{Mal}}, V) \to H^n(\Gamma, V)$$

is an isomorphism for all $n$ and all finite-dimensional $R$-representations $V$.

Proof. This follows by looking at the map $f : G(\eta \Gamma)^{R,\text{Mal}} \to \Gamma^{R,\text{Mal}}$ of simplicial pro-algebraic groups, which is a weak equivalence if and only if $\pi_n(B\Gamma)^{R,\text{Mal}} = 0$ for all $n \geq 2$. By [Pri3] Corollary 1.55, $f$ is a weak equivalence if and only if the morphisms

$$H^*(\Gamma^{R,\text{Mal}}, V) \to H^*(G(\eta \Gamma)^{R,\text{Mal}}, V)$$

are isomorphisms for all $R$-representations $V$. Since $H^*(G(\eta \Gamma)^{R,\text{Mal}}, V) = H^*(\eta \Gamma, V) = H^*(\Gamma, V)$, the result follows.

Lemma 3.15. Assume that $\Gamma$ is finitely presented, with $H^n(\Gamma, -)$ commuting with filtered direct limits of $\Gamma^{\rho,\text{Mal}}$-representations, and $H^n(\Gamma, V)$ finite-dimensional for all finite-dimensional $\Gamma^{\rho,\text{Mal}}$-representations $V$.

Then $\Gamma$ is good with respect to $\rho$ if and only if for any finite-dimensional $\Gamma^{\rho,\text{Mal}}$-representation $V$, and $\alpha \in H^n(\Gamma, V)$, there exists an injection $f : V \to W_\alpha$ of finite-dimensional $\Gamma^{\rho,\text{Mal}}$-representations, with $f(\alpha) = 0 \in H^n(\Gamma, W_\alpha)$.

Proof. The proof of [KPT1] Lemma 4.15 adapts to this generality.

Theorem 3.16. If $(X, x)$ is a pointed topological space with fundamental group $\Gamma$, equipped with a Zariski-dense representation $\rho : \Gamma \to R(k)$ to a reductive pro-algebraic group for which:

1. $\Gamma$ is good with respect to $\rho$,
2. $\pi_n(X, x)$ is of finite rank for all $n > 1$, and
3. the $\Gamma$-representation $\pi_n(X, x) \otimes Z k$ is an extension of $R$-representations (i.e. a $\Gamma^{\rho,\text{Mal}}$-representation),

then the canonical map

$$\pi_n(X, x) \otimes Z k \to \pi_n(X^{\rho,\text{Mal}}, x)$$

is an isomorphism for all $n > 1$.

Proof. This is [Pri3] Theorem 3.21. Alternatively, we could apply Theorem 3.10 to the fibration $(X, x) \to B\Gamma$.

3.2. Equivalent formulations.

Definition 3.17. Define $\mathcal{E}(R)$ to be the full subcategory of $\text{AGp} \downarrow R$ consisting of those morphisms $\rho : G \to R$ of pro-algebraic groups which are pro-unipotent extensions. Similarly, define $s\mathcal{E}(R)$ to consist of the pro-unipotent extensions in $s\text{AGp} \downarrow R$, and $\text{Ho}(s\mathcal{E}(R))$ to be full subcategory of $\text{Ho}(s\text{AGp})$ on objects $s\mathcal{E}(R)$.

Definition 3.18. Let $\text{cAlg}(R)_*$ be the category of of $R$-representations in cosimplicial $\mathbb{R}$-algebras, equipped with an augmentation to the structure sheaf $O(R)$ of $R$. A weak equivalence in $\text{cAlg}(R)_*$ is a map which induces isomorphisms on cohomology groups. We denote by $\text{Ho}(\text{cAlg}(R)_*)$ the localisation of $\text{cAlg}(R)_*$ at weak equivalences. Denote the respective opposite categories by $s\text{Aff}(R)_* = R \downarrow s\text{Aff}(R)$ and $\text{Ho}(s\text{Aff}(R)_*)$. 

Definition 3.19. Define $DGA_{\mathcal{G}}(R)_*$ to be the category of $R$-representations in non-negatively graded cochain $\mathbb{R}$-algebras, equipped with an augmentation to $O(R)$. Here, a cochain algebra is a cochain complex $A = \bigoplus_{i \in \mathbb{N}_0} A^i$ over $k$, equipped with a graded-commutative associative product $A^i \times A^j \to A^{i+j}$, and unit $1 \in A^0$.

A weak equivalence in $DGA_{\mathcal{G}}(R)_*$ is a map which induces isomorphisms on cohomology groups. We denote by $Ho(DGA_{\mathcal{G}}(R)_*)$ the localisation of $DGA_{\mathcal{G}}(R)_*$ at weak equivalences. Define $dg\text{Aff}(R)_*$ to be the category opposite to $DGA_{\mathcal{G}}(R)_*$, and $Ho(dg\text{Aff}(R)_*)$ opposite to $Ho(DGA_{\mathcal{G}}(R)_*)$.

Let $DGA_{\mathcal{G}}(R)_0$ be the full subcategory of $DGA_{\mathcal{G}}(R)_*$ whose objects $A$ satisfy $H^0(A) = k$. Let $Ho(DGA_{\mathcal{G}}(R)_*)_0$ be the full subcategory of $Ho(DGA_{\mathcal{G}}(R)_*)$ on the objects of $DGA_{\mathcal{G}}(R)_0$. Let $dg\text{Aff}(R)_0$ and $Ho(dg\text{Aff}(R)_*)_0$ be the opposite categories to $DGA_{\mathcal{G}}(R)_0$ and $Ho(DGA_{\mathcal{G}}(R)_*)_0$, respectively. Given $A \in c\text{Alg}(R)_*$, write $\text{Spec} A \in s\text{Aff}(R)_*$ for the corresponding object of the opposite category.

Definition 3.20. Define $dg\hat{\mathcal{N}}(R)$ to be the category of $R$-representations in finite-dimensional nilpotent non-negatively graded chain Lie algebras. Let $dg\hat{\mathcal{N}}(R)_*$ be the category of pro-objects in the Artinian category $dg\hat{\mathcal{N}}(R)$.

Let $dg\hat{\mathcal{P}}(R)$ be the category with the same objects as $dg\hat{\mathcal{N}}(R)_*$, and morphisms given by

$$\text{Hom}_{dg\hat{\mathcal{P}}(R)}(\mathfrak{g}, \mathfrak{h}) = \exp(H_0\mathfrak{h}) \times \exp(h_0^R) \text{Hom}_{Ho(dg\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}),$$

where $h_0^R$ (the Lie subalgebra of $R$-invariants in $h_0$) acts by conjugation on the set of homomorphisms. Composition of morphisms is given by $(u, f) \circ (v, g) = (u \circ f(v), f \circ g)$.

Definition 3.21. Let $s\hat{\mathcal{N}}(R)$ be the category of simplicial objects in $\hat{\mathcal{N}}(R)$, and let $s\hat{\mathcal{P}}(R)$ be the category with the same objects as $s\hat{\mathcal{N}}(R)$, and morphisms given by

$$\text{Hom}_{s\hat{\mathcal{P}}(R)}(\mathfrak{g}, \mathfrak{h}) = \exp(\pi_0\mathfrak{h}) \times \exp(h_0^R) \text{Hom}_{Ho(s\hat{\mathcal{N}}(R))}(\mathfrak{g}, \mathfrak{h}),$$

where composition of morphisms is given by $(u, f) \circ (v, g) = (u \circ f(v), f \circ g)$.

Definition 3.22. Define a functor $\hat{W} : dg\hat{\mathcal{N}}(R) \to dg\text{Aff}(R)$ by $O(\hat{W}\mathfrak{g}) := \text{Symm}(\mathfrak{g}^\vee[-1])$ the graded polynomial ring on generators $\mathfrak{g}^\vee[-1]$, with derivation defined on generators by $d_\mathfrak{g} + \Delta$, for $\Delta$ the Lie cobracket on $\mathfrak{g}^\vee$.

$\hat{W}$ has a left adjoint $G$, given by writing $\sigma A^\vee[1]$ for the brutal truncation (in non-negative degrees) of $A^\vee[1]$, and setting

$$G(A) = \text{Lie}(\sigma A^\vee[1]),$$

the free graded Lie algebra, with differential similarly defined on generators by $d_A + \Delta$, with $\Delta$ here being the coproduct on $A^\vee$.

We may then define $\hat{G} : Ho(dg\text{Aff}(R)_*) \to dg\hat{\mathcal{P}}(R)$ on objects by choosing, for $A \in DGA_{\mathcal{G}}(R)_0$, a quasi-isomorphism $B \to A$ with $B^0 = k$ (for an explicit construction of $B$, see [Pri3] Remark 4.35) and setting $\hat{G}(A) := G(B)$.

Definition 3.23. Given a cochain algebra $A \in DGA_{\mathcal{G}}(R)$, and a chain Lie algebra $\mathfrak{g} \in dg\hat{\mathcal{N}}(R)$, define the Maurer-Cartan space by

$$\text{MC}(A \hat{\otimes}^R \mathfrak{g}) := \{ \omega \in \prod_n A^{n+1} \hat{\otimes}^R \mathfrak{g}_n \mid d\omega + \frac{1}{2}[\omega, \omega] = 0 \},$$

where, for an inverse system $\{ \rho \}$, $\{ \rho \} \hat{\otimes} A := \varprojlim \rho^{-1}(\rho \otimes A)$, and $\{ \rho \} \hat{\otimes}^R A$ consists of $R$-invariants in this. Note that

$$\text{Hom}_{dg\text{Aff}(R)}(\text{Spec} A, \hat{W}\mathfrak{g}) \cong \text{MC}(A \hat{\otimes}^R \mathfrak{g}).$$
Definition 3.24. Given $A \in \text{DGAlg}(R)$ and $g \in \text{dgN}(R)$, we define the gauge group by

$$\text{Gg}(A \hat{\otimes}^R g) := \exp(\prod_n A^n \hat{\otimes}^R g_n).$$

Define a gauge action of $\text{Gg}(A \hat{\otimes}^R g)$ on $\text{MC}(A \hat{\otimes}^R g)$ by

$$g(\omega) := g \cdot \omega \cdot g^{-1} - (dg) \cdot g^{-1}.$$

Here, $a \cdot b$ denotes multiplication in the universal enveloping algebra $U(A \hat{\otimes}^R g)$ of the differential graded Lie algebra (DGLA) $A \hat{\otimes}^R g$. That $g(\omega) \in \text{MC}(A \hat{\otimes}^R g)$ is a standard calculation (see [Kon] or [Man]).

Proposition 3.25. For $A \in \text{DGAlg}(R)_*$ and $g \in \text{dgN}(R)$,

$$\text{Hom}_{\text{Ho}(\text{dgAff}(R)_*)}(\text{Spec } A, Wg) \cong \exp(H_0 g) \times \text{Gg}(A \hat{\otimes}^R g) \text{MC}(A \hat{\otimes}^R g),$$

where $Wg \in \text{dgAff}_*$ is the composition $R \to \text{Spec } k \to Wg$, and the morphism $\text{Gg}(A \hat{\otimes}^R g) \to \exp(H_0 g)$ factors through $\text{Gg}(O(R) \hat{\otimes}^R g) = g_0$.

Proof. The derived Hom space $R\text{Hom}_{\text{dgAff}(R)_*}(\text{Spec } A, Wg)$ is the homotopy fibre of

$$R\text{Hom}_{\text{dgAff}(R)}(\text{ Spec } A, Wg) \to R\text{Hom}_{\text{dgAff}(R)}(R, Wg),$$

over the unique element $0$ of $\text{MC}(O(R) \hat{\otimes}^R g)$. For a morphism $f : X \to Y$ of simplicial sets (or topological spaces), path components $\pi_0 F$ of the homotopy fibre over $0 \in Y$ are given by pairs $(x, \gamma)$, for $x \in X$ and $\gamma$ a homotopy class of paths from $0$ to $fx$, modulo the equivalence relation $(x, \gamma) \sim (x', \gamma')$ if there exists a path $\delta : x \to x'$ in $X$ with $\gamma * f \delta = \gamma'$. If $Y$ has a unique vertex $0$, this reduces to pairs $(x, \gamma)$, for $x \in X$ and $\gamma \in \pi_1(Y, 0)$, with $\delta$ acting as before.

Now, we can define an object $Vg \in \text{dgAff}(R)$ by

$$\text{Hom}_{\text{dgAff}(R)}(\text{Spec } A, Vg) \cong \text{Gg}(A \hat{\otimes}^R g),$$

and by [Pri3] Lemma 4.33, $Vg \times Wg$ is a path object for $Wg$ in $\text{dgAff}(R)$ via the maps

$$Wg \xrightarrow{(\text{id}, 1)} Wg \times Vg \xrightarrow{\phi} Wg,$$

where $\phi$ is the gauge action.

Thus the loop object $\Omega(Wg, 0)$ for $0 \in \text{MC}(A \hat{\otimes}^R g)$ is given by

$$\text{Hom}_{\text{dgAff}(R)}(\text{Spec } A, \Omega(Wg, 0)) = \{g \in \text{Gg}(A \hat{\otimes}^R g) : g(0) = 0\} = \exp(\ker d \cap \prod_n A^n \hat{\otimes}^R g_n).$$

Hence

$$\pi_1 R\text{Hom}_{\text{dgAff}(R)}(\text{Spec } A, \Omega(Wg, 0)) \cong H^{-i}(\prod_n A^n \hat{\otimes}^R g_n),$$

and in particular,

$$\pi_1(\text{RHom}_{\text{dgAff}(R)})(R, Wg, 0) = \pi_0 R\text{Hom}_{\text{dgAff}(R)}(\text{Spec } A, \Omega(Wg, 0)) \cong \exp(H_0 g).$$

This gives us a description of

$$\text{Hom}_{\text{Ho}(\text{dgAff}(R)_*)}(\text{Spec } A, Wg) = \pi_0 R\text{Hom}_{\text{dgAff}(R)}(\text{Spec } A, Wg)$$

as consisting of pairs $(x, \gamma)$ for $x \in \text{MC}(A \hat{\otimes}^R g)$ and $\gamma \in \exp(H_0 g)$, modulo the equivalence $(x, \gamma) \sim (\delta(x), \delta * \gamma)$ for $\delta \in \text{Gg}(A \hat{\otimes}^R g)$. In other words,

$$\text{Hom}_{\text{Ho}(\text{dgAff}(R)_*)}(\text{Spec } A, Wg) \cong \text{MC}(A \hat{\otimes}^R g) \times \text{Gg}(A \hat{\otimes}^R g) \exp(H_0 g),$$

as required. \qed
Corollary 3.26. \( W \) defines a functor \( \bar{W} : dg\mathcal{P}(R) \to \operatorname{Ho}(dg\text{Aff}(R)_*) \).

**Proof.** On objects, we map \( g \) to \( \bar{W}g \). Given a morphism \( f : g \to h \) in \( dg\mathcal{N}(R) \) and \( h \in \mathcal{H}_0h \), we can define an element \( \bar{W}(h, f) \) of
\[
\operatorname{Hom}_{\operatorname{Ho}(dg\text{Aff}(R)_*)}(\bar{W}g, \bar{W}h)
\]
by \( \[(\exp(h), \bar{W}f)\] \in \exp(\mathcal{H}_0g) \times^{Gg(\bar{A}^Rg)} \operatorname{Hom}_{\operatorname{Ho}(dg\text{Aff}(R))}(\bar{W}g, \bar{W}h) \).  

If \( f \) is a weak equivalence then \( \bar{W}(0, f) \) is a weak equivalence in \( dg\text{Aff}(R)_* \), which implies that \( \bar{W} \) must descend to a functor
\[
\bar{W} : dg\mathcal{P}(R) \to \operatorname{Ho}(dg\text{Aff}(R)_*),
\]
since \( \bar{W}(h, f) \) is a function of the homotopy class of \( f \). \( \square \)

**Definition 3.27.** Recall that the Thom-Sullivan (or Thom-Whitney) functor \( \operatorname{Th} \) from cosimplicial algebras to DG algebras is defined as follows. Let \( \Omega(|\Delta^n|) \) be the DG algebra of rational polynomial forms on the \( n \)-simplex, so
\[
\Omega(|\Delta^n|) = \mathbb{Q}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(1 - \sum_i t_i),
\]
for \( t_i \) of degree 0. The usual face and degeneracy maps for simplices yield \( \partial_i : \Omega(|\Delta^n|) \to \Omega(|\Delta^{n-1}|) \) and \( \sigma_i : \Omega(|\Delta^n|) \to \Omega(|\Delta^{n-1}|) \), giving a simplicial complex of DGAs. Given a cosimplicial algebra \( A \), we then set
\[
\operatorname{Th}(A) := \{ a \in \prod_n A^n \otimes \Omega(|\Delta^n|) : \partial_i a_n = \partial_i a_{n+1}, \sigma_j a_n = \sigma_j a_{n-1} \forall i, j \}.
\]

**Theorem 3.28.** We have the following diagram of equivalences of categories:
\[
\begin{array}{ccc}
\operatorname{Ho}(dg\text{Aff}(R)_*)_0 & \xrightarrow{\text{Spec } D} & \operatorname{Ho}(s\text{Aff}(R)_*)_0 \\
\bar{W} & \downarrow G & \bar{W} \exp \\
dg\mathcal{P}(R) & \xleftarrow{N} & s\mathcal{P}(R) & \xrightarrow{R \times \exp(-)} \operatorname{Ho}_s(s\mathcal{E}(R)),
\end{array}
\]
where \( N \) denotes Dold-Kan normalisation ([Pri3] Definition 4.11), \( D \) denormalisation ([Pri3] Definition 4.26), and \( \bar{W} \exp(g) \) is the classifying space of the simplicial group \( \exp(g) \). A homotopy inverse to \( D \) is given by the functor \( \operatorname{Th} \) of Thom-Sullivan cochains.

**Proof.** First, [Pri3] Propositions 4.27 and 4.12 ensure that \( \text{Spec } D \) and \( N \) are equivalences, while [Pri3] Theorem 4.39 implies that \( (\text{Spec } D) \circ \bar{W} = \bar{W} \circ N \). [HS] 4.1 shows that \( D \) and \( \operatorname{Th} \) are homotopy inverses. We now adapt the proof of [Pri3] Corollary 4.41.

The functor \( R \times \exp : s\mathcal{P}(R) \to \operatorname{Ho}_s(s\mathcal{E}(R)) \) maps \( g \) to the simplicial pro-algebraic group given in level \( n \) by \( R \times \exp(g_n) \). Given a morphism \( (f, u) \in \exp(\pi_0h) \times^{\exp(h)} \operatorname{Hom}_{\operatorname{Ho}(s\mathcal{N}(R))}(g, h) \), lift \( u \) to \( \bar{u} \in \exp(h_0) \), and construct a morphism
\[
\operatorname{ad}_{\bar{u}} \circ (R \times \exp(f))
\]
in \( s\mathcal{E}(R) \), were \( \operatorname{ad}_g(x) = gxg^{-1} \). Another choice \( \bar{u}' \) of \( \bar{u} \) amounts to giving \( v \in \exp(h_1) \) with \( \partial_0 v = \bar{u} \) and \( \partial_1 v = \bar{u}' \). Thus \( \operatorname{ad}_v \circ (R \times \exp(f)) \) gives a homotopy from \( \operatorname{ad}_{\bar{u}} \circ (R \times \exp(f)) \) to \( \operatorname{ad}_{\bar{u}'} \circ (R \times \exp(f)) \). This means that \( R \times \exp(-) : s\mathcal{P}(R) \to \operatorname{Ho}_s(s\mathcal{E}(R)) \) is a well-defined functor.

The existence of Levi decompositions ensures that \( R \times \exp(-) \) is essentially surjective and full (since every morphism in \( s\mathcal{E}(R) \) is the composite of an inner automorphism and a morphism preserving Levi decompositions). Since the choice of inner automorphism on \( R \times U \) is unique up to \( R \)-invariants \( U^R, R \times \exp(-) : s\mathcal{P}(R) \to \operatorname{Ho}_s(s\mathcal{E}(R)) \) is an equivalence (see [Pri3] Proposition 3.15 for a similar result).
Now, by [Pri3] Proposition 3.48, there is a canonical quasi-isomorphism \( \bar{W}G(X) \to X \), for all \( X \in dg\text{Aff}(R) \) with \( X_0 = \text{Spec} \, k \), and hence for all \( X \in dg\text{Aff}(R)_* \) with \( X_0 = \text{Spec} \, k \). With reasoning as in Definition 3.22, this means that
\[
\bar{W} : \text{dgP}(R) \to \text{Ho}(\text{dgAff}(R)_*)_0
\]
is essentially surjective, with \( G(Y) \) in the essential pre-image of \( Y \), although it does not guarantee that we can define \( G \) consistently on morphisms.

To establish that \( \bar{W} \) is full and faithful, it will suffice to show that for all \( A \in D\text{GAlg}(R) \) with \( A^0 = k \), the transformation
\[
\text{Hom}_{\text{dgP}(R)}(G(A), h) \to \text{Hom}_{\text{Ho}(\text{dgAff}(R)_*)}(\text{Spec} \, A, \bar{W}h)
\]
is an isomorphism. For \( A = k \), this is certainly true, since in both cases we get \( \exp(H_0h)/\exp(H_0h)^R \) for both Hom-sets (using Proposition 3.25). The morphism \( k \to A \) gives surjective maps from both Hom-sets above to \( \exp(H_0h)/\exp(H_0h)^R \), and by Proposition 3.25, the map on any fibre is just
\[
\text{Hom}_{\text{Ho}(\text{dgN}(R))}(G(A), h)/\exp(ker(h^R \to H_0h^R)) \to \text{MC}(A \hat{\otimes}^R h)/\ker(Gg(A \hat{\otimes}^R h) \to \exp(H_0h^R)).
\]

Now, \( G(A) \) is a hull for both functors on \( \text{dgN}(R) \) (in the sense of [Pri3] Proposition 3.43), so by the argument of [Pri3] Proposition 3.47, it suffices to show that \( \theta \) is an isomorphism whenever \( h \in N(R) \) (i.e. whenever \( h_i = 0 \) for all \( i > 0 \)). In that case,
\[
\text{Hom}_{\text{Ho}(\text{dgN}(R))}(G(A), h) = \text{Hom}_{\text{dgN}(R)}(G(A), h) = \text{MC}(A \hat{\otimes}^R h),
\]
and
\[
Gg(A \hat{\otimes}^R h) = \exp(A^0 \hat{\otimes}^R h) = \exp(k \hat{\otimes}^R h) = \exp(h^R) = \exp(h_0^R),
\]
so \( \theta \) is indeed an isomorphism. Hence \( \bar{W} \) is an equivalence, with quasi-inverse \( G \) on objects. \( \square \)

**Remark 3.29.** If we take a set \( T \) of points in \( X \), then the groupoid \( \Gamma : = T \times_{\{X\}} \pi_f X \) has objects \( T \), with morphisms \( \Gamma(x, y) \) corresponding to homotopy classes of paths from \( x \) to \( y \) in \( X \). If \( T = \{x\} \), note that \( \Gamma \) is just \( \pi_1(X, x) \).

Take a reductive pro-algebraic groupoid \( R \) (as in [Pri3] §2) on objects \( T \), and a morphism \( \rho : \Gamma \to R \) preserving \( T \). The relative Malcev completion \( G(X; T)^\rho,\text{Mal} \) is then a pro-unipotent extension of \( R \) (as a simplicial pro-algebraic groupoid — see [Pri3] §2.4). Then \( \varpi_1(X; T)^\rho,\text{Mal} \) is a groupoid on objects \( T \), with \( \varpi_1(X; T)^\rho,\text{Mal}(x, x) = \varpi_1(X, x)^\rho,\text{Mal} \). Likewise, \( \varpi_n(X; T)^\rho,\text{Mal} : = \pi_n^{-1}G(X; T)^\rho,\text{Mal} \) is a \( \varpi_1(X; T)^\rho,\text{Mal} \)-representation, with \( \varpi_1(X; T)^\rho,\text{Mal}(x) = \varpi_1(X, x)^\rho,\text{Mal} \). Here, \( \rho_x : \pi_1(X, x) \to R(x, x) \) is defined by restricting \( \rho \) to \( x \in T \).

If we set \( \text{dgAff}(R)_* : = (\coprod_{x \in T} R(x, -)) \downarrow \text{dgAff}(R) \) and \( \text{sAff}(R)_* : = (\coprod_{x \in T} R(x, -)) \downarrow \text{sAff}(R) \), where \( R(x, -) \) is the \( R \)-representation \( y \mapsto R(x, y) \), then Theorem 3.28 adapts to give equivalences
\[
\begin{array}{c}
\text{Ho}(\text{dgAff}(R)_*)_0 \quad \text{Spec} \, D
\\
\text{Ho}(\text{sAff}(R)_*)_0 \quad \bar{W}
\\
\text{dgP}(R) \quad \text{N}
\\
\text{sP}(R) \quad \text{R} \times \exp(-)
\\
\text{Ho}_* (\text{sE}(R)),
\end{array}
\]
where \( \text{Ho}_* (\text{sE}(R)) \) is the full subcategory of the homotopy category \( \text{Ho}(T \downarrow \text{sAGpd} \downarrow R) \) (of simplicial pro-algebraic groupoids under \( T \) and over \( R \)) whose objects are pro-unipotent extensions of \( R \). The objects of \( \text{sP}(R) \) are \( R \)-representations in \( s\mathcal{N} \), with morphisms given by
\[
\text{Hom}_{\text{sP}(R)}(h, k) = (\prod_{x \in T} \exp(\pi_0h(x))) \times \exp(h_0^R)^R \text{Hom}_{\text{Ho}(\text{sN}(R))}(h, k),
\]
where $h_0^R$ is the Lie algebra $\text{Hom}_R(\mathbb{R}, h_0)$ (with $\mathbb{R}$ regarded as a constant $R$-representation). The category $dg\mathcal{P}(R)$ is defined similarly.

**Definition 3.30.** Recall that $O(R)$ has the natural structure of an $R \times R$-representation, with the $R$-actions given by left and right multiplication.

**Definition 3.31.** Let $\mathbb{B}_\rho$ be the $R$-torsor on $X$ corresponding to the representation $\rho : \pi_1(X, x) \to R(\mathbb{R})$, and let $O(\mathbb{B}_\rho)$ be the $R$-representation $\mathbb{B}_\rho \times^R O(R)$ in local systems of $\mathbb{R}$-algebras on $X$ (with the $R$-representation structure given by the right action on $O(R)$).

**Proposition 3.32.** Under the equivalences of Theorem 3.28, the relative Malcev homotopy type $G(X, x)^{\rho, \text{Mal}}$ of a pointed topological space $(X, x)$ corresponds to the complex

$$(C^\bullet(X, O(\mathbb{B}_\rho))) \xrightarrow{x^*} O(R)) \in c\text{Alg}(R)_0^*$$

of $O(\mathbb{B}_\rho)$-valued chains on $X$.

**Proof.** This is essentially [Pri3] Theorem 3.55. □

**Definition 3.33.** Given a manifold $X$, denote the sheaf of real $C^\infty$ $n$-forms on $X$ by $\mathcal{A}^n$. Given a real sheaf $\mathcal{F}$ on $X$, write

$$A^n(X, \mathcal{F}) := \Gamma(X, \mathcal{F} \otimes_{\mathbb{R}} \mathcal{A}^n).$$

**Proposition 3.34.** If $k = \mathbb{R}$, then the relative Malcev homotopy type of a pointed manifold $(X, x)$ relative to $\rho : \pi_1(X, x) \to R(\mathbb{R})$ is given in $DGA_{\text{Alg}}(R)_*$ by $A^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{x^*} O(R)$.

**Proof.** This is essentially [Pri3] Proposition 4.50. □

**Remark 3.35.** If we take a set $T$ of points in $X$ and $\rho$ as in Remark 3.29, then Proposition 3.32 adapts to say that the relative Malcev homotopy type $G(X; T)^{\rho, \text{Mal}}$ corresponds to the complex

$$(C^\bullet(X, O(\mathbb{B}_\rho))) \xrightarrow{\prod_{x \in T} x^*} \prod_{x \in T} O(R)(x, -)) \in c\text{Alg}(R)_0^*.$$

Proposition 3.34 adapts to show that $(X; T)^{\rho, \text{Mal}}$ is given by

$$A^\bullet(X, O(\mathbb{B}_\rho)) \xrightarrow{\prod_{x \in T} x^*} \prod_{x \in T} O(R)(x, -) \in DGA_{\text{Alg}}(R)_*.$$

### 3.3. General homotopy types.

**Lemma 3.36.** For an $R$-representation $A$ in DG algebras, there is a cofibrantly generated model structure on the category $DG_{Z\text{-Mod}_{A}(R)}$ of $R$-representations in $\mathbb{Z}$-graded cochain $A$-modules, in which fibrations are surjections, and weak equivalences are isomorphisms on cohomology.

**Proof.** Let $S(n)$ denote the cochain complex $A[-n]$. Let $D(n)$ be the cone of id : $A[1 - n] \to A[1 - n]$, so the underlying graded vector space is just $A[1 - n] \oplus A[-n]$.

Define $I$ to be the set of canonical maps $S(n) \otimes V \to D(n) \otimes V$, for $n \in \mathbb{Z}$ and $V$ ranging over all finite-dimensional $R$-representations. Define $J$ to be the set of morphisms $0 \to D(n) \otimes V$, for $n \in \mathbb{Z}$ and $V$ ranging over all finite-dimensional $R$-representations. Then we have a cofibrantly generated model structure, with $I$ the generating cofibrations and $J$ the generating trivial cofibrations, by verifying the conditions of [Hov] Theorem 2.1.19. □

**Definition 3.37.** Let $D_{GZ_{\text{Alg}}}(R)$ be the category of $R$-representations in $\mathbb{Z}$-graded cochain $\mathbb{R}$-algebras. For an $R$-representation $A$ in algebras, we define $D_{GZ_{\text{Alg}}}(A)$ to be the comma category $A \downarrow D_{GZ_{\text{Alg}}}(R)$. Denote the opposite category by $d_{GZ_{\text{Aff}}}(A)(R)$. We will also sometimes write this as $d_{GZ_{\text{Aff Spec}}}(A)(R)$.
Lemma 3.38. There is a cofibrantly generated model structure on $DG_Z\text{Alg}_A(R)$, in which fibrations are surjections, and weak equivalences are quasi-isomorphisms.

Proof. This follows by applying [Hir] Theorem 11.3.2 to the forgetful functor $DG_Z\text{Alg}_A(R) \to DG_Z\text{Mod}_Q(R)$.

3.3.1. Derived pullbacks and base change.

Definition 3.39. Given a morphism $f : X \to Y$ in $dg\text{Aff}(R)$, the pullback functor $f^* : DG_Z\text{Alg}_Y(R) \to DG_Z\text{Alg}_X(R)$ is left Quillen, with right adjoint $f_*$. Denote the derived left Quillen functor by $Lf^* : \text{Ho}(DG_Z\text{Alg}_Y(R)) \to \text{Ho}(DG_Z\text{Alg}_X(R))$. Observe that $f_*$ preserves weak equivalences, so the derived right Quillen functor is just $Rf_* = f_*$. Denote the functor opposite to $Lf^*$ by $\times^R X : \text{Ho}(dg\text{Aff}_Y(R)) \to \text{Ho}(dg\text{Aff}_X(R))$.

Lemma 3.40. If $f : \text{Spec} B \to \text{Spec} A$ is a flat morphism in $\text{Aff}(R)$, then $Lf^* = f^*$.

Proof. This is just the observation that flat pullback preserves weak equivalences. $Lf^*C$ is defined to be $f^*C$, for $C \to A$ a cofibrant approximation, but we then have $f^*C \to f^*C$ a weak equivalence, so $Lf^*C = f^*C$.

Proposition 3.41. If $S \in DG_Z\text{Alg}_A(R)$, and $f : A \to B$ is any morphism in $DG\text{Alg}(R)$, then cohomology of $Lf^*S$ is given by the hyper tor groups

$$H^i(Lf^*S) = \text{Tor}^A_i(S, B).$$

Proof. Take a cofibrant approximation $C \to S$, so $Lf^*S \cong f^*C$. Thus $A \to C$ is a retraction of a transfinite composition of pushouts of generating cofibrations. The generating cofibrations are filtered direct limits of projective bounded complexes, so $C$ is a retraction of a filtered direct limit of projective bounded cochain complexes. Since cohomology and hyper tor both commute with filtered direct limits (the latter following since we may choose a Cartan-Eilenberg resolution of the colimit in such a way that it is a colimit of Cartan-Eilenberg resolutions of the direct system), we may apply [Wei] Application 5.7.8 to see that $C$ is a resolution computing the hyper tor groups of $S$.

Proposition 3.42. If $S \in DG_Z\text{Alg}_A(R)$ is flat, and $f : A \to B$ is any morphism in $\text{Alg}(R)$, with either $S$ bounded or $f$ of finite flat dimension, then

$$Lf^*S \simeq f^*S.$$ 

Proof. If $S$ is bounded, then $Lf^*S \simeq S \otimes_A^L B$, which is just $S \otimes_A B$ when $S$ is also flat. If instead $f$ is of finite flat dimension, then [Wei] Corollary 10.5.11 implies that $H^*(S \otimes_A B) = \text{Tor}^A_*(S, B)$, as required.

Definition 3.43. Given an $R$-representation $Y$ in schemes define $DG_Z\text{Alg}_Y(R)$ to be the category of $R$-equivariant quasi-coherent $Z$-graded cochain algebras on $Y$. Define a weak equivalence in this category to be a map giving isomorphisms on cohomology sheaves (over $Y$), and define $\text{Ho}(DG_Z\text{Alg}_Y(R))$ to be the homotopy category obtained by localising at weak equivalences. Define the categories $dg\text{Aff}_Y(R), \text{Ho}(dg\text{Aff}_Y(R))$ to be the respective opposite categories.

Definition 3.44. Given a quasi-compact, quasi-affine scheme $X$, let $j : X \to \hat X$ be the open immersion $X \to \text{Spec} \Gamma(X, \mathcal{O}_X)$. Take a resolution $\mathcal{O}_X \to \mathcal{E}_X^\bullet$ of $\mathcal{O}_X$ in $DG_Z\text{Alg}_X(R)$, flabby with respect to Zariski cohomology (for instance by applying the Thom-Sullivan functor $\text{Th}$ to the cosimplicial algebra $\mathcal{E}^\bullet(\mathcal{O}_X)$ arising from a Čech resolution). Define $Rj_*\mathcal{O}_X$ to be $j_*\mathcal{E}_X^\bullet \in DG_Z\text{Alg}_X(R)$.

Proposition 3.45. The functor $j^* : DG_Z\text{Alg}_{Rj_*\mathcal{O}_X}(R) \to DG_Z\text{Alg}_X(R)$ induces an equivalence $\text{Ho}(DG_Z\text{Alg}_{Rj_*\mathcal{O}_X}(R)) \to \text{Ho}(DG_Z\text{Alg}_{Rj_*\mathcal{O}_X}(R))$.

For any $R$-representation $B$ in algebras, this extends to an equivalence $\text{Ho}(DG_Z\text{Alg}_{Rj_*\mathcal{O}_X}(R) \downarrow Rj_*\mathcal{O}_X \otimes B) \to \text{Ho}(DG_Z\text{Alg}_{Rj_*\mathcal{O}_X}(R) \downarrow \mathcal{O}_X \otimes B)$. 
Proof. Since $j$ is flat, $j^*$ preserves quasi-isomorphisms, so $j^*$ descends to a morphism of homotopy categories. If $C \in \mathcal{C}$ is a quasi-isomorphism, then $j_* \mathcal{C}(\mathcal{A})$ is a quasi-isomorphism, as is the map $j^* j_* \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$, since
\[ \mathcal{H}^i(j^* j_* \mathcal{C}(\mathcal{A})) = j^* \mathcal{R}^i j_* (\mathcal{A}) = \mathcal{H}^i(\mathcal{A}), \]
as $j^* \mathcal{R}^i j_* \mathcal{F} = 0$ for $i > 0$ and $\mathcal{F}$ a quasi-coherent sheaf (concentrated in degree 0), $X$ being quasi-affine.

Now, the composite morphism
\[ R j_* \mathcal{O}_X \to j_* j^* (R j_* \mathcal{O}_X) \to j_* \mathcal{C}(j^*(R j_* \mathcal{O}_X)) \]
is a quasi-isomorphism, since $j^*(R j_* \mathcal{O}_X) \to \mathcal{O}_X$ is a quasi-isomorphism. Cofibrant objects $\mathcal{M} \in DG_{Z} Mod_{R j_* \mathcal{O}_X}(R)$ are retracts of $I$-cells, which admit (ordinal-indexed) filtrations whose graded pieces are copies of $(R j_* \mathcal{O}_X)[i]$, so we deduce that for cofibrant modules $\mathcal{M}$, the map
\[ \mathcal{M} \to j_* \mathcal{C}(j^* \mathcal{M}) \]
is a quasi-isomorphism. Since cofibrant algebras are a fortiori cofibrant modules, $\mathcal{B} \to j_* \mathcal{C}(j^* \mathcal{B})$ is a quasi-isomorphism for all cofibrant $\mathcal{B} \in DG_{Z} Alg_{R j_* \mathcal{O}_X}(R)$, which completes the proof in the case when $\mathcal{C}_X = \mathcal{C}(\mathcal{O}_X)$.

For the general case, note that we have quasi-isomorphisms $Th \mathcal{C}(\mathcal{O}_X) \to Th \mathcal{C}(\mathcal{C}_X) \to Th \mathcal{C}(\mathcal{C}_X)$, giving quasi-isomorphisms $j_* Th \mathcal{C}(\mathcal{O}_X) \to j_* Th \mathcal{C}(\mathcal{C}_X)$, and hence right Quillen equivalences
\[ DG_{Z} Alg_{j_* Th \mathcal{C}(\mathcal{O}_X)}(R) \leftarrow DG_{Z} Alg_{j_* Th \mathcal{C}(\mathcal{C}_X)}(R) \rightarrow DG_{Z} Alg_{j_* Th \mathcal{C}(\mathcal{C}_X)}(R). \]

Lemma 3.46. Let $G$ be an affine group scheme, with a reductive subgroup scheme $H$ acting on a reductive pro-algebraic group $R$. Then the model categories $dg_{Z} Aff_{G}(R \times H)$ and $dg_{Z} Aff_{G/H}(R)$ are equivalent.

Proof. This is essentially the observation that $H$-equivariant quasi-coherent sheaves on $G$ are equivalent to quasi-coherent sheaves on $G/H$. Explicitly, define $U : dg_{Z} Aff_{G/H}(R) \to dg_{Z} Aff_{G}(R \times H)$ by $U(Z) = Z \times_{G/H} G$. This has a left adjoint $F(Y) = Y/H$. We need to show that the unit and co-unit of this adjunction are isomorphisms.

The co-unit is given on $Z \in dg_{Z} Aff_{G/H}(R)$ by
\[ Z \leftarrow FU(Z) = (Z \times_{G/H} G)/H \cong Z \times_{G/H} (G/H) \cong Z, \]
so is an isomorphism.

The unit is $Y \to UF(Y) = (Y/H) \times_{G/H} G$, for $Y \in dg_{Z} Aff_{G}(R \times H)$. Now, there is an isomorphism $Y \times_{G/H} G \cong Y \times H$, given by $(y, \pi(y) \cdot h^{-1}) \mapsto (y, h)$, for $\pi : Y \to G$. This map is $H$-equivariant for the left $H$-action on $Y \times_{G/H} G$, and the diagonal $H$-action on $Y \times H$. Thus
\[ UF(Y) = (Y \times_{G/H} G)/(H \times 1) \cong (Y \times H)/H \cong Y, \]
with the final isomorphism given by $(y, h) \mapsto y \cdot h^{-1}$. 

3.3.2. Extensions.

Definition 3.47. Given $B \in DG_{Z} Alg_{A}(R)$, define the cotangent complex
\[ L_{B/A}^\bullet = \text{Ho}(DG_{Z} Mod_{B}(R)) \]
by taking a factorisation $A \to C \to B$, with $A \to C$ a cofibration and $C \to B$ a trivial fibration. Then set $L_{B/A}^\bullet := \Omega_{C/A}^{\bullet} \otimes_B I/I^2$, where $I = \text{ker}(C \otimes_A B \to B)$. Note that $L_{B/A}^\bullet$ is independent of the choices made, as it can be characterised as the evaluation at $B$.
of the derived left adjoint to the functor $M \mapsto B \oplus M$ from DG $B$-modules to $B$-augmented DG algebras over $A$.

**Lemma 3.48.** Given a surjection $A \to B$ in $DG_{\mathbb{Z}}\text{Alg}(R)$, with square-zero kernel $I$, and a morphism $f : T \to C$ in $DG_{\mathbb{Z}}\text{Alg}_A(R)$, the hyperext group

$$\text{Ext}^1_{T/R}(L_{I/A}^{ullet}, T \otimes^L_A I \xrightarrow{f} C \otimes^L_A I)$$

of the cone complex is naturally isomorphic to the weak equivalence class of triples $(\theta, f', \gamma)$, where $\theta : T' \otimes^L_A B \to T \otimes^L_A B$ is a weak equivalence, $f' : T' \to C$ a morphism, and $\gamma$ a homotopy between the morphisms $(f \otimes_A B) \circ \theta, (f' \otimes_A B) : T' \otimes^L_A B \to C \otimes^L_A B$.

**Proof.** This is a slight generalisation of a standard result, and we now sketch a proof. Assume that $A \to T$ is a cofibration, and that $T \to C$ is a fibration (i.e. surjective). We first consider the case $\gamma = 0$, considering objects $T'$ (flat over $A$) such that $\theta : T' \otimes_A B \to T \otimes_A B$ is an isomorphism and $(f \otimes_A B) \circ \theta = (f' \otimes_A B)$.

Since $T$ is cofibrant over $A$, the underlying graded ring $UT$ is a retract of a polynomial ring, so $UT' \cong UT$. The problem thus reduces to deforming the differential $d$ on $T$. If we denote the differential of $T'$ by $d'$, then fixing an identification $UT = UT'$ gives $d' = d + \alpha$, for $\alpha : UT \to UT \otimes_A I[1]$ a derivation with $d\alpha + \alpha d = 0$. In order for $f : T' \to C$ to be a chain map, we also need $f\alpha = 0$. Thus

$$\alpha \in Z^1\text{HOM}_{T/R}(\Omega_{T/A}, \ker(f) \otimes_A I),$$

where $\text{HOM}(U, V)$ is the $\mathbb{Z}$-graded cochain complex given by setting $\text{HOM}(U, V)^n$ to be the space of graded morphisms $U \to V[n]$ (not necessarily respecting the differential).

Another choice of isomorphism $UT \cong UT'$ (fixing $T \otimes_A B$) amounts to giving a derivation $\beta : UT \to UT \otimes_A I$, with $id + \beta$ the corresponding automorphism of $UT$. In order to respect the augmentation $f$, we need $f\beta = 0$. This new choice of isomorphism sends $\alpha$ to $\alpha + d\beta$, so the isomorphism class is

$$[\alpha] \in \text{Ext}^1_{T/R}(\Omega_{T/A}, \ker(f) \otimes_A I).$$

Since $A \to T$ is a cofibration and $f$ a fibration, this is just hyperext

$$\text{Ext}^1_{T/R}(L_{I/A}^{ullet}, T \otimes^L_A I \xrightarrow{f} C \otimes^L_A I)$$

of the cone complex. Since this expression is invariant under weak equivalences, it follows that it gives the weak equivalence class required.

### 4. Structures on relative Malcev homotopy types

Now, fix a real reductive pro-algebraic group $R$, a pointed connected topological space $(X, x)$, and a Zariski-dense morphism $\rho : \pi_1(X, x) \to R(\mathbb{R})$.

**Definition 4.1.** Given a pro-algebraic group $K$ acting on $R$ and on a scheme $Y$, define $dg_{\mathbb{Z}}\text{Aff}_Y(R)_*(K)$ to be the category $(Y \times R) \downarrow dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)$ of objects under $R \times Y$. Note that this is not the same as $dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)_* = (Y \times R \times K) \downarrow dg_{\mathbb{Z}}\text{Aff}_Y(R \times K)$.

#### 4.1. Homotopy types

Motivated by Definitions 1.3, 1.27, 1.36 and 1.42, we make the following definitions:

**Definition 4.2.** An algebraic Hodge filtration on a pointed Malcev homotopy type $(X, x)^{\rho, \text{Mal}}$ consists of the following data:

1. an algebraic action of $U_1$ on $R$,
2. an object $(X, x)^\rho_\text{Mal}_\mathbb{Z} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^\bullet}(R)_*(S))$, where the $S$-action on $R$ is defined via the isomorphism $S/G_m \cong U_1$, while the $R \times S$-action on $R$ combines multiplication by $R$ with conjugation by $S$.
3. an isomorphism $(X, x)^{\rho, \text{Mal}} \cong (X, x)_\mathbb{Z} \times_{C^\bullet, 1} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)$. 


Definition 4.3. An algebraic mixed Hodge structure on a pointed Malcev homotopy type \((X,x)^{\rho,\text{Mal}}\) consists of the following data:

1. an object \((X,x)^{\rho,\text{Mal}}_T \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_C^\times(R)_*(\mathbb{G}_m))\),
2. an isomorphism \((X,x)^{\rho,\text{Mal}} \cong (X,x)^{\rho,\text{Mal}}_T \times_{\mathbb{A}^1,1} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)\).

Note that under the equivalence \(dg_{\mathbb{Z}}\text{Aff}(R) \simeq dg_{\mathbb{Z}}\text{Aff}_m(R \times \mathbb{G}_m)\) of Lemma 3.46, \((X,x)^{\rho,\text{Mal}}\) corresponds to the derived pullback \((X,x)^{\rho,\text{Mal}}_T \times_{\mathbb{C}^\times} \mathbb{G}_m\).

Definition 4.4. An algebraic mixed Hodge structure \((X,x)^{\rho,\text{MHS}}\) on a pointed Malcev homotopy type \((X,x)^{\rho,\text{Mal}}\) consists of the following data:

1. an algebraic action of \(U_1\) on \(R\),
2. an object \((X,x)^{\rho,\text{MHS}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^\times}(R)_*(\mathbb{G}_m \times S))\), where \(S\) acts on \(R\) via the \(U_1\)-action, using the canonical isomorphism \(U_1 \cong S/\mathbb{G}_m\),
3. an object \(\text{gr}(X,x)^{\rho,\text{MHS}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(S))\),
4. an isomorphism \((X,x)^{\rho,\text{MHS}} \cong (X,x)^{\rho,\text{MHS}} \times_{\mathbb{A}^1 \times C^\times,(1,1)} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)\),
5. an isomorphism (called the opposedness isomorphism) \(\theta^2(\text{gr}(X,x)^{\rho,\text{MHS}}) \times C^\times \cong (X,x)^{\rho,\text{MHS}} \times_{\mathbb{A}^1,1} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^\times}(R)_*(\mathbb{G}_m \times S))\),

for the canonical map \(\theta : \mathbb{G}_m \times S \to S\) given by combining the inclusion \(\mathbb{G}_m \hookrightarrow S\) with the identity on \(S\).

Definition 4.5. Given an algebraic mixed Hodge structure \((X,x)^{\rho,\text{MHS}}\) on \((X,x)^{\rho,\text{Mal}}\), define \(\text{gr}^W(X,x)^{\rho,\text{MHS}} := (X,x)^{\rho,\text{MHS}} \times_{\mathbb{A}^1,0} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^\times}(\mathbb{G}_m \times R)_*(S))\), noting that this is isomorphic to \(\theta^2(\text{gr}(X,x)^{\rho,\text{MHS}}) \times C^\times\). We also define \(\text{gr}(X,x)^{\rho,\text{Mal}} := (X,x)^{\rho,\text{Mal}} \times_{\mathbb{A}^1,1} \text{Spec } \mathbb{R}\), noting that this is an algebraic Hodge filtration on \((X,x)^{\rho,\text{Mal}}\).

Definition 4.6. A real splitting of the mixed Hodge structure \((X,x)^{\rho,\text{MHS}}\) is a \(\mathbb{G}_m \times S\)-equivariant isomorphism
\[\mathbb{A}^1 \times \text{gr}(X,x)^{\rho,\text{MHS}} \times C^\times \cong (X,x)^{\rho,\text{MHS}},\]
in \(\text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^\times}(R)_*(\mathbb{G}_m \times S))\), giving the opposedness isomorphism on pulling back along \(\{0\} \to \mathbb{A}^1\).

Definition 4.7. An algebraic mixed twistor structure \((X,x)^{\rho,\text{MTS}}\) on a pointed Malcev homotopy type \((X,x)^{\rho,\text{Mal}}\) consists of the following data:

1. an object \((X,x)^{\rho,\text{MTS}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{\mathbb{A}^1 \times C^\times}(R)_*(\mathbb{G}_m \times \mathbb{G}_m))\),
2. an object \(\text{gr}(X,x)^{\rho,\text{MTS}} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*(\mathbb{G}_m))\),
3. an isomorphism \((X,x)^{\rho,\text{MTS}} \cong (X,x)^{\rho,\text{MTS}} \times_{\mathbb{A}^1,0} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}(R)_*)\),
4. an isomorphism (called the opposedness isomorphism) \(\theta^2(\text{gr}(X,x)^{\rho,\text{MTS}}) \times C^\times \cong (X,x)^{\rho,\text{MTS}} \times_{\mathbb{A}^1,0} \text{Spec } \mathbb{R} \in \text{Ho}(dg_{\mathbb{Z}}\text{Aff}_{C^\times}(R)_*(\mathbb{G}_m \times \mathbb{G}_m))\),

for the canonical diagonal map \(\theta : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m\).
Definition 4.8. Given an algebraic mixed twistor structure \((X, x)^{\rho, \text{Mal}}\) on \((X, x)^{\rho, \text{Mal}}\), define
\[
\text{gr}^W(X, x)^{\rho, \text{Mal}}_{\text{MTS}} := (X, x)^{\rho, \text{Mal}}_{\text{MTS}} \times^R \mathbb{A}^1_0 \text{Spec} \mathbb{R} \in \text{Ho}(R \times C^* dgZ \text{Aff}_{C^*} (\mathbb{G}_m \times R \times \mathbb{G}_m)),
\]
noting that this is isomorphic to \(\theta^t(\text{gr}(X, x)^{\rho, \text{Mal}}_{\text{MTS}}) \times C^*\). We also define \((X, x)^{\rho, \text{Mal}}_T := (X, x)^{\rho, \text{Mal}}_{\text{MTS}} \times^R \mathbb{A}^1_1 \text{Spec} \mathbb{R}\), noting that this is an algebraic twistor filtration on \((X, x)^{\rho, \text{Mal}}\).

Remark 4.9. As in Remark 3.29, we might want to consider many basepoints, or none. The definitions above then have analogues \((X; T)^{\rho, \text{Mal}}_F, (X; T)^{\rho, \text{Mal}}_T, (X; T)^{\rho, \text{Mal}}_{\text{MHS}}, (X; T)^{\rho, \text{Mal}}_{\text{MTS}}, \) given by replacing the \(R\)-representation \(R\) with the representation \(\coprod_{x \in T} R(x, -)\), as in Remark 3.35.

4.2. Splittings over \(S\). We now work with the \(S\)-equivariant map \(\text{row}_1 : \text{SL}_2 \to C^*\) as defined in §1.1.1.

Definition 4.10. An \(S\)-splitting (or \(\text{SL}_2\)-splitting) of a mixed Hodge structure \((X, x)^{\rho, \text{Mal}}_{\text{MHS}}\) on a relative Malcev homotopy type is a \(\mathbb{G}_m \times S\)-equivariant isomorphism
\[
\mathbb{A}^1 \times \text{gr}(X, x)^{\rho, \text{Mal}}_{\text{MHS}} \times \text{SL}_2 \cong \text{row}_1^*(X, x)^{\rho, \text{Mal}}_{\text{MHS}},
\]
in \(\text{Ho}(dgZ \text{Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_* (\mathbb{G}_m \times S)),\) giving \(\text{row}_1^*\) of the opposedness isomorphism on pulling back along \(\{0\} \to \mathbb{A}^1\).

An \(S\)-splitting (or \(\text{SL}_2\)-splitting) of a mixed twistor structure \((X, x)^{\rho, \text{Mal}}_{\text{MTS}}\) on a relative Malcev homotopy type is a \(\mathbb{G}_m \times \mathbb{G}_m\)-equivariant isomorphism
\[
\mathbb{A}^1 \times \text{gr}(X, x)^{\rho, \text{Mal}}_{\text{MTS}} \times \text{SL}_2 \cong \text{row}_1^*(X, x)^{\rho, \text{Mal}}_{\text{MTS}},
\]
in \(\text{Ho}(dgZ \text{Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_* (\mathbb{G}_m \times \mathbb{G}_m)),\) giving \(\text{row}_1^*\) of the opposedness isomorphism on pulling back along \(\{0\} \to \mathbb{A}^1\).

Lemma 4.11. Let \(S'\) be \(S\) or \(\mathbb{G}_m\). Take flat fibrant objects
\[
Y \in \text{dgZ Aff}_{\mathbb{A}^1 \times \text{SL}_2}(R)_* (\mathbb{G}_m \times S'), \quad Z \in \text{dgZ Aff}(R)_* (\mathbb{G}_m \times S'),
\]
together with a surjective quasi-isomorphism \(\phi^\natural : 0^* \mathcal{O}_Y \to \mathcal{O}_Z \otimes \mathcal{O}_{\text{SL}_2}\) in \(\text{dgZ Aff}_{\text{SL}_2}(R)_* (\mathbb{G}_m \times S')\) equipped with weak equivalences \(f : \text{row}_1^* X \to Y\) and \(g : 0^* X \to Z \times C^*\) with \(\phi \circ \text{row}_1^* g = 0^* f\) is either \(0\) or a principal homogeneous space for the group
\[
\text{Ext}^0(L^\bullet(1), \ker(\phi^\natural : \mathcal{O}_Z \otimes \mathcal{O}_{\text{SL}_2}) \to (W_{-1} \mathcal{O}_{\mathbb{A}^1}) \otimes (\mathcal{O}_R) \otimes \mathcal{O}_{\text{SL}_2})^G_{\mathbb{G}_m \times R \times S'},
\]
where \(L^\bullet\) is the cotangent complex of \(Y \cup_{\mathbb{A}^1 \times \text{SL}_2} (Z \times C^*)\) over \((\mathbb{A}^1 \times \text{SL}_2) \cup (\{0\} \times C^*)\), and \(\text{Ext}^0\) is taken over \((Z \times C^*)\).

Proof. The data \(Y, Z, \phi\) determine the pullback of \(X\) to
\[
(\mathbb{A}^1 \times \text{SL}_2) \cup (\{0\} \times \text{SL}_2) \cap (\{0\} \times C^*).
\]
Since \(\phi^\natural\) is surjective, we may define
\[
\mathcal{O}_Y \times (\mathcal{O}_Z \otimes \mathcal{R}(C^*)) \to \mathcal{O}(R) \otimes ((\mathcal{O}_{\mathbb{A}^1} \otimes \mathcal{O}_{\text{SL}_2}) \times \mathcal{O}_{\text{SL}_2} \mathcal{R}(C^*))
\]
over
\[
(\mathcal{O}(\mathbb{A}^1) \otimes \mathcal{O}(\text{SL}_2)) \times \mathcal{O}(\text{SL}_2) \mathcal{R}(C^*),
\]
which we wish to lift to \(\mathcal{R}(C^*)\), making use of Proposition 3.45.

Now, the morphism \(\mathcal{R}(C^*) \to (O(\mathbb{A}^1) \otimes O(\text{SL}_2)) \times O(\text{SL}_2) \mathcal{R}(C^*)\) is surjective, with square-zero kernel \((W_{-1}O(\mathbb{A}^1)) \otimes O(\text{SL}_2)(-1)[-1]\), where \(W_{-1}O(\mathbb{A}^1) = \ker(\mathcal{O}(\mathbb{A}^1) \to \mathbb{R})\), so Proposition 3.48 gives the required result. □
Corollary 4.12. The weak equivalence class of $S$-split mixed Hodge structures $(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}}$ with $\text{gr}(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}} = (R \rightarrow Z)$ is canonically isomorphic to

$$\text{Ext}^0_Z(\mathbb{L}_Z, (W_1 O(A^1)) \otimes (\theta_Z \rightarrow z_* O(R)) \otimes O(SL_2)(-1))^{G_m \times R \times S}.$$ 

The weak equivalence class of $S$-split mixed twistor structures $(X, x)^{\rho_{\text{Mal}}}_{\text{MTS}}$ with $\text{gr}(X, x)^{\rho_{\text{Mal}}}_{\text{MTS}} = (R \rightarrow Z)$ is canonically isomorphic to

$$\text{Ext}^0_Z(\mathbb{L}_Z, (W_1 O(A^1)) \otimes (\theta_Z \rightarrow z_* O(R)) \otimes O(SL_2)(-1))^{G_m \times R \times S}.$$ 

Proof. Set $Y = A^1 \times Z \times SL_2$ in Lemma 4.11, and note that the cone of $O(A^1) \rightarrow \mathbb{R}$ is quasi-isomorphic to $W_1 O(A^1)$. The class of possible extensions is non-empty, since $A^1 \times Z \times C^*$ is one possibility for $(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}}$ (resp. $(X, x)^{\rho_{\text{Mal}}}_{\text{MTS}}$). This gives a canonical basepoint for the principal homogeneous space, and hence the canonical isomorphism. \qed

4.3. Grouplike structures.

Definition 4.13. Given $A \in DGA_{alg}(R)$, define the category of $R$-equivariant dg pro-algebraic groups $G_\bullet$ over $A$ to be opposite to the category of $R$-equivariant DG Hopf algebras over $A$. Explicitly, this consists of objects $Q \in DGA_{alg}(R)$ equipped with morphisms $Q \rightarrow Q \otimes A Q$, (comultiplication), $Q \rightarrow A$ (coidentity) and $Q \rightarrow Q$ (coinverse), satisfying the usual axioms.

A morphism $f : G_\bullet \rightarrow K_\bullet$ of dg pro-algebraic groups is said to be a quasi-isomorphism if it induces an isomorphism $H^*(O(K)) \rightarrow H^*(O(G))$ on cohomology of the associated DG Hopf algebras.

Definition 4.14. Given $G \in sAGp$, define the dg pro-algebraic group $NG$ over $R$ by setting $O(NG) = D^*(O(G))$, where $D^*$ is left adjoint to the denormalisation functor for algebras. The comultiplication on $O(NG)$ is then defined using the fact that $D^*$ preserves coproducts, so $D^*(O(G) \otimes O(G)) = O(NG) \otimes O(NG)$, where $(O(G) \otimes O(G))^n = O(G)^n \otimes O(G)^n$ but $(O(NG) \otimes O(NG))^n = \bigoplus_{i+j=n} O(NG)^i \otimes O(NG)^j$.

Examples 4.15. Given $g \in dg\hat{N}(R \times S)$, we may form an $S$-equivariant dg pro-algebraic group $\text{exp}(g)$ over $R$ by letting $O(\text{exp}(g))$ represent the functor

$$\text{exp}(g)(A) := \exp\{\{g \in \prod_n g_n \otimes A^n : (d \otimes 1)g_n = (1 \otimes d)g_{n-1}\}\},$$

for DG algebras $A$. Note that the underlying dg algebra is given by $O(\text{exp}(g)) = R[g \vee]$, with comultiplication dual to the Campbell-Baker-Hausdorff formula.

For any DG algebra $B$, observe that the $R$-action on $g$ provides an $R(H^0 B)$-action on $\text{exp}(g)(B)$, so we can then define the $S$-equivariant dg pro-algebraic group $R \times \text{exp}(g)$ to represent the functor

$$A \mapsto R(H^0 A) \times \text{exp}(g)(A),$$

noting that $O(R \times \text{exp}(g)) \cong O(R) \otimes O(\text{exp}(g))$ as a DG algebra.

If $g \in s\hat{N}(R \times S)$, note that $N(R \times \text{exp}(g)) \cong R \times \text{exp}(NG)$, since both represent the same functor.

Definition 4.16. Define a grouplike mixed Hodge structure on a pointed Malcev homotopy type $(X, x)^{\rho_{\text{Mal}}}$ to consist of the following data:

1. an algebraic action of $U_1$ on $R$,
2. a flat $G_m \times S$-equivariant dg pro-algebraic group $G(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}}$ over $O(A^1) \otimes RO(C^*)$, equipped with an $S$-equivariant map $G(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}} \rightarrow A^1 \times R \times \text{Spec} RO(C^*)$ of dg pro-algebraic groups over $A^1 \times \text{Spec} RO(C^*)$, where $S$ acts on $R$ via the $U_1$-action.
3. an object $\text{gr}(X, x)^{\rho_{\text{Mal}}}_{\text{MHS}} \in dg\hat{N}(R \times S)$. 
4.18 Remark. \(U\) and \(gr\) are independent of the choice of \(\mathbb{A}^1\times \text{Spec } \mathbb{R}O(C^\ast)\), \((1, I)\) \(\text{Spec } \mathbb{R}\) of pro-algebraic dg groups on \(\text{Spec } \mathbb{R}\), respecting the \(R\)-augmentations, where \(I : \text{Spec } \mathbb{R} \to \text{SL}_2 \to \text{Spec } \mathbb{R}O(C^\ast)\) is the identity matrix. 

4.19 Proposition. We can adapt Definition 4.13 in the spirit of Remark 3.29 by defining an \(\mathbb{R}\)-equivariant dg pro-algebraic groupoid with an \(\mathbb{R}\)-split MHS \((X, x)^{\mathbb{R}\text{-Mal}}\), replacing dg pro-algebraic groups with dg pro-algebraic groupoids on objects \(T\), noting that \(\text{Ob } T = T\).

Definition 4.17. Define a grouplike mixed twistor structure similarly, dispensing with the \(U_1\)-action on \(R\), and replacing \(S\) with \(\mathbb{G}_m\).

Remark 4.18. We can adapt Definition 4.13 in the spirit of Remark 3.29 by defining an \(\mathbb{R}\)-equivariant dg pro-algebraic groupoid \(G\) over \(A\) to consist of a set \(\text{Ob } G\) of objects, together with \(O(G)(x, y) \in \text{DGA}_{\mathbb{A}}(R)\) for all \(x, y \in \text{Ob }\), equipped with morphisms \(O(G)(x, z) \to O(G)(x, y) \otimes A O(G)(y, z)\) (comultiplication), \(O(G)(x, x) \to A\) (coidentity) and \(O(G)(x, y) \to O(G)(y, x)\) (coinverse), satisfying the usual axioms.

Given a reductive pro-algebraic groupoid with an \(S\)-action, and \(\mathfrak{g} \in \text{dg}\mathcal{N}(R \times S)\), we then define the \(S\)-equivariant dg pro-algebraic group \(R \times \exp(\mathfrak{g})\) to have objects \(\text{Ob } R\), with

\[ (R \times \exp(\mathfrak{g}))(x, y) = R(x, y) \times \exp(\mathfrak{g}(y)), \]

and multiplication as in [Pri3] Definition 2.15.

Definitions 4.16 and 4.17 then adapt to multipointed Malcev homotopy types \((X: T)^{\mathbb{R}\text{-Mal}}\), replacing dg pro-algebraic groups with dg pro-algebraic groupoids on objects \(T\), noting that \(\text{Ob } T = T\).

Proposition 4.19. Take an \(S\)-split MHS \((X, x)^{\mathbb{R}\text{-Mal}}\) (resp. an \(S\)-split MTS \((X, x)^{\mathbb{R}\text{-MTS}}\)) on a relative Malcev homotopy type, and assume that \(\text{gr}(X, x)^{\mathbb{R}\text{-Mal}} \in \text{Ho}(\text{dg}_2 \text{Aff}(R), (\mathbb{S}_0))\) (resp. \(\text{gr}(X, x)^{\mathbb{R}\text{-MTS}} \in \text{Ho}(\text{dg}_2 \text{Aff}(R), (\mathbb{S}_0))\)).

Then there is a canonical grouplike MHS (resp. grouplike MTS) on \((X, x)^{\mathbb{R}\text{-Mal}}\), independent of the choice of \(S\)-splitting.

Moreover, the induced pro-MHS \(R_\mathbb{A}(G(X, x)^{\mathbb{R}\text{-Mal}})\) (resp. MTS \(R_\mathbb{A}(G(X, x)^{\mathbb{R}\text{-MTS}})\)) on the abelianisation of the pro-unipotent radical of \(G(X, x)^{\mathbb{R}\text{-Mal}}\) is dual to the complex given by the cokernel of

\[ \mathbb{R}[-1] \to O(X)^{\mathbb{R}\text{-Mal}}[-1] \text{ resp. } \mathbb{R}[-1] \to O(X)^{\mathbb{R}\text{-MTS}}[-1], \]

where \(X^{\mathbb{R}\text{-Mal}}\) = Spec \(O(X)^{\mathbb{R}\text{-Mal}}\).

Proof. We will prove this for mixed Hodge structure; the case of mixed twistor structures is entirely similar.

Choose a representative \(Z\) for \(\text{gr}(X, x)^{\mathbb{R}\text{-Mal}}\) with \(Z_0 = \text{Spec } \mathbb{R}\), and set \(\mathfrak{g} = \text{gr}(X, x)^{\mathbb{R}\text{-Mal}} := G(Z)\) (for \(G\) as in Definition 3.22). Then \(Z \to \mathcal{W}\mathfrak{g}\) is a weak equivalence, making \(O(\mathcal{W}\mathfrak{g})\) into a cofibrant representative for \(Z\), so by Corollary 4.12, \((X, x)^{\mathbb{R}\text{-Mal}}\) corresponds to a class

\[ \nu \in \text{Ext}^0(\Omega(O(\mathcal{W}\mathfrak{g})/\mathbb{R}), (W_{-1}O(\mathbb{A}^1)) \otimes (O(Z) \to z_\ast O(R)) \otimes O(SL_2)(-1))^{G_m \times R \times S}. \]

Now, \(\Omega(O(\mathcal{W}\mathfrak{g})/\mathbb{R}) \cong \mathfrak{g}^\ast[-1]\), so we may choose a representative

\[ (\alpha', \gamma') : \mathfrak{g}^\ast[-1] \to (W_{-1}O(\mathbb{A}^1)) \otimes (O(Y) \times O(R)[-1]) \otimes O(SL_2)(-1) \]

for \(\nu\), with \([d, \alpha] = 0, [d, \gamma] = z^\ast \alpha'\).

Studying the adjunction \(\mathcal{W} \dashv G\), we see that \(\alpha'\) is equivalent to an \(R \times S\)-equivariant Lie coalgebra derivation \(\alpha : \mathfrak{g}^\ast \to W_{-1}O(\mathbb{A}^1) \otimes \mathfrak{g}^\ast \otimes O(SL_2)(-1)\) with \([d, \alpha] = 0\). This generates a derivation \(\alpha : O(R \times \exp(\mathfrak{g})) \to (W_{-1}O(\mathbb{A}^1)) \otimes O(R \times \exp(\mathfrak{g})) \otimes O(SL_2)(-1).\)
\(\gamma'\) corresponds to an element \(\gamma \in (\mathfrak{g}_0 \hat{\otimes} (W_{-1}O(\mathbb{A}^1)) \otimes O(\text{SL}_2)(-1)))^G_m \times S\), and conjugation by this gives another such derivation \([\gamma, -]\), so we then set \(O(G(x, x)_{\rho, \text{Mal}}^{\text{MHS}})\) to be the quasi-isomorphism class of the dg Hopf algebra over \(O(\mathbb{A}^1) \otimes RO(C^*)\) given by the graded Hopf algebra

\[O(\mathbb{A}^1) \otimes O(R \ltimes \exp(\mathfrak{g})) \otimes (O(\text{SL}_2) + O(\text{SL}_2)(-1))\epsilon\]

(where \(\epsilon\) is of degree 1 and \(\epsilon^2 = 0\), with differential \(d_{(\alpha, \gamma)} := d_{O(R \ltimes \exp(\mathfrak{g}))} + (\text{id} \otimes \text{id} \otimes N + \alpha + [\gamma, -])\epsilon\).

Explicitly, \(G(x, x)_{\rho, \text{Mal}}^{\text{MHS}}\) represents the group-valued functor on \(\text{DGA}_G \times RO(C^*) (G_m \times S)\) given by mapping \(A\) to the subgroup of \((R(A^0) \ltimes \exp(\mathfrak{g})A^0)^S\), consisting of \((r, g)\) such that

\[d_A \circ (r, g) = (r, g) \circ d_{(\alpha, \gamma)} : O(R \ltimes \exp(\mathfrak{g})) \to A[1]\]

or equivalently \((d_A \circ (r, g) - (r, g) \circ d_{(\alpha, \gamma)}) \circ (r, g)^{-1} = 0\), so

\[d(r g) \cdot g^{-1} = r \alpha(g)g^{-1}r^{-1} + (\gamma rg - rg \gamma)g^{-1}r^{-1} \in (\text{Lie } R) \hat{\otimes} A \oplus (\mathfrak{g} \hat{\otimes} A)^1,\]

where \(d\) is the total differential \(d_A - d_q\). This reduces to

\[dg \cdot g^{-1} + r^{-1} \cdot dr = \alpha(g) \cdot g^{-1} + \text{ad}_{\gamma} - \text{ad}_g(\gamma).\]

To see that this is well-defined, another choice of representative for \(\nu\) would be of the form \((\alpha + [d, h], \gamma + dk)\), for a Lie coalgebra derivation \(h : \mathfrak{g}^\vee \to (W_{-1}O(\mathbb{A}^1)) \otimes \mathfrak{g}^\vee \otimes O(\text{SL}_2)(-1)[1]\), and \(k \in (\mathfrak{g}_1 \hat{\otimes} (W_{-1}O(\mathbb{A}^1)) \otimes O(\text{SL}_2)(-1))^G_m \times S\). The morphism \(\text{id} + h \epsilon + [k, -] \epsilon\) then provides a quasi-isomorphism between the two representatives of \(O(G(x, x)_{\rho, \text{Mal}}^{\text{MHS}})\).

The evaluations of \(\alpha\) and \(\gamma\) at \(0 \in \mathbb{A}^1\) are both 0 (since \(W_{-1}O(\mathbb{A}^1) = \ker 0^*\)), so there is a canonical isomorphism

\[\psi : 0^* G(X, x)_{\rho, \text{Mal}}^{\text{MHS}} \cong (R \ltimes \exp(\mathfrak{g})) \times \text{Spec } RO(C^*).\]

Meanwhile, pulling back along the canonical morphism \(r_1 : \text{SL}_2 \to \text{Spec } RO(C^*)\) gives an isomorphism \(r_1^* G(X, x)_{\rho, \text{Mal}}^{\text{MHS}} \cong \mathbb{A}^1 \times (R \ltimes \exp(\mathfrak{g})) \times \text{SL}_2\), so

\[(1, I)^* G(X, x)_{\rho, \text{Mal}}^{\text{MHS}} \cong (R \ltimes \exp(\mathfrak{g}))\]

and combining this with the pullback along \(I \to \text{SL}_2\) of our choice of \(\text{SL}_2\)-splitting gives a quasi-isomorphism

\[\phi : (1, I)^* G(X, x)_{\rho, \text{Mal}}^{\text{MHS}} \simeq G(X, x)_{\rho, \text{Mal}}.\]

Now, \((R\eta G(X, x)_{\rho, \text{Mal}}^{\text{MHS}})_{\text{ab}}\) is dual to the complex

\[O(\mathbb{A}^1) \otimes \text{coker } (k \to O(Z))[\epsilon] \otimes (O(\text{SL}_2) + O(\text{SL}_2)(-1))\epsilon\]

with differential \(d_Z + (\text{id} \otimes \text{id} \otimes N + \alpha')\epsilon\). Under the characterisation of Lemma 4.12, this is quasi-isomorphic to the cokernel \(\text{coker } (k \to O(X)_{\rho, \text{Mal}}^{\text{MHS}})\) of complexes, with \(\phi\) and \(\psi\) recovering the structure maps of the ind-MHS.

Finally, another choice of \(S\)-splitting amounts to giving a homotopy class of automorphisms \(u\) of \(\mathbb{A}^1 \times \text{gr}(X, x)_{\rho, \text{Mal}}^{\text{MHS}} \times \text{SL}_2\), giving the identity on pulling back along \(0 \to \mathbb{A}^1\).

Since \(Z \simeq \text{gr}(X, x)_{\rho, \text{Mal}}^{\text{MHS}}\) and \(\eta : Z \to \text{WG}(Z)\) is a fibrant replacement for \(Z\), \(u\) gives rise to a homotopy class of morphisms \(v : \mathbb{A}^1 \times Z \times \text{SL}_2 \to \text{WG}(Z)\), with \(0^* v = \eta\). Via the adjunction \(G \dashv \text{W}\) this gives a homotopy automorphism \(U : \mathbb{A}^1 \times G(Z) \times \text{SL}_2 \to \mathbb{A}^1 \times G(Z) \times \text{SL}_2\) with \(0^* U = \text{id}\). This gives a quasi-isomorphism between the respective constructions of \((G(X, x)_{\rho, \text{Mal}}^{\text{MHS}}, \psi, \phi)\). \(\square\)
Theorem 4.20. If the $S$-action on $H^0O(\text{gr}X^{\rho,\text{Mal}})$ is of non-negative weights, then the grouplike MHS (resp. grouplike MTS) of Proposition 4.19 gives rise to ind-MHS (resp. ind-MTS) on the relative Malcev homotopy groups $\varpi_n(X, x)^{\rho,\text{Mal}}$ for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(X, x)^{\rho,\text{Mal}})$.

These structures are compatible with the action of $\varpi_1$ on $\varpi_n$, the Whitehead bracket and the Hurewicz maps $\varpi_n(X^{\rho,\text{Mal}}) \to H^n(X, O(\mathbb{P}_\rho))^\vee$ ($n \geq 2$) and $R_u\varpi_1(X^{\rho,\text{Mal}}) \to H^1(X, O(\mathbb{P}_\rho))$, for $\mathbb{P}_\rho$ as in Definition 3.31.

Proof. Again we give the proof for MHS only, as the MTS case follows by replacing $S$ with $\mathbb{G}_m$ and Proposition 1.40 with Proposition 1.48.

Choose a representative $Z$ for $\text{gr}(X, x)^{\rho,\text{Mal}}_\text{MHS}$ with $Z_0 = \text{Spec} \mathbb{R}$, and $O(Z)$ of non-negative weights. [To see that this is possible, take a minimal model $m$ for $\overline{G}(\text{gr}(X, x)^{\rho,\text{Mal}}_\text{MHS})$ as in [Pri3] Proposition 4.7, and note that $m/[m,m] \cong H^0(\text{gr}X^{\rho,\text{Mal}})^\vee$ is of non-negative weights, so $m$ is of non-negative weights, and therefore $O(Wm)$ is of non-negative weights, so $Wm$ is a possible choice for $Z$.] Set $g = \text{gr}(X, x)^{\rho,\text{Mal}}_\text{MHS} = G(Z)$.

Since $RO(C^*)$ is a dg algebra over $O(C)$, we may regard it as a quasi-coherent sheaf on $C$, and consider the quasi-coherent dg algebra $j^{-1}RO(C^*)$ on $C^*$, for $j : C^* \to C$.

Define
\[
\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} := \text{Spec} \mathcal{H}^0(j_*O(G(X, x)^{\rho,\text{Mal}}_\text{MHS})),
\]
which is an affine group object over $\mathbb{A}^1 \times C^*$, as $j^{-1}RO(C^*)$ is a resolution of $O_{C^*}$. Since $\text{row}_1 : \text{SL}_2 \to C^*$ is flat,
\[
\text{row}_1^*\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} = \text{Spec} \mathcal{H}^0(\text{row}_1^*O(G(X, x)^{\rho,\text{Mal}}_\text{MHS})).
\]

Now the choice of $S$-splitting gives
\[
\chi : \text{row}_1^*O(G(X, x)^{\rho,\text{Mal}}_\text{MHS}) \cong \mathbb{A}^1 \times (R \times \exp(g)) \times \text{Spec} (\text{row}_1^*RO(C^*)),
\]
and $\text{row}_1^*RO(C^*)$ is a resolution of $O(\text{SL}_2)$, so
\[
\text{row}_1^*\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} \cong \mathbb{A}^1 \times (R \times \exp(g)) \times \text{SL}_2,
\]
whose structure sheaf is flat on $\mathbb{A}^1 \times \text{SL}_2$, and has non-negative weights with respect to the $\mathbb{G}_m \times 1$-action. Lemma 1.16 then implies that the structure sheaf of $\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS}$ is flat over $\mathbb{A}^1 \times C^*$, with non-negative weights.

Set $\text{gr}(X, x)^{\rho,\text{Mal}}_\text{MHS} := (R \times \exp(H_0g))$. The morphisms $\phi$ and $\psi$ from the proof of Proposition 4.19 now induce an $S$-equivariant isomorphism
\[
\text{Spec} \mathbb{R} \times_{\mathbb{A}^1,0} \varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} \cong \text{gr}\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} \times C^*,
\]
and an isomorphism
\[
\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS} \times_{\mathbb{A}^1 \times C^*,(1,1)} \text{Spec} \mathbb{R} \cong \varpi_1(X, x)^{\rho,\text{Mal}},
\]
giving the data of a flat algebraic MHS on $O(\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS})$, of non-negative weights. By Proposition 1.40, this is the same as an ind-MHS of non-negative weights.

Next, we consider the dg Lie coalgebra over $O(\mathbb{A}^1) \otimes RO(C^*)$ given by
\[
\mathcal{C}(G(X, x)^{\rho,\text{Mal}}_\text{MHS}) := \Omega(G(X, x)^{\rho,\text{Mal}}_\text{MHS}/RO(C^*)) \otimes_{O(G(X, x)^{\rho,\text{Mal}}_\text{MHS}),1} RO(C^*),
\]
which has non-negative weights with respect to the $\mathbb{G}_m \times 1$-action. Pulling back along $j$ gives a dg Lie coalgebra $j^{-1}\mathcal{C}(G(X, x)^{\rho,\text{Mal}}_\text{MHS})$ over $O(\mathbb{A}^1) \otimes j^{-1}RO(C^*)$, so the cohomology sheaves $\mathcal{H}^*(j^{-1}\mathcal{C}(G(X, x)^{\rho,\text{Mal}}_\text{MHS}))$ form a graded Lie coalgebra over $O(\mathbb{A}^1) \otimes O_{C^*}$.

The isomorphism $\chi$ above implies that these sheaves are flat over $\mathbb{A}^1 \times C^*$, and therefore that $\mathcal{H}^0(j^{-1}\mathcal{C}(G(X, x)^{\rho,\text{Mal}}_\text{MHS}))$ is just the Lie coalgebra of $\varpi_1(X, x)^{\rho,\text{Mal}}_\text{MHS}$. For $n \geq 2$, we set
\[
\varpi_n(X, x)^{\rho,\text{Mal}}_\text{MHS} := \mathcal{H}^{n-1}(\text{row}_1^*\mathcal{C}(G(X, x)^{\rho,\text{Mal}}_\text{MHS})),
\]
noting that these have a conjugation action by \( \varpi_1(X,x)_{\text{Mal}}^{\rho,\text{MHS}} \) and a natural Lie bracket.

Setting \( \text{gr}\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}} := (\text{H}_{n-1}(g))^\vee \), the isomorphisms \( \phi \) and \( \psi \) induce \( S \)-equivariant isomorphisms
\[
\text{Spec } \mathbb{R} \times A^1,0 \xrightarrow{\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}}} \xrightarrow{\text{gr}\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}}} C^\ast ,
\]
and isomorphisms
\[
\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}} \times A^1 \times C^\ast ,\{1,1\} \xrightarrow{\text{Spec } \mathbb{R}} \cong \varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}},
\]
so Proposition 4.14 gives the data of an non-negatively weighted ind-MHS on \( \varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}} \), compatible with the \( \varpi_1 \)-action and Whitehead bracket.

Finally, the Hurewicz map comes from
\[
R_\alpha G(X,x)_{\text{Mal}}^{\rho,\text{Mal}} \rightarrow (R_\alpha G(X,x)_{\text{Mal}}^{\rho,\text{Mal}})^{ab} \cong \text{coker } (\mathbb{R} \rightarrow O(X_{\text{Mal}}^{\rho,\text{Mal}}))[−1]^\vee ,
\]
which is compatible with the ind-MHS, by the final part of Proposition 4.19. Thus the Hurewicz maps
\[
\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}} \rightarrow H^n(X,O(\mathbb{R})) \quad \text{and } \quad R_\alpha \varpi_1(X_{\text{Mal}}^{\rho,\text{Mal}}) \rightarrow H^1(X,O(\mathbb{R}))
\]
preserve the ind-MHS.

In Proposition 4.19 and Theorem 4.20, the only rôle of the \( \mathcal{S} \)-splitting is to ensure that the algebraic MHS is flat. We now show how a choice of \( \mathcal{S} \)-splitting gives additional data.

**Theorem 4.21.** A choice of \( \mathcal{S} \)-splitting for \( (X,x)_{\text{MHS}}^{\rho,\text{Mal}} \) (resp. \( (X,x)_{\text{MHS}}^{\rho,\text{Mal}} \)) gives a morphism
\[
O(\varpi_1(X,x)_{\text{Mal}}^{\rho,\text{Mal}}) \otimes \mathcal{S} \cong \text{gr}^W O(\varpi_1(X,x)_{\text{Mal}}^{\rho,\text{Mal}}) \otimes \mathcal{S}
\]
of (real) quasi-MHS (resp. quasi-\( \text{MTS} \)) in Hopf algebras, and isomorphisms
\[
(\varpi_n(X,x)_{\text{MHS}}^{\rho,\text{Mal}})^\vee \otimes \mathcal{S} \cong \text{gr}^W (\varpi_n(X,x)_{\text{Mal}}^{\rho,\text{Mal}})^\vee \otimes \mathcal{S}
\]
of (real) quasi-MHS (resp. quasi-\( \text{MTS} \)), inducing the identity on \( \text{gr}^W \), and compatible with the Whitehead bracket.

**Proof.** The choice of \( \mathcal{S} \)-splitting gives an isomorphism
\[
\text{row}^*G(X,x)_{\text{Mal}}^{\rho,\text{MHS}} \cong A^1 \times (R \ltimes \exp(g)) \times \text{row}^*\text{Spec } RO(C^\ast )
\]
in Proposition 4.19. The isomorphisms now follow from Lemma 1.18 and the constructions of Theorem 4.20. \( \square \)

**Remark 4.22.** This leads us to ask what additional data are required to describe the ind-MHS on homotopy groups in terms of the Hodge structure \( \text{gr}(X,x)_{\text{Mal}}^{\rho,\text{MHS}} \). If we set \( g = \tilde{G}(\text{gr}(X,x)_{\text{Mal}}^{\rho,\text{MHS}}) \), then we can let \( \mathcal{D}^\ast := \text{Der}(R \ltimes \exp(g), R \ltimes \exp(g)) \) be the complex of DG Hopf algebra derivations on \( O(R \ltimes \exp(g)) \). This has a canonical \( \mathcal{S} \)-action (inherited from \( R \) and \( g \)), and the proof of Proposition 4.19 gives
\[
[\beta] := [\alpha + [\gamma,−]] \in H^0(W_{−1},\gamma^0(\mathcal{D}^\ast \otimes \mathcal{S})),
\]
for \( \gamma^0 \) as in Definition 1.24. This determines the mixed Hodge structure, by Corollary 4.12, and \( \gamma^0(\mathcal{D}^\ast \otimes \mathcal{S}) \cong \text{R} \Gamma_{\text{wH}}(\mathcal{D}^\ast \otimes \mathcal{S}) \), by Remark 1.26.

This gives a derivation \( N + \beta : O(R \ltimes \exp(g)) \otimes \mathcal{S} \rightarrow O(R \ltimes \exp(g)) \otimes S(−1) \), and this diagram is a resolution of \( D^\ast O(G(X,x)) \), making \( O(G(X,x)) \) into a mixed Hodge complex. As in §2.4, we think of \( N + \beta \) as the monodromy operator at the Archimedean place. This will be constructed explicitly in §8.

Moreover, for any \( \mathcal{S} \)-split MHS \( V \) arising as an invariant of \( O(G(X,x)) \), the induced map \( N + \beta : (\text{gr}^W V) \otimes \mathcal{S} \rightarrow (\text{gr}^W V) \otimes S(−1) \) just comes from conjugating the surjective map \( \text{id} \otimes N : V \otimes \mathcal{S} \rightarrow V \otimes S(−1) \) with respect to the splitting isomorphism \( (\text{gr}^W V) \otimes \mathcal{S} \cong V \otimes \mathcal{S} \). Therefore \( N + \beta \) is surjective, and \( V = \ker(N + \beta) \).

All these results have analogues for mixed twistor structures.
Remark 4.23. If we have a multipointed MHS (resp. MTS) \((X; T)^{\rho; Mal}_{\text{MHS}}\) (resp. \((X; T)^{\rho; Mal}_{\text{MTS}}\)) as in Remark 4.9, then Proposition 4.19 and Theorems 4.20 and 4.21 adapt to give \(S\)-split multipointed grouplike MHS (resp. MTS) as in Remark 4.18, together with \(S\)-split ind-MHS (resp. ind-MTS) on the algebras \(O(x; x, y)^{\rho; Mal}_{\text{MHS}}\), compatible with the pro-algebraic groupoid structure. The \(S\)-split ind-MHS (resp. ind-MTS) on \((x, y)^{\rho; Mal}_{\text{MHS}}\) are then compatible with the co-action

\[
(\varpi_n(x, y)^{\rho; Mal})^\vee \to O(\varpi_1(x, y)^{\rho; Mal}) \otimes (\varpi_n(x, y)^{\rho; Mal})^\vee.
\]

In the proof of Proposition 4.19, \(g = \varpi_0\) becomes an \(R\)-representation, giving \(g_x\) for all \(x \in T\). For objects \(G(x; T)^{\rho; Mal}_{\text{MHS}}\) is then defined on \(G_m \times S\)-equivariant DGAs \(A\) over \(O(A^1) \otimes RO(C^*)\) by setting, for \(x, y \in T\), \(G(x; y)^{\rho; Mal}_{\text{MHS}}(A)\) to be the subset of \((R(x, y)(A^0) \times \exp(\varpi_0 A))^S\), consisting of \((r, g)\) such that

\[
dg \cdot g^{-1} + r^{-1} \cdot dr = \alpha(g) \cdot g^{-1} + \text{ad}_{r-1} \gamma_x - \text{ad}_g(\gamma_y).
\]

4.4. MHS representations. Take a pro-unipotent extension \(G \to R\) of pro-algebraic groups with kernel \(U\), together with a compatible ind-MHS on the Hopf algebra \(O(G)\). This gives rise to \(G_m \times S\)-equivariant affine group objects \(U_{\text{MHS}} \to G_{\text{MHS}}\) over \(A^1 \times C^*\), given by

\[
G_{\text{MHS}} = \text{Spec} \xi(O(G), \text{MHS}), \quad U_{\text{MHS}} = \text{Spec} \xi(O(U), \text{MHS}),
\]

and this gives a morphism \(G_{\text{MHS}} \to A^1 \times R \times C^*\) with kernel \(U_{\text{MHS}}\).

Now, since \(U = \exp(u)\) is pro-unipotent, we can express \(G_{\text{MHS}} \to A^1 \times R \times C^*\) as a composition of extensions by locally free abelian groups. On pulling back to the affine scheme \(\mathbb{A}^1 \times \text{SL}_2\), the argument of [Pri3] Proposition 2.17 adapts to give a \(G_m \times S\)-equivariant section

\[
\sigma_G : \mathbb{A}^1 \times R \times \text{SL}_2 \to \text{row}^{\star}_G G_{\text{MHS}},
\]

since \(R\) and \(G_m \times S\) (linearly) reductive. This section is unique up to conjugation by \(\Gamma(\mathbb{A}^1 \times \text{SL}_2, U_{\text{MHS}})G_{m \times S}\).

This is equivalent to giving a retraction \(\sigma^*_G : O(G) \otimes S \to O(R) \otimes S\) of quasi-MHS in Hopf algebras over \(S\), unique up to conjugation by \(\exp(W_0^0(u \otimes S))\).

Now, applying the derivation \(N\) gives a morphism

\[
(\sigma^*_G + N \sigma^*_G \xi) : O(G) \otimes (S \oplus S(-1)e) \to O(R) \otimes (S \oplus S(-1)e)
\]

of quasi-MHS in Hopf algebras over \(S \oplus S(-1)\), where \(e^2 = 0\). The argument above (considering affine group schemes over \(\mathbb{A}^1 \times \text{Spec}(O(SL_2) \oplus O(SL_2)(-1)e)\) adapts to show that there exists \(\gamma_G \in \Gamma(\mathbb{A}^1 \times \text{SL}_2, U_{\text{MHS}}(-1)_{\mathbb{A}^1 \times \text{SL}_2})\) with

\[
\sigma_G + N \sigma_G \xi = \text{ad}_{1 + \gamma_G e} \circ \sigma_G : \mathbb{A}^1 \times R \times \text{Spec}(O(SL_2) \oplus O(SL_2)(-1)e) \to G_{\text{MHS}}.
\]

Then observe that \(N - [\gamma_G, -] : u \otimes S \to u \otimes S(-1)\) is \(R\)-equivariant, and denote this derivation by \(\alpha_G\).

If instead we started with an ind-MTS on \(O(G)\), then the construction above would give corresponding data for \(G_{\text{MTS}} = \text{Spec} \xi(O(G), \text{MTS})\), replacing \(S\) with \(G_m\) throughout.

Definition 4.24. For \(G \to R\) as above, let \((u, \alpha_G)_{\text{MHS}}\) (resp. \((u, \alpha_G)_{\text{MTS}}\)) be the Lie algebra \(\text{row}^*_G \xi(u, MHS)(-1)\) (resp. \(\text{row}^*_G \xi(u, MTS)(-1)\)) over \(O(A^1) \otimes RO(C^*)\).

Definition 4.25. Given a pro-nilpotent DG Lie algebra \(L^\bullet\) in non-negative cochain degrees, define the Deligne groupoid \(\text{Del}(L)\) to have objects \(MC(L) \subset L^1\) (see Definition 3.23), with morphisms \(\omega \to \omega'\) consisting of \(g \in Gg(L) = \exp(L^0)\) with \(g \ast \omega = \omega'\), for the gauge action of Definition 3.24.
Since Theorem 4.20 gives the Hopf algebra \( O(\varpi_1(x, x)^{\rho, \text{Mal}}) \) an ind-MHS or ind-MTS independent of the choice of \( S \)-splitting, we now show how to describe MHS and MTS representations of \( \varpi_1(x, x)^{\rho, \text{Mal}} \) in terms of \( (x, x)^{\rho, \text{Mal}} \).

**Proposition 4.26.** For \( G \to R \) and \( (x, x)^{\rho, \text{Mal}} \) as above, the set \( \text{Hom}(\varpi_1(x, x)^{\rho, \text{Mal}}, G)_{\text{HMS}} \) (resp. \( \text{Hom}(\varpi_1(x, x)^{\rho, \text{Mal}}, G)_{\text{MTS}} \)) of morphisms

\[
O(G) \to O(\varpi_1(x, x)^{\rho, \text{Mal}})
\]

of ind-MHS (resp. ind-MTS) in Hopf algebras extending \( \rho \) is isomorphic to the fibre of the morphism

\[
\mathcal{D}(O(X^{\rho, \text{Mal}}_{\text{MHS}}) \otimes \mathbb{R}^{\times S} (u, \alpha_{G})_{\text{MHS}}) \xrightarrow{x^*} \mathcal{D}(u, \alpha_{G})_{\text{MHS}}^{\mathbb{R}^{\times S}}
\]

(resp.

\[
\mathcal{D}(O(X^{\rho, \text{Mal}}_{\text{MTS}}) \otimes \mathbb{R}^{\times S} (u, \alpha_{G})_{\text{MTS}}) \xrightarrow{x^*} \mathcal{D}(u, \alpha_{G})_{\text{MTS}}^{\mathbb{R}^{\times S}}
\]

over \( \gamma_{G} \). Here, the morphism of Deligne groupoids is induced by \( x : \mathbb{A}^1 \times R \times C^* \to X^{\rho, \text{Mal}}_{\text{MHS}} \) (resp. \( x : \mathbb{A}^1 \times R \times C^* \to X^{\rho, \text{Mal}}_{\text{MTS}} \)).

**Proof.** We will prove this for the MHS case only; the MTS case can be recovered by replacing \( S \) with \( G_{m} \).

An element of \( \text{Hom}(\varpi_1(x, x)^{\rho, \text{Mal}}, G)_{\text{HMS}} \) is just a \( G_{m} \times S \)-equivariant morphism \( \psi : G(X, x)^{\rho, \text{Mal}}_{\text{MHS}} \to G_{\text{MHS}} \) of pro-unipotent extensions of \( \mathbb{A}^1 \times R \times C^* \). The proof of Proposition 4.19 gives a choice \( \sigma : R \times SL_2 \to G(X, x)^{\rho, \text{Mal}}_{\text{MHS}} \) of section, and the argument above shows that there must exist \( u \in \Gamma(\mathbb{A}^1 \times SL_2, U_{\text{HMS}})_{G_{m} \times S} = W_{0}^{\gamma_{0}}U(S) \) with \( \psi \circ \sigma = \text{ad}_{u} \circ \sigma_{G} \).

Then \( \text{ad}_{u}^{-1} \circ \psi \) preserves the Levi decompositions, giving an \( R \)-equivariant morphism \( f : g \otimes S \to u \otimes S \) of pro-Lie algebras in quasi-MHS over \( S \), with

\[
\psi(r \cdot g) = u \cdot r \cdot f(g) \cdot u^{-1},
\]

for \( r \in R, \: g \in \exp(g) \).

We also need \( \psi \) to commute with \( N \). Looking at \( N \circ \psi = \psi \circ N \) restricted to \( R \), we need

\[
[u f(\gamma) u^{-1}, u ru^{-1}] = [\gamma_{G} + \alpha_{G}(u)u^{-1}, uru^{-1}]
\]

for \( r \in R, \: \gamma \) as in the proof of Proposition 4.19. Equivalently, there exists \( b \in W_{0}^{\gamma_{0}}(u^{R} \otimes S(-1)) \) with \( u^{-1} \gamma_{G} u + u^{-1} \alpha_{G}(u) = f(\gamma) + b \) Looking at \( \psi \) on \( g \in \exp(g) \) then gives the condition that \( \alpha_{G} \circ f(g) = f \circ \alpha_{G} = [b, f(g)] \).

A different choice of \( \psi \) would be of the form \( uv \), for \( v \in \Gamma(\mathbb{A}^1 \times SL_2, U_{\text{HMS}})_{G_{m} \times R} \), and we then have to replace \( (f, b) \) with \( (\text{ad}_{u}^{-1} f, v^{-1} b v + v^{-1} \alpha_{G}(v)) \).

We now proceed by developing an equivalent description of the Deligne groupoids. Define a \( G_{m} \times R \times S \)-equivariant DG algebra \( A \) over \( O(\mathbb{A}^1) \otimes RO(C^*) \) by

\[
A^n = (O(\mathbb{A}^1) \otimes O(\hat{W}g)^n) \otimes O(SL_2) \oplus (O(\mathbb{A}^1) \otimes O(\hat{W}g)^{n-1} \otimes O(SL_2)(-1)e),
\]

with differential \( d_W \pm (N + \alpha) \). Then

\[
\text{MC}(A \otimes O(\mathbb{A}^1) \otimes RO(C^*)) (u, \alpha_{G})_{\text{MHS}}^{G_{m} \times R} \otimes RO(C^*)
\]

consists of pairs

\[
(f, b) \in \text{Hom}(g, \text{row}^{'i} \xi(u, \text{MHS})) \times \Gamma(\mathbb{A}^1 \times SL_2, \text{row}^{'i} \xi(u, \text{MHS})(-1)))^{G_{m} \times R},
\]

satisfying the the Maurer-Cartan conditions. These are equivalent to saying that \( f \) is a Lie algebra homomorphism, and that \( \alpha_{G} \circ f(g) = f \circ \alpha_{G}(g) = [b, f(g)] \).

Meanwhile,

\[
\text{MC}((u, \alpha_{G})_{\text{MHS}})_{G_{m} \times S} = W_{0}^{\gamma_{0}}(u \otimes S(-1)),
\]

\[
\text{Gg}((u, \alpha_{G})_{\text{MHS}})_{G_{m} \times S} = W_{0}^{\gamma_{0}}U(S),
\]

\[
\text{Gg}(A \otimes O(\mathbb{A}^1) \otimes RO(C^*) (u, \alpha_{G})_{\text{MHS}}) = W_{0}^{\gamma_{0}}U^R(S).
\]
There is a morphism $A \to O(\mathbb{A}^1) \otimes O(R) \otimes RO(C^*)$ determined on generators by $\gamma : g^\gamma \to O(\mathbb{A}^1) \otimes O(SL_2)(-1)$ in level 1, for $\gamma$ as in the proof of Proposition 4.19.

Thus the fibre of the Deligne groupoids

$\mathcal{D}el(A \otimes_{O(\mathbb{A}^1) \otimes RO(C^*)} \gamma_x \otimes_{O(\mathbb{A}^1) \otimes RO(C^*)} (u, \alpha_G)_{\text{MHS}}) \to \mathcal{D}el((u, \alpha_G)_{\text{MHS}})^{G_m \times S})$

over $\gamma_G$ consists of $(f, b)$ as above, together with $u \in W_0^\gamma U(S)$ mapping $\gamma_G$ to $f(\gamma) + b$ under the gauge action of Definition 3.24. Morphisms in this groupoid are given by $W^\gamma \circ \text{map}$ and trivial morphisms.

Various quasi-isomorphisms $(\overline{\gamma})$ and there is a canonical quasi-isomorphism $\mathcal{D}el((u, \alpha_G)_{\text{MHS}})^{G_m \times S})$ by [GM] Theorem 2.4, giving the required result.

5. Mixed Hodge structures on relative Malcev homotopy types of compact Kähler manifolds

Fix a compact Kähler manifold $X$ and a point $x \in X$.

5.1. Real homotopy types.

**Definition 5.1.** Define the Hodge filtration on the real homotopy type $(X \otimes \mathbb{R}, x)$ by $(X \otimes \mathbb{R}, x)_F := (\text{Spec} \mathbb{R} \times C^* \xrightarrow{\tau} \text{Spec} j^* \hat{A}^*(X)) \in \text{Ho}(C^* \downarrow d\mathbb{Z} \text{Aff} C\text{-}(S))$, for $j : C^* \to C$ and $\hat{A}^*(X)$ as in Definition 2.1.

**Definition 5.2.** Define the algebraic mixed Hodge structure $(X \otimes \mathbb{R}, x)_{\text{MHS}}$ on $(X \otimes \mathbb{R}, x)$ to be $\text{Spec}$ of the Rees algebra associated to the good truncation filtration $W_r = \tau^{\leq r} j^* \hat{A}: \text{Spec} \mathbb{R} \to \text{Spec} H^* \otimes \partial(C^*)$.

Define $(\text{gr}(X \otimes \mathbb{R})_{\text{MHS}}, 0)$ to be the unique morphism $\text{Spec} \mathbb{R} \to \text{Spec} H^* \otimes \partial(C^*)$, determined by the isomorphism $H^0(X, \mathbb{R}) \cong \mathbb{R}$. Now

$(X \otimes \mathbb{R}, x)_{\text{MHS}} \times_{\mathbb{A}^1} \{0\} \xrightarrow{\tau} \text{Spec} \text{gr}^W j^* \hat{A}^*(X))$,

and there is a canonical quasi-isomorphism $\text{gr}^W j^* \hat{A}^*(X) \to \mathcal{H}^* (j^* \hat{A}^*(X))$. As in the proof of Corollary 2.9, this is $S$-equivariantly isomorphic to $H^* (X, \mathbb{R}) \otimes \partial(C^*)$, giving the opposedness quasi-isomorphism

$(X \otimes \mathbb{R}, x) \times_{\mathbb{A}^1} \{0\} \xrightarrow{\text{gr}(X \otimes \mathbb{R})_{\text{MHS}}, 0} C^*$.

**Proposition 5.3.** The algebraic MHS $(X \otimes \mathbb{R}, x)_{\text{MHS}}$ splits on pulling back along row$_1 : SL_2 \to C^*$. Explicitly, there is an isomorphism

$(X \otimes \mathbb{R}, x)_{\text{MHS}} \times_{\mathbb{A}^1} \text{row}_1 : SL_2 \cong \mathbb{A}^1 \times (\text{gr}(X \otimes \mathbb{R})_{\text{MHS}}, 0) \times C^*$

in $\text{Ho}(\mathbb{A}^1 \times SL_2 \downarrow d\mathbb{Z} \text{Aff}_{\mathbb{A}^1 \times SL_2} (G_m \times S))$, whose pullback to $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

**Proof.** Corollary 2.9 establishes the corresponding splitting for the Hodge filtration $(X \otimes \mathbb{R}, x)_F$, and good truncation commutes with everything, giving the splitting for $(X \otimes \mathbb{R}, x)_{\text{MHS}}$. The proof of Corollary 2.9 ensures that pulling the $S$-splitting back to $0 \in \mathbb{A}^1$ gives row$_1^*$ applied to the opposedness isomorphism.
Corollary 5.4. There are natural pro-MHS on the homotopy groups \( \pi_n(X \otimes \mathbb{R}, x) \).

Proof. Apply Theorem 4.20 in the case \( R = 1 \), noting that Proposition 5.3 gives the requisite \( S \)-splitting. \( \square \)

Corollary 5.5. For \( S \) as in Example 1.23, and for all \( n \geq 1 \), there are \( S \)-linear isomorphisms

\[
\pi_n(X \otimes \mathbb{R}, x)^{\vee} \otimes_{\mathbb{R}} S \cong \pi_n(H^*(X, \mathbb{R}))^{\vee} \otimes_{\mathbb{R}} S,
\]

of quasi-MHS, compatible with Whitehead brackets and Hurewicz maps. The graded map associated to the weight filtration is just the pullback of the standard isomorphism \( \text{gr}_W \pi_n(X \otimes \mathbb{R}, x) \cong \pi_n(H^*(X, \mathbb{R})) \) (coming from the opposeness isomorphism).

Proof. The \( S \)-splitting of Proposition 5.3 allows us to apply Theorem 4.21, giving isomorphisms

\[
\pi_n(X \otimes \mathbb{R}, x)^{\vee} \otimes_{\mathbb{R}} S \cong \pi_n(\text{gr}(X \otimes \mathbb{R})_{\text{MHS}}, 0)^{\vee} \otimes_{\mathbb{R}} S
\]
of quasi-MHS.

The definition of \( \text{gr}(X \otimes \mathbb{R})_{\text{MHS}} \) implies that \( \pi_n(\text{gr}(X \otimes \mathbb{R})_{\text{MHS}}, 0) = \pi_{n-1}G(H^*(X, \mathbb{R})) \), giving the required result. \( \square \)

5.1.1. Comparison with Morgan. We now show that our mixed Hodge structure on homotopy groups agrees with the mixed Hodge structure given in [Mor] for simply connected varieties.

Proposition 5.6. The mixed Hodge structures on homotopy groups given in Corollary 5.4 and [Mor] Theorem 9.1 agree.

Proof. In [Mor] \( \S \)6, a minimal model \( \mathcal{M} \) was constructed for \( \mathfrak{A}^*(X, \mathbb{C}) \), equipped with a bigraded (i.e. a \( \mathbb{G}_m \times \mathbb{G}_m \)-action). The associated quasi-isomorphism \( \psi: \mathcal{M} \to \mathfrak{A}^*(X, \mathbb{C}) \) satisfies \( \psi(\mathcal{M}^{pq}) \subset \tau^{\leq p+q}F^p\mathfrak{A}^*(X, \mathbb{C}) \). Thus \( \psi \) is a map of bifiltered DGAs. It is also a quasi-isomorphism of DGAs, but we need to show that it is a quasi-isomorphism of bifiltered DGAs. By [Mor] Lemma 6.2b, \( \psi \) maps \( H^*(\mathcal{M}^{pq}) \) isomorphically to \( H^{pq}(X, \mathbb{C}) \), so the associated Rees algebras are quasi-isomorphic.

Equivalently, this says that we have a \( \mathbb{G}_m \times \mathbb{G}_m \)-equivariant quasi-isomorphism

\[
\xi (\mathcal{A}^*(X) \otimes \mathcal{O}(\mathcal{C})) O(\mathcal{A}_{C}^1; \tau) \simeq \xi (\mathcal{M}; F, W).
\]

over the subscheme \( \mathbb{A}^1 \times \mathbb{A}^1 \subset \mathbb{A}^1 \times \mathbb{C}^* \) given by \( u - iv = 1 \) as in Lemma 1.12. Now, Lemma 3.46 gives equivalences

\[
DG_{\mathcal{G}} \mathcal{A}_{\mathcal{G}}^{\mathcal{A}^1 \times \mathcal{C}^*} (\mathbb{G}_m \times S) \cong DG_{\mathcal{G}} \mathcal{A}_{\mathcal{G}}^{\mathcal{A}^1 \times \mathcal{C}^*} (\mathbb{G}_m \times S) \cong DG_{\mathcal{G}} \mathcal{A}_{\mathcal{G}}^{\mathcal{A}^1 \times \mathcal{C}^*} (\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{C}_m \times \mathbb{C}_m)
\]

so \( \xi (\mathcal{M}; F, W) \otimes \mathcal{O}(\mathbb{G}_m, \mathcal{C}) \) is quasi-isomorphic to \( \xi (\mathcal{A}^*(X) \otimes \mathcal{O}(\mathcal{C})) O(\mathcal{C}^*; \tau) \), which is just the pullback \( p^*O((X \otimes \mathbb{R})_{\text{MHS}}) \) along \( p: \mathcal{A}^* \to \mathcal{C}^* \). Equivalently, \( \mathcal{M} \) is a \( \mathcal{C}^* \)-splitting (rather than an SL-2-splitting) of the MHS on \( O(X \otimes \mathbb{R}) \).

Note that \( \mathcal{M}^{\mathcal{D}} = \mathbb{C} \), so there is a unique map \( \mathcal{M} \to \mathbb{C} \), and thus \( \xi (\mathcal{M}; F, W) \otimes \mathcal{O}(\mathbb{G}_m, \mathcal{C}) \) is quasi-isomorphic to \( p^*O((X \otimes \mathbb{R})_{\text{MHS}}) \) in \( DG\mathcal{A}^{\mathcal{A}^1 \times \mathcal{C}^*} (\mathbb{G}_m \times S) \downarrow O(\mathcal{A}_{C}^1 \times \mathcal{C}^*) \). Since \( p \) factors through \( \text{row}_1: \text{SL}_2 \to \mathcal{C}^* \) by Lemma 1.16, we have a morphism \( q: \mathcal{R}O(C^*) \to O(\mathcal{C}^*) \), and the construction of Proposition 4.19 then gives a quasi-isomorphism

\[
q^*G(X \otimes \mathbb{R}, x)_{\text{MHS}} \simeq \xi (\exp(G(\mathcal{M})); W, F)
\]
of \( \mathbb{G}_m \times \text{S-splitting} \) pro-nilpotent Lie algebras over \( \mathcal{A}_C \times \mathcal{C}^* \).

Taking homotopy groups as in the proof of Theorem 4.20, we see that

\[
q^*\pi_n(X \otimes \mathbb{R}, x)_{\text{MHS}} \cong \xi (H_{n-1}G(\mathcal{M})); W, F).
\]
Now, under the equivalences of Theorem 3.28, \( H_{n-1}(G(M))^{\vee} = H^n(L_{M/R} \otimes L_M R) \). Since \( M \) is cofibrant, this is just \( H^n(\Omega(M/R) \otimes_M R) \). Finally, \( M \) is minimal, so the complex \( \Omega(M/R) \otimes_M R \) is isomorphic to the indecomposables \( I \) of \( M \), with trivial differential. This means that \( H_{n-1}(G(M))^{\vee} \cong I^n \), and
\[
\xi(\varpi_n(X \otimes \mathbb{C}, x)^{\vee}; W, F) = p^* \xi(\varpi_n(X \otimes \mathbb{R}, x)^{\vee}, \text{MHS}) \cong \xi(I^n; W, F),
\]
so the Hodge and weight filtrations from Theorem 4.20 and [Mor] agree. \( \square \)

5.2. Relative Malcev homotopy types.

5.2.1. The reductive fundamental groupoid is pure of weight 0.

**Lemma 5.7.** There is a canonical action of the discrete group \( U_1^G \) on the real reductive pro-algebraic completion \( \varpi_1(X, x)^{\text{red}} \) of the fundamental group \( \pi_1(X, x) \).

**Proof.** By Tannakian duality, this is equivalent to establishing a \( U_1^G \)-action on the category of real semisimple local systems on \( X \). This is just the unitary part of the \( C^\infty \)-action on complex local systems from [Sim3]. Given a real \( C^\infty \) vector bundle \( V \) with a flat connection \( D \), there is an essentially unique pluriharmonic metric, giving a unique decomposition \( D = d^+ + \partial \) of \( D \) into antisymmetric and symmetric parts. In the notation of [Sim3], \( d^+ = \partial + \partial \) and \( \partial = \partial + \partial \). Given \( t \in U_1^G \), we define \( t \circ D \) by \( d^+ + t \circ \partial = \partial + \partial + t \partial + t^{-1} \partial \) (for \( \circ \) as in Definition 2.2), which preserves the metric. \( \square \)

5.2.2. Variations of Hodge structure. The following results are taken from [Pri1] §2.3.

**Definition 5.8.** Given a discrete group \( \Gamma \) acting on a pro-algebraic group \( G \), define \( \Gamma \)-\( G \) to be the maximal quotient of \( G \) on which \( \Gamma \) acts algebraically. This is the inverse limit
\[
\lim_{\alpha} G_{\alpha}
\]
with \( G_{\alpha} \) algebraic (i.e. of finite type), for which the \( \Gamma \)-action descends to \( G_{\alpha} \). Equivalently, \( O(\Gamma G) \) is the sum of those finite-dimensional \( \Gamma \)-representations of \( O(G) \) which are closed under comultiplication.

**Definition 5.9.** Define the quotient group \( \text{VHS}_{\varpi_1}(X, x) \) of \( \varpi_1(X, x) \) by
\[
\text{VHS}_{\varpi_1}(X, x) := U_1^{\delta} \varpi_1(X, x)^{\text{red}}.
\]

**Remarks 5.10.** This notion is analogous to the definition given in [Pri5] of the maximal quotient of the \( l \)-adic pro-algebraic fundamental group on which Frobenius acts algebraically. In the same way that representations of that group corresponded to semisimple subsystems of local systems underlying Weil sheaves, representations of \( \text{VHS}_{\varpi_1}(X, x) \) will correspond to local systems underlying variations of Hodge structure (Proposition 5.12).

**Proposition 5.11.** The action of \( U_1 \) on \( \text{VHS}_{\varpi_1}(X, x) \) is algebraic, in the sense that
\[
U_1 \times \text{VHS}_{\varpi_1}(X, x) \to \text{VHS}_{\varpi_1}(X, x)
\]
is a morphism of schemes.

It is also an inner action, coming from a morphism
\[
U_1 \to (\text{VHS}_{\varpi_1}(X, x))/Z(\text{VHS}_{\varpi_1}(X, x))
\]
of pro-algebraic groups, where \( Z \) denotes the centre of the group.

**Proof.** In the notation of Definition 5.8, write \( \varpi_1(X, x) = \lim_{\alpha} G_{\alpha} \). As in [Sim3] Lemma 5.1, the map
\[
\text{Aut}(G_{\alpha}) \to \text{Hom}(\pi_1(X, x), G_{\alpha})
\]
is a closed immersion of schemes, so the map
\[
U_1^G \to \text{Aut}(G_{\alpha})
\]
is analytic, hence continuous. This means that it defines a one-parameter subgroup, so is algebraic. Therefore the map
\[ U_1 \times VHS_{2}(X, x) \to VHS_{2}(X, x) \]
is algebraic, as \( VHS_{2}(X, x) = \varprojlim G_{\alpha} \).

Since \( V_{1}(X, x)_{\text{red}} \) is a reductive pro-algebraic group, \( G_{\alpha} \) is a reductive algebraic group. This implies that the connected component \( \text{Aut}(G_{\alpha})_{0} \) of the identity in \( \text{Aut}(G_{\alpha}) \) is given by
\[ \text{Aut}(G_{\alpha})_{0} = G_{\alpha}(x, x)/Z(G_{\alpha}). \]

Since
\[ VHS_{2}(X, x)/Z(VHS_{2}(X, x)) = \varprojlim G_{\alpha}/Z(G_{\alpha}), \]
we have an algebraic map
\[ U_{1} \to VHS_{2}(X, x)/Z(VHS_{2}(X, x)) \]
as required. \( \square \)

**Proposition 5.12.** The following conditions are equivalent:

1. \( V \) is a representation of \( VHS_{2}(X, x) \);
2. \( V \) is a representation of \( V_{1}(X, x)_{\text{red}} \) such that \( t \otimes V \cong V \) for all \( t \in U_{1}^{\delta} \);
3. \( V \) is a representation of \( V_{1}(X, x)_{\text{red}} \) such that \( t \otimes V \cong V \) for some non-torsion \( t \in U_{1}^{\delta} \).

Moreover, representations of \( VHS_{2}(X, x) \times U_{1} \) correspond to weight 0 variations of Hodge structure on \( X \).

**Proof.**

1. \( \implies \) 2. If \( V \) is a representation of \( VHS_{2}(X, x) \), then it is a representation of \( V_{1}(X, x)_{\text{red}} \), so is a semisimple representation of \( V_{1}(X, x) \). By Lemma 5.11, \( t \in U_{1}^{\delta} \) is an inner automorphism of \( VHS_{2}(X, x) \), coming from \( g \in VHS_{2}(X, x) \), say. Then multiplication by \( g \) gives the isomorphism \( t \otimes V \cong V \).

2. \( \implies \) 3. Trivial.

3. \( \implies \) 1. Let \( M \) be the monodromy group of \( V \); this is a quotient of \( V_{1}(X, x)_{\text{red}} \). The isomorphism \( t \otimes V \cong V \) gives an element \( g \in \text{Aut}(M) \), such that \( g \) is the image of \( t \) in \( \text{Hom}(\pi_{1}(X, x), M) \), using the standard embedding of \( \text{Aut}(M) \) as a closed subscheme of \( \text{Hom}(\pi_{1}(X, x), M) \). The same is true of \( g^{n}, t^{n} \), so the image of \( U_{1} \) in \( \text{Hom}(\pi_{1}(X, x), M) \) is just the closure of \( \{ g^{n} \}_{n \in \mathbb{Z}} \), which is contained in \( \text{Aut}(M) \), as \( \text{Aut}(M) \) is closed. For any \( s \in U_{1}^{\delta} \), this gives us an isomorphism \( s \otimes V \cong V \), as required.

Finally, a representation of \( VHS_{2}(X, x) \times U_{1} \) gives a semisimple local system \( V = \ker(D : \mathcal{V} \to \mathcal{V} \otimes \rho_{0} \mathcal{A}^{1}) \) (satisfying one of the equivalent conditions above), together with a coassociative coaction \( \mu : (\mathcal{V}, D) \to (\mathcal{V} \otimes O(U_{1}), t \otimes D) \) of ind-finite-dimensional local systems, for \( t = a + ib \in O(U_{1}) \otimes \mathbb{C} \). This is equivalent to giving a decomposition \( \mathcal{V} \otimes \mathbb{C} = \bigoplus_{p+q=0} \mathcal{V}^{pq} \) with \( \mathcal{V}^{pq} = \mathcal{V}^{qp} \), and with the decomposition \( D = \partial + \bar{\partial} + t \theta + t^{-1} \bar{\theta} \) (as in Lemma 5.7) satisfying
\[ \partial : \mathcal{V}^{pq} \to \mathcal{V}^{pq} \otimes \mathcal{A}^{10}, \quad \bar{\partial} : \mathcal{V}^{pq} \to \mathcal{V}^{p+1,q-1} \otimes \mathcal{A}^{01}, \]
which is precisely the condition for \( V \) to be a VHS. Note that if we had chosen \( V \) not satisfying one of the equivalent conditions, then \( (\mathcal{V} \otimes O(U_{1}), t \otimes D) \) would not yield an ind-finite-dimensional local system. \( \square \)

**Lemma 5.13.** The obstruction \( \varphi \) to a surjective map \( \alpha : V_{1}(X, x)_{\text{red}} \to R \), for \( R \) algebraic, factoring through \( VHS_{2}(X, x) \) lies in \( H^{1}(X, \text{adB}_{\alpha}) \), for \( \text{adB}_{\alpha} \) the vector bundle associated to the adjoint representation of \( \alpha \) on the Lie algebra of \( R \). Explicitly, \( \varphi \) is given by \( \varphi = [i \theta - i \bar{\theta}] \), for \( \theta \in A^{10}(X, \text{adB}_{\alpha}) \) the Higgs form associated to \( \alpha \).
Theorem 5.14. If $R$ is any quotient of $\VHS_{\varpi_1}(X,x)_{\Bbb R}^{\text{red}}$, then there is an algebraic mixed Hodge structure $(X,x)^{\rho,\text{Mal}}_{\text{MHS}}$ on the relative Malcev homotopy type $(X,x)^{\rho,\text{Mal}}$, where $\rho$ denotes the quotient map to $R$.

There is also an $S$-equivariant splitting

$$\mathbb{A}^1 \times (\text{gr}(X,x)^{\rho,\text{Mal}}_{\text{MHS}}, 0) \times SL_2 \simeq (X^{\rho,\text{Mal}}, x)_{\text{MHS}} \times \mathbb{R}^* \times \text{row}_1 SL_2$$

on pulling back along $\text{row}_1 : SL_2 \to C^*$, whose pullback over $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. By Proposition 5.11, we know that representations of $R$ all correspond to local systems underlying polarised variations of Hodge structure, and that the $U^\delta_1$-action on $\VHS_{\varpi_1}(X,x)_{\Bbb R}^{\text{red}}$ descends to an inner algebraic action on $R$, via $U_1 \to R/Z(R)$. This allows us to consider the semi-direct products $R \rtimes U_1$ and $R \rtimes S$ of pro-algebraic groups, making use of the isomorphism $U_1 \cong S$.

The $R$-representation $O(\mathbb{B}_\rho) = \mathbb{B}_\rho \times^R O(R)$ in local systems of $\mathbb{R}$-algebras on $X$ thus has an algebraic $U_1$-action, denoted by $(t,v) \mapsto t \circ v$ for $t \in U_1, v \in O(\mathbb{B}_\rho)$, and we define an $S$-action on the de Rham complex

$$\mathcal{A}^*(X, O(\mathbb{B}_\rho)) = \mathcal{A}^*(X, \mathbb{R}) \otimes^R O(\mathbb{B}_\rho)$$

by $\lambda \boxdot (a \otimes v) := (\lambda \circ a) \otimes \left(\frac{\lambda}{\mathbb{A}} \circ v\right)$, noting that the $\circ$ and $\boxdot$ actions commute. This gives an action on the global sections

$$\mathcal{A}^*(X, O(\mathbb{B}_\rho)) := \Gamma(X, \mathcal{A}^*(X, O(\mathbb{B}_\rho))).$$

It follows from [Sim3] Theorem 1 that there exists a harmonic metric on every semisimple local system $\mathcal{V}$, and hence on $O(\mathbb{B}_\rho)$. We then decompose the connection $D$ as $D = d^+ + \partial$ into antisymmetric and symmetric parts, and let $D^c := i \circ d^+ - i \circ \partial$. To see that this is independent of the choice of metric, observe that for $C = -1 \in U_1$ acting on $\varpi_1(X,x)^{\text{red}}$, antisymmetric and symmetric parts are the 1- and $-1$-eigenvectors.

Now, we define the DGA $\tilde{A}^*(X, O(\mathbb{B}_\rho))$ on $C$ by

$$\tilde{A}^*(X, O(\mathbb{B}_\rho)) := \left(A^*(X, O(\mathbb{B}_\rho)) \otimes_R O(C), uD + vD^c\right),$$

and we denote the differential by $\tilde{D} := uD + vD^c$. Note that the $\boxdot S$-action makes this $S$-equivariant over $C$. Thus $\tilde{A}(X, O(\mathbb{B}_\rho)) \in DGA_{\text{alg}}(R \rtimes S)$, and we define the Hodge filtration by

$$(X^{\rho,\text{Mal}}_x, x) := (R \times C^* \xrightarrow{\phi} (\text{Spec} \tilde{A}(X, O(\mathbb{B}_\rho))) \times_C C^*) \in dg_{\mathbb{Z}} \text{Aff}_{C^*}(R)(S)$$.
making use of the isomorphism $O(\mathbb{B}_p)_x \cong O(R)$.

We then define the mixed Hodge structure $(X_{\text{MHS}}^\rho, x)$ by
\[(A^1 \times R \times C^\ast \to (\text{Spec}(\mathbb{A}(\mathbb{B}_p)), \tau)) \times C^\ast) \in dg_Z \text{Aff}_{\mathbb{A}^1 \times C^\ast}(R)_\ast(\mathbb{G}_m \times S),\]
with $(\text{gr}X_{\text{MHS}}^\rho, 0)$ given by
\[(R \to \text{Spec} H^\ast(X, O(\mathbb{B}_p))) \in dg_Z \text{Aff}(R)_\ast(S).\]

The rest of the proof is now the same as in §5.1, using the principle of two types from [Sim3] Lemmas 2.1 and 2.2. Corollary 2.9 adapts to give the quasi-isomorphism
\[(X_{\text{MHS}}^\rho, x) \times_{C^\ast, \text{row}_1} SL_2 \cong (\text{gr}X_{\text{MHS}}^\rho, 0) \times SL_2,
\]
which gives the splitting.

Observe that this theorem easily adapts to multiple basepoints, as considered in Remark 4.9.

Remark 5.15. Note that the filtration $W$ here and later is not related to the weight tower $W^s F^0$ of [KPT2] §3, which does not agree with the weight filtration of [Mor]. $W^s F^0$ corresponded to the lower central series filtration $G_{n\mathfrak{g}}$ on $\mathfrak{g} := R_n(G(X)^{\text{alg}})$, given by $G_{1\mathfrak{g}} = \mathfrak{g}$ and $G_n\mathfrak{g} = [G_{n-1}\mathfrak{g}, \mathfrak{g}]$, by the formula $W^s F^0 = \mathfrak{g}/G_{n+1}\mathfrak{g}$. Since this is just the filtration $G(\text{Fil})$ coming from the filtration $\text{Fil}_-A^\ast = 0, \text{Fil}_0 A^\ast = R, \text{Fil}_1 A^\ast = A^\ast$ on $A^\ast$, it amounts to setting higher cohomology groups to be pure of weight 1; [KPT2] Proposition 3.2.6(4) follows from this observation, as the graded pieces $\text{gr}^i_W F^0$ defined in [KPT2] Definition 3.2.3 are just $\text{gr}^i_{G(\text{Fil})} \mathfrak{g}$.

**Corollary 5.16.** In the scenario of Theorem 5.14, the homotopy groups $\pi_n(X, \mathbb{B}_p, x)$ for $n \geq 2$, and the Hopf algebra $O(\pi_1(X, \mathbb{B}_p, x))$ carry natural ind-MHS, functorial in $(X, x)$, and compatible with the action of $\pi_1$ on $\pi_n$, the Whitehead bracket and the Hurewicz maps $\pi_n(X, \mathbb{B}_p, x) \to H^n(X, O(\mathbb{B}_p)) \vee$.

Moreover, there are $S$-linear isomorphisms
\[
\pi_n(X, \mathbb{B}_p, x) \cong O(\pi_1(X, \mathbb{B}_p, x) \otimes S \cong O(R \times \pi_1(H^s(X, O(\mathbb{B}_p))) \otimes S
\]
of quasi-MHS. The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\text{gr}^i_W \pi_1(X, \mathbb{B}_p, x) \cong \pi_1(H^s(X, O(\mathbb{B}_p)))$.

Here, $\pi_1(H^s(X, O(\mathbb{B}_p)))$ are the homotopy groups $H_{s-1}(G(H^s(X, O(\mathbb{B}_p))))$ associated to the $R \times S$-equivariant DGA $H^s(X, O(\mathbb{B}_p))$ (as constructed in Definition 3.22), with the induced real Hodge structure.

**Proof.** Theorem 5.14 provides the data required by Theorems 4.20 and 4.21 to construct $S$-split ind-MHS on homotopy groups.

Remark 5.17. If we have a set $T$ of several basepoints, then Remark 4.23 gives $S$-split ind-MHS on the algebras $O(\pi_1(X, x, y)\rho, \mathbb{B}_p)$, compatible with the pro-algebraic groupoid structure. The $S$-split ind-MHS on $(\pi_n(X, x, y)\rho, \mathbb{B}_p)\vee$ are then compatible with the co-action
\[(\pi_n(X, x, y)\rho, \mathbb{B}_p)\vee \to O(\pi_1(X, x, y)\rho, \mathbb{B}_p) \otimes (\pi_n(X, x, y)\rho, \mathbb{B}_p)\vee.
\]

**Remark 5.18.** Corollary 5.16 confirms the first part of [Ara] Conjecture 5.5. If $V$ is a $k$-variation of Hodge structure on $X$, for a field $k < \mathbb{R}$, and $R$ is the Zariski closure of $\pi_1(X, x) \to \text{GL}(V_x)$, the conjecture states that there is a natural ind-$k$-MHS on the $k$-Hopf algebra $O(\pi_1(X, x, y)\rho, \mathbb{B}_p)$. Applying Corollary 5.16 to the Zariski-dense real representation $\rho \mathbb{R} : \pi_1(X, x) \to R(\mathbb{R})$ gives a real ind-MHS on the real Hopf algebra $O(\pi_1(X, x, y)\rho, \mathbb{B}_p) = O(\pi_1(X, x, y)\rho, \mathbb{B}_p) \otimes_k \mathbb{R}$. The weight filtration is just given
by the lower central series on the pro-unipotent radical, so descends to \( k \), giving an \( \text{ind}-k \)-MHS on the \( k \)-Hopf algebra \( \varpi_1(\pi^\text{Mal}_1, x) \).

If \( V \) is a variation of Hodge structure on \( X \), and \( R = \text{GL}(V_x) \), then Corollary 5.16 recovers the \( \text{ind-MHS} \) on \( \varpi_1(\pi^\text{Mal}_1, x) \) first described in [Hai4] Theorem 13.1. If \( T \) is a set of basepoints, and \( R \) is the algebraic groupoid \( R(x, y) = \text{Iso}(V_x, V_y) \) on objects \( T \), then Remark 5.17 recovers the \( \text{ind-MHS} \) on \( \varpi_1(\pi^\text{Mal}_1; T) \) first described in [Hai4] Theorem 13.3.

**Corollary 5.19.** If \( \pi_1(X, x) \) is algebraically good with respect to \( R \) and the homotopy groups \( \pi_n(X, x) \) have finite rank for all \( n \geq 2 \), with each \( \pi_1(X, x) \)-representation \( \pi_n(X, x) \otimes \mathbb{Z} \mathbb{R} \) an extension of \( R \)-representations, then Theorem 4.20 gives mixed Hodge structures on \( \pi_n(X, x) \otimes \mathbb{R} \) for all \( n \geq 2 \), by Theorem 3.16.

Before stating the next proposition, we need to observe that for any morphism \( f : X \to Y \) of compact Kähler manifolds, the induced map \( \pi_1(X, x) \to \pi_1(Y, f(x)) \) gives rise to a map \( \varpi_1(X, x)^\text{red} \to \varpi_1(Y, f(x)^\text{red} \) of reductive pro-algebraic fundamental groups. This is not true for arbitrary topological spaces, but holds in this case because semisimplicity is preserved by pullbacks between compact Kähler manifolds, since Higgs bundles pull back to Higgs bundles.

**Proposition 5.20.** If we have a morphism \( f : X \to Y \) of compact Kähler manifolds, and a commutative diagram

\[
\begin{array}{ccc}
\pi_1(X, x) & \xrightarrow{f} & \pi_1(Y, f(x)) \\
\rho \downarrow & & \downarrow \theta \\
R & \xrightarrow{\theta} & R'
\end{array}
\]

of groups, with \( R, R' \) real reductive pro-algebraic groups to which the \( U_1^k \)-actions descend and act algebraically, and \( \rho, \theta \) Zariski-dense, then the natural map \( (X^\text{Mal}, x) \to \theta^\sharp(Y^\text{Mal}, f(x)) \) extends to a natural map

\[
(X^\text{MHS}_1, x) \to \theta^\sharp(Y^\text{MHS}_1, f(x))
\]

of algebraic mixed Hodge structures.

**Proof.** This is really just the observation that the construction \( \hat{A}^\bullet(X, V) \) is functorial in \( X \).

Note that, combined with Theorem 3.10, this gives canonical MHS on homotopy types of homotopy fibres.

6. Mixed twistor structures on relative Malcev homotopy types of compact Kähler manifolds

Let \( X \) be a compact Kähler manifold.

**Theorem 6.1.** If \( \rho : (\pi_1(X, x))^\text{red} \to R \) is any quotient, then there is an algebraic mixed twistor structure on the relative Malcev homotopy type \( (X, x)^\text{Mal}_1 \), functorial in \( (X, x) \), which splits on pulling back along row \( 1 : \text{SL}_2 \to C^* \), with the pullback of the splitting over \( 0 \in \mathbb{A}^1 \) given by the opposedness isomorphism.

**Proof.** For \( O(\mathbb{B}_\rho) \) as in Definition 3.31, we define a \( \mathbb{G}_m \)-action on the de Rham complex

\[
\mathfrak{A}^\bullet(X, O(\mathbb{B}_\rho)) = \mathfrak{A}^\bullet(X, \mathbb{R}) \otimes_{\mathbb{R}} O(\mathbb{B}_\rho)
\]

by taking the \( \sim \)-action of \( \mathbb{G}_m \) on \( \mathfrak{A}^\bullet(X, \mathbb{R}) \), acting trivially on \( O(\mathbb{B}_\rho) \).
There is an essentially unique harmonic metric on $O(B)$, and we decompose the connection $D$ as $D = d^+ + \vartheta$ into antisymmetric and symmetric parts, and let $D^c := i \circ d + i \circ \vartheta$. Now, we define the DGA $\tilde{A}(X, O(B))$ on $C$ by
$$\tilde{A}^*(X, O(B)) := (A^*(X, O(B)) \otimes_R O(C), uD + vD^c),$$
and we denote the differential by $\tilde{D} := uD + vD^c$. Note that the $*$-action of $\mathbb{G}_m$ makes this $\mathbb{G}_m$-equivariant over $C$. Thus $\tilde{A}(X, O(B)) \in DGA_{C(R \times \mathbb{G}_m)}$. The construction is now the same as in Theorem 5.14, except that we only have a $\mathbb{G}_m$-action, rather than an $S$-action.

Observe that this theorem easily adapts to multiple basepoints, as considered in Remark 4.9.

**Corollary 6.2.** In the scenario of Theorem 6.1, the homotopy groups $\varpi_n(X^\rho, \text{Mal}, x)$ for $n \geq 2$, and the Hopf algebra $O(\varpi_1(X^\rho, \text{Mal}, x))$ carry natural ind-MTS, functorial in $(X, x)$, and compatible with the action of $\varpi_1$ on $\varpi_n$, the Whitehead bracket and the Hurewicz maps $\varpi_n(X^\rho, \text{Mal}, x) \to H^n(X, O(B))$.

Moreover, there are $S$-linear isomorphisms
$$\varpi_n(X^\rho, \text{Mal}, x) \otimes S \cong \pi_n(H^*(X, O(B))) \otimes S$$
$$O(\varpi_1(X^\rho, \text{Mal}, x)) \otimes S \cong O(R \times \pi_1(H^*(X, O(B)))) \otimes S$$
of quasi-MTS. The associated graded map from the weight filtration is just the pullback of the standard isomorphism $\text{gr} \circ \varpi_1(X^\rho, \text{Mal}) \cong \pi_1(H^*(X, O(B)))$.

**Proof.** Theorem 6.1 provides the data required by Theorems 4.20 and 4.21 to construct $S$-split ind-MTS on homotopy groups. □

### 6.1. Unitary actions.

Although we only have a mixed twistor structure (rather than a mixed Hodge structure) on general Malcev homotopy types, $\varpi_1(X, x)^{\text{red}}$ has a discrete unitary action, as in Lemma 5.7. We will extend this to a discrete unitary action on the mixed twistor structure. On some invariants, this action will become algebraic, and then we have a mixed Hodge structure as in Lemma 1.35.

For the remainder of this section, assume that $R$ is any quotient of $\varpi_1(X, x)^{\text{red}}$ to which the action of the discrete group $U_1^\delta$ descends, but does not necessarily act algebraically, and let $\rho : \pi_1(X, x) \to R$ be the associated representation.

**Proposition 6.3.** The mixed twistor structure $(X^\rho_{\text{MTS}}, x)$ of Theorem 6.1 is equipped with a $U_1^\delta$-action, satisfying the properties of Lemma 1.35 (except algebraicity of the action). Moreover, there is a $U_1^\delta$-action on $\varpi_1(X^\rho_{\text{MTS}}, 0)$, such that the $\mathbb{G}_m \times \mathbb{G}_m$-equivariant splitting
$$A^1 \times \varpi_1(X^\rho_{\text{MTS}}, 0) \times \text{SL}_2 \cong (X^\rho_{\text{MTS}}, x) \times \text{C}^*, \text{row}_1 \text{SL}_2$$
of Theorem 6.1 is also $U_1^\delta$-equivariant.

**Proof.** Since $U_1^\delta$ acts on $R$, it acts on $O(B)$, and we denote this action by $v \mapsto t \otimes v$, for $t \in U_1^\delta$. We may now adapt the proof of Theorem 5.14, defining the $U_1^\delta$-action on $\varpi^*(X, R) \otimes_R O(B)$ by setting $t \otimes (a \otimes v) := (t \circ a) \otimes (t^2 \otimes v)$ for $t \in U_1^\delta$. □

**Remark 6.4.** Note that taking $R = (\pi_1(X, x))^{\text{red}}$ satisfies the conditions of the Proposition. Taking the fibre over $\left(\frac{1}{i}, \frac{1}{i}\right)$ in $\text{SL}_2(R)$ of the $S$-splitting from Theorem 6.1 gives the formality result of [KPT2], namely $X^\rho_{\text{MTS}} \cong X^\rho_{\text{MTS}}/\{1, i\}^*$ since $-id + dc = -2i\bar{\vartheta}$. Now, $(i, 1)$ is not a stable point for the $S$-action, but has stabiliser $1 \times \mathbb{G}_m \subset \mathbb{C}$. In [KPT2], it is effectively shown that this action of $\mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^*$ lifts to a discrete action on $X^\rho_{\text{MTS}}/\{1, i\}$. From...
our algebraic $\mathbb{G}_m$-action and discrete $U_1$-action on $X^\rho_{\text{Mal}}$, we may recover the restriction of this action to $U_1 \subset \mathbb{C}^*$, with $i^2$ acting as the composition of $t \in \mathbb{G}_m(\mathbb{C})$ and $t \in U_1$.

Another type of Hodge structure defined on $X^\rho_{\text{Mal}}$ was the real Hodge structure (i.e. $S$-action) of [Pri1]. This corresponded to taking the fibre of the splitting over $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, giving an isomorphism $X^\rho_{\text{Mal}} \cong \mathbb{R}X^\rho_{\text{MTS}}$, and then considering the $S$-action on the latter. However, that Hodge structure was not in general compatible with the Hodge filtration.

Now, Proposition 6.3 implies that the mixed twistor structures on homotopy groups given in Theorem 4.20 have discrete $U_1$-actions. By Lemma 1.35, we know that this will give a mixed Hodge structure whenever the $U_1$-action is algebraic.

6.1.1. Evaluation maps. For a group $\Gamma$, let $\mathbb{S}(\Gamma)$ denote the category of $\Gamma$-representations in simplicial sets.

**Definition 6.5.** Given $X \in \mathbb{S}(R(A))$, define $C^\bullet(X, \mathcal{O}(R) \otimes A) \in \text{cAlg}(R)$ by

$$C^n(X, \mathcal{O}(R) \otimes A) := \text{Hom}_{R(A)}(X_n, A \otimes \mathcal{O}(R)).$$

**Lemma 6.6.** Given a real algebra $A$, the functor $s\text{Aff}_A(R) \to \mathbb{S}(R(A))$ given by $Y \mapsto Y(A)$ is right Quillen, with left adjoint $X \mapsto \text{Spec} C^\bullet(X, \mathcal{O}(R) \otimes A)$.

**Proof.** This is essentially the same as [Pri3] Lemma 3.52, which takes the case $A = \mathbb{R}$. □

Recall from [GJ] Lemma VI.4.6 that there is a right Quillen equivalence $\text{holim}_{R(A)} : \mathbb{S}(R(A)) \to \mathbb{S} \downarrow \text{BR}(A)$, with left adjoint given by the covering system functor $X \mapsto \tilde{X}$.

**Definition 6.7.** Given $f : X \to \text{BR}(A)$, define

$$C^\bullet(X, \mathcal{O}(\mathbb{B}_f)) := C^\bullet(\tilde{X}, \mathcal{O}(R) \otimes A).$$

**Lemma 6.8.** Given a real algebra $A$, the functor $s\text{Aff}_A(R) \to \mathbb{S} \downarrow \text{BR}(A)$ given by $Y \mapsto \text{holim}_{R(A)} Y(A)$ is right Quillen, with left adjoint

$$(X \xrightarrow{f} \text{BR}(A)) \mapsto \text{Spec} C^\bullet(X, \mathcal{O}(\mathbb{B}_f)).$$

**Proof.** The functor $s\text{Aff}(R) \to \mathbb{S}(R(A))$ given by $Y \mapsto Y(A)$ is right Quillen, with left adjoint as in Lemma 6.6. Composing this right Quillen functor with $\text{holim}_{R(A)}$ gives the right Quillen functor required. □

6.1.2. Analyticity.

**Lemma 6.9.** There is a group homomorphism

$$\sqrt{h} : \pi_1(X, x) \to R(U^\text{an}_1),$$

invariant with respect to the $U^\delta_1$-action given by combining the actions on $R$ and $U^\text{an}_1$, such that $1^* \sqrt{h} = \rho : \pi_1(X, x) \to R(\mathbb{R})$, for $1 : \text{Spec} \mathbb{R} \to U^\text{an}_1$.

**Proof.** This is just the unitary action from Lemma 5.7, given on connections by $\sqrt{h}(t)(d^+, \vartheta) = (d^+, t \circ \vartheta)$, for $t \in U_1$. The analyticity of the isomorphism between de Rham and Betti spaces from [Sim4] then shows that the map $U_1 \times \pi_1(X, x) \to R$ is analytic. For more details, see [Pri4] Proposition 1.5, taking $C = -1 \in U_1$. □

Informally, this gives an analyticity property of the discrete $U_1$-action, and we now wish to show a similar analyticity property for the $U^\delta_1$-action on the mixed twistor structure $(X^\rho_{\text{Mal}}, x)_{\text{MTS}}$ of Proposition 6.3. Recalling that $X_T = X_{\text{MTS}} \times_{R^1} \{1\}$, we want an analytic map

$$(X, x) \times U_1 \to \mathcal{R} \text{holim}(X^\rho_{\text{Mal}}, x)_T$$
over $C^*$. The following is essentially [Pri1] §3.3.2:

**Proposition 6.10.** For the $U^\delta_1$-actions on $\text{gr}X^{\rho,\text{Mal}}_{\text{MTS}}$ of Proposition 6.3 and on $U^\text{an}_1$, there is a $U^\delta_1$-invariant map

$$h \in \text{Hom}_{\text{Ho}(\mathbb{S}_0 \downarrow BR(U^\text{an}_1))}(\text{Sing}(X, x), R \text{holim}(X, x)^{\rho,\text{Mal}}_{T}(U^\text{an}_1_{C^*})),\$$

extending the map $h : X \rightarrow BR(U^\text{an}_1)$ corresponding to the group homomorphism $h : \pi_1(X, x) \rightarrow R(U^\text{an}_1)$ given by $h(t) = \sqrt{t}(t^2)$, for $\sqrt{t}$ as in Lemma 6.9 and $t \in U_1$. Here, $(X^{\rho,\text{Mal}}, x)_{T}(U^\text{an}_1_{C^*}) := \text{Hom}_{C^*}(U^\text{an}_1, (X^{\rho,\text{Mal}}, x)_{T}).$

Moreover, for $1 : \text{Spec } \mathbb{R} \rightarrow U^\text{an}_1$, the map

$$1^* h : \text{Sing}(X, x) \rightarrow (R \text{holim}(X^{\rho,\text{Mal}}, x)^{T}(U^\text{an}_1_{C^*}) \times_{BR(U^\text{an}_1)} BR(\mathbb{R}))$$

in $\text{Ho}(\mathbb{S}_0 \downarrow BR(\mathbb{R}))$ is just the canonical map

$$\text{Sing}(X, x) \rightarrow R \text{holim}(X^{\rho,\text{Mal}}(\mathbb{R}), x).$$

**Proof.** By Lemma 6.8, this is equivalent to giving a $U^\delta_1$-equivariant morphism

$$\text{Spec } C^*(\text{Sing}(X), O(\mathbb{B}_h)) \rightarrow (X^{\rho,\text{Mal}}, x)^{T} \times_{C^*} U^\text{an}_1$$

in $\text{Ho}((R \times U^\text{an}_1) \downarrow \text{Aff}_1(R))$, noting that for the trivial map $f : \{x\} \rightarrow BR(U^\text{an}_1)$, we have $C^*(\{x\}, O(\mathbb{B}_f)) = O(R) \otimes O(U^\text{an}_1)$, so $x \rightarrow X$ gives a map $R \times U^\text{an}_1 \rightarrow \text{Spec } C^*(\text{Sing}(X), O(\mathbb{B}_h))$.

Now, the description of the $U_1$-action in Lemma 6.9 shows that the local system $O(\mathbb{B}_h)$ on $X$ has a resolution given by

$$(\mathcal{O}^*(X, O(\mathbb{B}_p)) \otimes_{\mathbb{R}} O(U^\text{an}_1), d^t + t^{-2} \circ \vartheta),$$

for $t$ the complex co-ordinate on $U_1$, so $C^*(\text{Sing}(X), O(\mathbb{B}_h)) \xrightarrow{\mathcal{A}^*} O(R) \otimes O(U^\text{an}_1)$ is quasi-isomorphic to $E^* \xrightarrow{x} O(R) \otimes O(U^\text{an}_1)$, where

$$E^* := D(A^*(X, O(\mathbb{B}_p)) \otimes_{\mathbb{R}} O(U^\text{an}_1), d^t + t^{-2} \circ \vartheta),$$

for $D$ the denormalisation functor.

Now, $O(U_1)$ is the quotient of $O(S)$ given by $\mathbb{R}[u, v]/(u^2 + v^2 - 1)$, where $t = u + iv$, and then

$$uD + vD^c = t \circ d^t + \bar{t} \circ \vartheta = t \circ (d^t + t^{-2} \circ \vartheta).$$

Thus $t \circ$ gives a $U_1^\delta$-equivariant quasi-isomorphism from $R \times U^\text{an}_1 \xrightarrow{x} \text{Spec } E^*$ to $(X_T, x)^{\rho,\text{Mal}} \times_{C^*} U^\text{an}_1$, as required. 

**Corollary 6.11.** For all $n$, the map $\pi_n(X, x) \times U_1 \rightarrow \varpi_n(X^{\rho,\text{Mal}}, x)_{T}$, given by composing the map $\pi_n(X, x) \rightarrow \varpi_n(X^{\rho,\text{Mal}}, x)$ with the $U^\delta_1$-action on $(X^{\rho,\text{Mal}}, x)_{T}$, is analytic.

**Proof.** Proposition 6.10 gives a $U^\delta_1$-invariant map

$$\pi_n(h) : \pi_n(X, x) \rightarrow \pi_n(R \text{holim}(X^{\rho,\text{Mal}}, x)^{T}(U^\text{an}_1_{C^*})).$$

It therefore suffices to prove that

$$\pi_n(R \text{holim}(X^{\rho,\text{Mal}}, x)^{T}(U^\text{an}_1_{C^*})) = \varpi_n(X^{\rho,\text{Mal}}, x)^{T}(U^\text{an}_1_{C^*}).$$

Observe that the morphism $S \rightarrow C^*$ factors through $\text{row}_1 : \text{SL}_2 \rightarrow C^*$, via the map $S \rightarrow \text{SL}_2$ given by the $S$-action on the identity matrix. This gives us a factorisation
of $U_1^{an} \to C^*$ through $\text{SL}_2$, using the maps $U_1^{an} \to U_1 \subset S$. It thus gives a morphism $U_1^{an} \to \text{Spec} \mathcal{R}O(C^*)$, so the $\text{SL}_2$-splitting of Theorem 6.1 gives an equivalence

$$(X^{\rho,\text{Mal}}, x)_T \times_{\mathbb{R}} U_1^{an} \simeq (\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0) \times U_1^{an}. $$

Similarly, we may pull back the grouplike MTS $G(X, x)^{\rho,\text{Mal}}_{\text{MTS}}$ from Proposition 4.19 to a dg pro-algebraic group over $U_1^{an}$, and the $\text{SL}_2$-splitting then gives us an isomorphism

$$(\varpi_n(X, x)^{\rho,\text{Mal}}_T \times_{C^*} U_1^{an}) \cong \varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0) \times U_1^{an},$$

compatible with the equivalence above.

Thus it remains only to show that

$$\pi_n(\mathcal{R} \text{holim}(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)(U_1^{an})) = \varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)(U_1^{an}).$$

Now, write $\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}} \simeq \tilde{W}N\mathfrak{g}$ under the equivalences of Theorem 3.28, for $\mathfrak{g} \in s\tilde{N}(R)$. By [Pri3] Lemma 3.53, the left-hand side becomes $\pi_n(\tilde{W}(R \ltimes \exp(\mathfrak{g}))(U_1^{an}))$, which is just $\pi_{n-1}((R \ltimes \exp(\mathfrak{g}))(U_1^{an}))$, giving $(R \ltimes \exp(\pi_0 \mathfrak{g}))(U_1^{an})$ for $n = 1$, and $(\pi_{n-1} \mathfrak{g})(U_1^{an})$ for $n \geq 2$. Meanwhile, the right-hand side is $(R \ltimes \exp(H_0 \mathfrak{N} \mathfrak{g}))(U_1^{an})$ for $n = 1$, and $(H_{n-1} \mathfrak{N} \mathfrak{g})(U_1^{an})$ for $n \geq 2$. Thus the required isomorphism follows from the Dold-Kan correspondence.

Hence (for $R$ any quotient of $(\varpi_1(X, x))^{\text{red}}$ to which the $U_1^{\delta}$-action descends), we have:

**Corollary 6.12.** If the group $\varpi_n(X, x)^{\rho,\text{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(X, x)$, then the former carries a natural $S$-split mixed Hodge structure, which extends the mixed twistor structure of Corollary 6.2. This is functorial in $(X, x)$ and compatible with the $G$-action of $\varpi_n$ on $\varpi_n$, the Whitehead bracket, the $R$-action, and the Hurewicz maps $\varpi_n(X, x)^{\rho,\text{Mal}} \to H^n(X, O(\mathbb{B}_p))$.\$

**Proof.** The splittings of Theorem 4.21 and Proposition 6.3 combine with Corollary 6.11 to show that the map

$$\pi_n(X, x) \times U_1 \to \varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)$$

is analytic. Since the splitting also gives an isomorphism $\varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0) \cong \varpi_n(X, x)^{\rho,\text{Mal}}$, we deduce that $\pi_n(X, x)$ spans $\varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)$, so the $U_1$ action on $\varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)$ is analytic.

Since any finite-dimensional analytic $U_1$-action is algebraic, this gives us an algebraic $U_1$-action on $\varpi_n(\text{gr} X^{\rho,\text{Mal}}_{\text{MTS}}, 0)$. Retracing our steps through the splitting isomorphisms, this implies that the $U_1$-action on $\varpi_n(X, x)^{\rho,\text{Mal}}_{\text{MTS}}$ is algebraic. As in Lemma 1.35, this gives an algebraic $G \times S$-action on $\text{row}^* \varpi_n(X^{\rho,\text{Mal}}_{\text{MTS}})$, so we have a mixed Hodge structure. That this is $S$-split follows from Proposition 6.3, since the $S$-splitting of the MTS in Corollary 6.2 is $U_1$-equivariant.

**Remark 6.13.** Observe that if $\pi_1(X, x)$ is algebraically good with respect to $R$ and the homotopy groups $\pi_n(X, x)$ have finite rank for all $n \geq 2$, with the local system $\pi_n(X, -) \otimes_{\mathbb{R}} \mathbb{R}$ an extension of $R$-representations, then Theorem 3.16 implies that $\varpi_n(X^{\rho,\text{Mal}}, x) \cong \pi_n(X, x) \otimes \mathbb{R}$, ensuring that the hypotheses of Corollary 6.12 are satisfied.

7. Variations of mixed Hodge and mixed twistor structures

Fix a compact Kähler manifold $X$.

**Definition 7.1.** Define the sheaf $\mathcal{A}^\bullet(X)$ of DGAs on $X \times C$ by

$$\mathcal{A}^\bullet = (\mathcal{A}^* \otimes_{\mathbb{R}} O(C), ud + vd^c),$$

for co-ordinates $u, v$ as in §1.1. We denote the differential by $\tilde{d} := ud + vd^c$. Note that $\Gamma(X, \mathcal{A}^\bullet) = A^\bullet(X)$, as given in Definition 2.1.
Definition 7.2. Define a real $\mathcal{C}^\infty$ family of mixed Hodge (resp. mixed twistor) structures $\mathcal{E}$ on $X$ to be of a finite locally free $S$-equivariant (resp. $\mathbb{G}_m$-equivariant) $j^{-1}\mathcal{S}^0_X$-sheaf on $X \times \mathbb{C}^*$ equipped with a finite increasing filtration $W_i\mathcal{E}$ by locally free $S$-equivariant (resp. $\mathbb{G}_m$-equivariant) subbundles such that for all $x \in X$, the pullback of $\mathcal{E}$ to $x$ corresponds under Proposition 1.40 to a mixed Hodge structure (resp. corresponds under Corollary 1.48 to a mixed twistor structure).

Lemma 7.3. A (real) variation of mixed Hodge structures (in the sense of [SZ]) on $X$ is equivalent to a real $\mathcal{C}^\infty$ family of mixed twistor structures $\mathcal{E}$ on $X$, equipped with a flat $S$-equivariant $\tilde{\partial}$-connection
\[
\tilde{D} : \mathcal{E} \rightarrow \mathcal{E} \otimes_{j^{-1}\mathcal{S}^0_X} j^{-1}\mathcal{S}^1_X,
\]
compatible with the filtration $W$.

Proof. Given a real VMHS $\mathcal{V}$, we obtain a $\mathcal{C}^\infty$ family $\mathcal{E} := \xi(\mathcal{V} \otimes \mathcal{S}^0, \mathcal{F})$ of mixed Hodge structures (in the notation of Corollary 1.8), and the connection $D : \mathcal{V} \otimes \mathcal{S}^0 \rightarrow \mathcal{V} \otimes \mathcal{S}^1$ gives $\tilde{D} = \xi(D, \mathcal{F})$. $S$-equivariance of $\tilde{D}$ is equivalent to the condition
\[
D : F^p(\mathcal{V} \otimes \mathcal{S}^0 \otimes \mathcal{C}) \rightarrow F^p(\mathcal{V} \otimes \mathcal{S}^0 \otimes \mathcal{C}) \otimes_{\mathcal{S}^0_X \otimes \mathcal{C}} \mathcal{S}^{01} \oplus F^{p+1}(\mathcal{V} \otimes \mathcal{S}^0 \otimes \mathcal{C}) \otimes_{\mathcal{S}^0_X \otimes \mathcal{C}} \mathcal{S}^{10},
\]
corresponding to a Hodge filtration on $\mathcal{V} \otimes \mathcal{O}_X$, with $D : F^p(\mathcal{V} \otimes \mathcal{O}_X) \rightarrow F^{p-1}(\mathcal{V} \otimes \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X$. □

Definition 7.4. Adapting [Sim2] §1 from complex to real structures, we define a (real) variation of mixed twistor structures (or VMTS) on $X$ to consist of a real $\mathcal{C}^\infty$ family of mixed twistor structures $\mathcal{E}$ on $X$, equipped with a flat $\mathbb{G}_m$-equivariant $\tilde{\partial}$-connection
\[
D : \mathcal{E} \rightarrow \mathcal{E} \otimes_{j^{-1}\mathcal{S}^0_X} j^{-1}\mathcal{S}^1_X,
\]
compatible with the filtration $W$.

Definition 7.5. Given an ind-MHS (resp. ind-MTS) structure on a Hopf algebra $O(\Pi)$, define an MHS (resp. MTS) representation of $G$ to consist of an MHS (resp. MTS) $V$, together with a morphism
\[
V \rightarrow V \otimes O(\Pi)
\]
of ind-MHS (resp. ind-MTS), co-associative with respect to the Hopf algebra comultiplication.

Fix a representation $\rho : \pi_1(X, x) \rightarrow R$ as in Theorem 5.14.

Theorem 7.6. For $\varrho : \pi_1(X, x) \rightarrow \text{VHS}X(\pi_1(X, x))$ (resp. $\varrho : \pi_1(X, x) \rightarrow \omega_1(X, x)^{\text{red}}$) the category of MHS (resp. MTS) representations of $\omega_1(X_{\text{Mal}, x})$ is equivalent to the category of real variations of mixed Hodge structure (resp. variations of mixed twistor structure) on $X$. Under this equivalence, the forgetful functor to real MHS (resp. MTS) sends a real VMHS (resp. VMTS) $\mathcal{V}$ to $\mathcal{V}_X$.

For $R$ any quotient of $\text{VHS}X(\pi_1(X, x))$ (resp. $\omega_1(X, x)^{\text{red}}$) and $\rho : \pi_1(X, x) \rightarrow R$, MHS (resp. MTS) representations of $\omega_1(X_{\text{Mal}, x})$ correspond to real VMHS (resp. VMTS) $\mathcal{V}$ whose underlying local systems are extensions of $R$-representations.

Proof. We will prove this for VMHS. The proof for VMTS is almost identical, replacing $S$ with $\mathbb{G}_m$, and Proposition 1.40 with Proposition 1.48.

Given an MHS representation $\psi : \omega_1(X_{\text{Mal}, x}) \rightarrow \text{GL}(V)$ for an MHS $V$, let $S_0V$ be the maximal semisimple subrepresentation of $V$, and define the increasing filtration $S_iV$ inductively by the property that $(S_iV/S_{i-1}V)$ is the maximal semisimple subrepresentation of $V/(S_{i-1}V)$. Then let $\text{GL}^S(V) \leq \text{GL}(V)$ consist of automorphisms respecting the filtration $S$. Then $\psi$ induces a morphism $R \rightarrow \prod_i \text{GL}(\text{gr}_i^S V) = \text{GL}^S(V)^{\text{red}}$, and we set
\[ G = \text{GL}^S(V) \times \text{GL}^S(V)_{\text{red}} \text{R}. \] The Hopf algebra \( O(G) \) then inherits an ind-MHS structure from \( V \), and \( U := \ker(G \to R) \) is the matrix group \( I + S_{-1}\text{End}(V) \).

The \( S \)-splitting of \( \varpi_1(X^{\rho,\text{Mal}}, x) \) gives a section \( R \times \text{Spec} \, S \to \varpi_1(X^{\rho,\text{Mal}}, x) \) compatible with the ind-MHS, which combines with \( \psi \) to give a section \( \sigma_G : R \times \text{Spec} \, S \to G \). As in §4.4, this gives rise to \( \gamma_G \in \gamma^0(S_{-1}\text{End}(V) \otimes S(-1)) \) with \( \sigma_G + N\sigma_G e = \text{ad}_{1+\gamma_G} \circ \sigma_G \). If we set

\[ V' := \ker(\text{id} \otimes N - \gamma_G : V \otimes S \to V \otimes S(-1)), \]

then it follows that \( V' \) is a real \( R \)-representation, with \( V' \to V' \otimes O(R) \) a morphism of quasi-MHS. Since \( \gamma_G \) is nilpotent, \( V \otimes S \cong V' \otimes S \) and \( \text{gr}^S V = \text{gr}^S V' \). Since \( O(\varpi_1(X^{\rho,\text{Mal}}, x)) \) is of non-negative weights, with \( O(\varpi_1(X^{\rho,\text{Mal}}, x)_{\text{red}}) = O(R) \) of weight 0, this also implies \( \text{gr}^W V \cong \text{gr}^W V' \). Thus \( V' \) is a MHS, and \( V' \) is an MHS representation of \( R \).

Proposition 1.40 then associates to \( V' \) a locally free \( \mathcal{O}_{\mathbb{A}^1} \otimes \mathcal{O}_{\mathcal{C}^*} \)-module \( E' \) on \( X \times \mathbb{A}^1 \times C^* \), equipped with a \( \mathbb{G}_m \times S \)-action on \( E' \otimes \mathcal{O}_{\mathcal{X}}^0 \), compatible with the \( \mathcal{O}(\mathbb{A}^1) \otimes \mathcal{O}(\text{SL}_2) \)-multiplication. The fibre at \((1, 1) \in \mathbb{A}^1 \times C^*\) is the local system \( V'' \) associated to the \( R \)-representation \( V' \). If \( D' = \text{id} \otimes d : E' \otimes \mathcal{O}_{\mathcal{X}}^0 \to E' \otimes \mathcal{O}_{\mathcal{X}}^1 \) is the associated connection, then by Proposition 5.12 the element \((s, \lambda) \in \mathbb{G}_m(R) \times S(\mathbb{R})\) sends \( D' \) to \( \frac{1}{\lambda} \otimes D' \).

Proposition 1.40 then gives a MHS on the \( C^\infty \)-family \( V'' := V'' \otimes \mathcal{O}_{\mathcal{X}}^0 \), with \( E' \otimes \mathcal{O}_{\mathcal{X}}^0 = \xi(V'', \text{MHS}) \), and the connection \( D' : V'' \to V'' \otimes \mathcal{O}_{\mathcal{X}}^1 \) preserves \( W \), with

\[
\partial : F^p F^q (V'' \otimes \mathbb{C}) \to \mathcal{O}_{\mathcal{X}}^{10} \otimes F^p F^{q+1} (V'' \otimes \mathbb{C}),
\]

\[
\theta : F^p F^q (V'' \otimes \mathbb{C}) \to \mathcal{O}_{\mathcal{X}}^{10} \otimes F^{p-1} F^{q+1} (V'' \otimes \mathbb{C}).
\]

This implies that the filtration \( F \) descends to \( V'' \otimes \mathcal{O}_{\mathcal{X}}^0 = \ker(\partial + \theta) \), that the connection \( D' \) satisfies Griffiths transversality, and that \( \text{gr}^W V' \) is a VHS. Thus \( V' \) is a semisimple VMHS on \( X \).

We next put a quasi-MHS on the DG Lie algebra

\[ A^*(X, S_{-1}\text{End}(V')), \]

with weight filtration and Hodge filtration given by

\[
W_n A^*(X, S_{-1}\text{End}(V')) = \sum_{i+j=n} \tau^{-i} A^*(X, W_j S_{-1}\text{End}(V'))
\]

\[
F^p A^m (X, S_{-1}\text{End}(V') \otimes \mathbb{C}) = \sum_{i+j=p} A^{i, m-i} (X, F^i S_{-1}\text{End}(V') \otimes \mathbb{C}).
\]

Now, the derivation

\[ \alpha_G = \text{id} \otimes N - [\gamma_G, -] : S_{-1}\text{End}(V) \otimes S \to S_{-1}\text{End}(V) \otimes S(-1) \]

corresponds under the isomorphism \( V \otimes S \cong V' \otimes S \) to the derivation

\[ \text{id} \otimes N : S_{-1}\text{End}(V') \otimes S \to S_{-1}\text{End}(V') \otimes S(-1), \]

so the morphism of Deligne groupoids from Proposition 4.26 is

\[ \mathcal{D}el(W_{0}^0 (A^*(X, S_{-1}\text{End}(V')) \otimes S N \to S(-1))) \to \mathcal{D}el(W_{0}^0 (S_{-1}\text{End}(V') \otimes S N \to S(-1))). \]

Objects of the first groupoid are elements

\[ (\omega, \eta) \in \gamma^0 A^*(X, W_{-1} S_{-1}\text{End}(V') \otimes S) \times \gamma^0 A^0 (X, W_0 S_{-1}\text{End}(V')) \otimes S \]

satisfying \([D', \omega] + \omega^2 = 0, [D' + \omega, \eta] + N \omega = 0\). This is equivalent to giving a \((d + N)\)-connection

\[ D = (\text{id} \otimes d \otimes \text{id}) + (\text{id} \otimes d \otimes N) + \omega \otimes \eta : V' \otimes \mathcal{O}^0 \otimes S \to V' \otimes \mathcal{O}^1 \otimes S \oplus V' \otimes \mathcal{O}^0 \otimes S(-1) \]

with the composite

\[ D^2 : V' \otimes \mathcal{O}^0 \otimes S \to V' \otimes \mathcal{O}^2 \otimes S \oplus V' \otimes \mathcal{O}^1 \otimes S(-1). \]
vanishing. If we let \( V = \ker D \), then this gives \( V \otimes \mathcal{A}^0 \otimes S \cong V' \otimes \mathcal{A}^0 \otimes S \), with \( \text{gr}^W V = \text{gr}^W V' \), so it follows that \( V \) is a VMHS.

A morphism in the second groupoid from \( x^*(\omega, \eta) \) to \( \gamma_G \) is

\[
g \in \operatorname{id} + W_0 \gamma^0(S_{-1} \operatorname{End}(V')) \otimes S
\]

with the property that \( gD_x g^{-1} = \alpha_G + \gamma_g \). Since \( V = \ker(\alpha + \gamma_G : V' \otimes S \to V' \otimes S(-1)) \), this means that \( g \) is an isomorphism \( \nabla_x \to V \) of MHS.

Thus the MHS representation \( V \) gives rise to a VMHS \( \nabla \) equipped with an isomorphism \( \nabla_x \cong V \) of MHS.

Conversely, given a VMHS \( \nabla \) with \( \nabla_x = V \), let \( \nabla' \) be its semisimplification. Since \( \nabla' \) is a semisimple VMHS, the corresponding \( R \)-representation on \( \nabla' = \nabla_x \) is an MHS representation, giving \( \sigma : R \times \text{Spec} S \to \text{GL}(\nabla') \). We may then adapt Proposition 1.25 to get an isomorphism \( \nabla' \otimes \mathcal{A}^0 \otimes S \cong \nabla' \otimes \mathcal{A}^0 \otimes S \) of \( C^\infty \)-families of quasi-MHS, since \( \mathcal{A}^0 \) is flabby. We may therefore consider the difference

\[
D - D' : \nabla' \otimes \mathcal{A}^0 \otimes S \to \nabla' \otimes \mathcal{A}^1 \otimes S \oplus \nabla' \otimes \mathcal{A}^0 \otimes S(-1)
\]

between the \((d + N)\)-connections associated to \( \nabla \) and \( \nabla' \). We may now reverse the argument above to show that this gives an object of the Deligne groupoid, and hence an MHS representation \( \mathcal{W}_1(X_{\rho,\text{Mal}}, x) \to \text{GL}(V) \).

For \( \rho \) as in Theorem 7.6, we now have the following.

**Corollary 7.7.** There is a canonical algebra \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) in ind-VMHS (resp. ind-VMTS) on \( X \times X \), with \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}})_{x,x} = \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}, x) \). This has a comultiplication

\[
pr_{13}^{-1} \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \to pr_{12}^{-1} \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \otimes pr_{23}^{-1} \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}})
\]

on \( X \times X \times X \), a co-identity \( \Delta^{-1} \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \to \mathbb{R} \) on \( X \) (where \( \Delta(x) = (x, x) \)) and a co-inverse \( \tau^{-1} \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \to \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) (where \( \tau(x, y) = (y, x) \)), all of which are morphisms of algebras in ind-VMHS (resp. ind-VMTS).

There are canonical ind-VMHS (resp. ind-VMTS) \( \Pi^n(X_{\rho,\text{Mal}}) \) on \( X \) for all \( n \geq 2 \), with \( \Pi^n(X_{\rho,\text{Mal}})_{x,x} = \mathcal{W}_n(X_{\rho,\text{Mal}}, x) \).

**Proof.** The left and right actions of \( \mathcal{W}_1(X_{\rho,\text{Mal}}, x) \) on itself make \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}, x) \) into an ind-VMHS (resp. ind-VMTS) representation of \( \mathcal{W}_1(X_{\rho,\text{Mal}}, x)^2 \), so it corresponds under Theorem 7.6 to an ind-VMHS (resp. ind-VMTS) \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) with the required properties. Theorem 5.16 makes \( \mathcal{W}_n(X_{\rho,\text{Mal}}, x) \) into an ind-VMHS-representation of \( \mathcal{W}_1(X_{\rho,\text{Mal}}, x) \), giving \( \Pi^n(X_{\rho,\text{Mal}}) \).

Note that for any VMHS (resp. VMTS) \( \nabla \), this means that we have a canonical morphism \( pr_2^{-1} \nabla \to pr_1^{-1} \nabla \otimes \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) of ind-VMHS (resp. ind-VMTS) on \( X \times X \), for \( \rho \) as in Theorem 7.6.

**Remark 7.8.** Using Remarks 4.18 and 5.17, we can adapt Theorem 7.6 to any MHS/MTS representation \( V \) of the groupoid \( \mathcal{W}_1(X_{\rho,\text{Mal}} ; T) \) with several basepoints (i.e. require that \( V(x) \to \mathcal{O}(\mathcal{W}_1(X_{\rho,\text{Mal}} ; x, y)) \otimes V(y) \) be a morphism of ind-MHS/MTS). This gives a VMHS/VMTS \( \nabla \), with canonical isomorphisms \( \nabla_x \cong V(x) \) of MHS/MTS for all \( x \in T \).

Corollary 7.7 then adapts to multiple basepoints, since there is a natural representation of \( \mathcal{W}_1(X_{\rho,\text{Mal}} ; T) \times \mathcal{W}_1(X_{\rho,\text{Mal}} ; T) \) given by \( (x, y) \mapsto \mathcal{O}(\mathcal{W}_1(X_{\rho,\text{Mal}} ; x, y)) \). This gives a canonical Hopf algebra \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) in ind-VMHS/VMTS on \( X \times X \), with \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}, x, y) = \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}, x, y) \) for all \( x, y \in T \). Since this construction is functorial for sets of basepoints, we deduce that this is the VMHS/VMTS \( \mathcal{O}(\mathcal{W}_1 X_{\rho,\text{Mal}}) \) of Corollary 7.7 (which is therefore independent of the basepoint \( x \)). This generalises [Hat4] Corollary 13.11 (which takes \( R = \text{GL}(\nabla_x) \) for a VHS \( \nabla \)).
Likewise, the representation \( x \mapsto \varpi_n(X^{\rho,\text{Mal}}, x)^\vee \) of \( \varpi_1(X^{\rho,\text{Mal}}, T) \) gives an ind-VMHS/VMTS \( \Pi^n(X^{\rho,\text{Mal}}) \) (independent of \( x \)) on \( X \) with \( \Pi^n(X^{\rho,\text{Mal}})_x = \varpi_n(X^{\rho,\text{Mal}}, x)^\vee \), for all \( x \in X \).

**Remark 7.9.** [Ara] introduces a quotient \( \varpi_1(X, x)^{\text{alg}}_k \to \pi_1(X, x)^{\text{hodge}}_k \) over any field \( k \subset \mathbb{R} \), characterised by the property that representations of \( \pi_1(X, x)^{\text{hodge}}_k \) correspond to local systems underlying \( k \)-VMHS on \( X \).

Over any field \( k \subset \mathbb{R} \), there is a pro-algebraic group \( \text{MT}_k \) over \( k \), whose representations correspond to mixed Hodge structures over \( k \). If \( \rho : \varpi_1(X, x)^{\text{alg}}_k \to \text{VHS}_k \varpi_1(X, x)_k \) is the largest quotient of the pro-\( k \)-algebraic completion with the property that the surjection \( \varpi_1(X, x)^{\text{alg}}_k \to \varpi_1(X, x)^{\text{alg}}_k \otimes_k \mathbb{R} \) factors through \( \text{VHS}_k \varpi_1(X, x)_k \), then Theorem 5.16 and Remark 5.18 give an algebraic action of \( \text{MT}_k \) on \( \varpi_1(X, x)^{\rho,\text{Mal}}_k \); with representations of \( \varpi_1(X, x)^{\rho,\text{Mal}}_k \times \text{MT}_k \) being representations of \( \varpi_1(X, x)^{\rho,\text{Mal}}_k \) in \( k \)-MHS. Theorem 7.6 implies that these are precisely \( k \)-VMHS on \( X \), so [Ara] Lemma 2.8 implies that \( \varpi_1(X, x)^{\rho,\text{Mal}}_k = \pi_1(X, x)^{\rho,\text{Mal}}_k \).

For any quotient \( \rho' : \text{VHS}_k \varpi_1(X, x)_k \to R \) (in particular if \( R \) is the image of the monodromy representation of a \( k \)-VHS), Theorem 7.6 then implies that \( \varpi_1(X, x)^{\rho',\text{Mal}}_k \) is a quotient of \( \pi_1(X, x)^{\rho,\text{Mal}}_k \), proving the second part of [Ara] Conjecture 5.5.

Note that this also implies that if \( V \) is a local system on \( X \) whose semisimplification \( V^{ss} \) underlies a VHS, then \( V \) underlies a VMHS (which need not be compatible with the VHS on \( V^{ss} \)).

**Example 7.10.** One application of the ind-VMHS on \( \mathcal{O}(\varpi_1 X^{\rho,\text{Mal}}) \) from Corollary 7.7 is to look at deformations of the representation associated to a VHS \( V \). Explicitly, \( V \) gives representations \( \rho_x : \text{VHS}_k \varpi_1(X, x) \to \text{GL}(V_x) \) for all \( x \in X \), and for any Artinian local \( \mathbb{R} \)-algebra \( A \) with residue field \( \mathbb{R} \), we consider the formal scheme \( F_{\rho_x}(A) \) given by

\[
F_{\rho_x}(A) = \text{Hom}(\pi_1(X, x), \text{GL}(V_x \otimes A)) \times \text{Hom}(\varpi_1(X, x), \text{GL}(V_x)) \{ \rho_x \}.
\]

Now, \( \text{GL}(V_x \otimes A) = \text{GL}(V_x) \times \exp(\mathfrak{gl}(V_x) \otimes \mathfrak{m}(A)) \), where \( \mathfrak{m}(A) \) is the maximal ideal of \( A \). If \( R(x) \) is the image of \( \rho_x \), and \( \rho'_x : \text{VHS}_k \varpi_1(X, x) \to R(x) \) is the induced morphism, then

\[
F_{\rho_x}(A) = \text{Hom}(\varpi_1(X, x)^{\rho'_x,\text{Mal}}, R(x) \times \exp(\mathfrak{gl}(V_x) \otimes \mathfrak{m}(A))_{\rho'_x}.
\]

Thus \( F_{\rho_x} \) is a formal subscheme contained in the germ at 0 of \( O(\varpi_1(X, x)^{\rho'_x,\text{Mal}}) \otimes \mathfrak{gl}(V_x) \), defined by the conditions

\[
f(a \cdot b) = f(a) \star (\text{ad}_{\rho'_x}(a)(f(b)))
\]

for \( a, b \in \varpi_1(X, x)^{\rho'_x,\text{Mal}} \), where \( \star \) is the Campbell-Baker-Hausdorff product \( a \star b = \log(\exp(a) \cdot \exp(b)) \).

Those same conditions define a family \( F(\rho) \) on \( X \) of formal subschemes contained in \( (\Delta^{-1}\mathcal{O}(\varpi_1 X^{\rho,\text{Mal}})) \otimes \mathfrak{gl}(V) \), with \( F(\rho)_x = F_{\rho_x} \). If \( F = \text{Spf} \mathcal{B} \), the VMHS on \( \mathcal{O}(\varpi_1 X^{\rho,\text{Mal}}) \) and \( V \) then give \( \mathcal{B} \) the natural structure of a \( (\text{pro-Artinian algebra in}) \) pro-VMHS. This generalises [ES] to real representations, and also adapts easily to \( S \)-equivariant representations in more general groups than \( \text{GL}_n \). Likewise, if we took \( V \) to be any variation of twistor structures, the same argument would make \( \mathcal{B} \) a pro-VMTS.

### 7.1. Enriching VMTS.

Say we have some quotient \( R \) of \( \varpi_1(X, x)^{\text{red}} \) to which the action of the discrete group \( U_1^\delta \) descends, but does not necessarily act algebraically, and let \( \rho : \pi_1(X, x) \to R \) be the associated representation. Corollary 6.2 puts an ind-MTS on the Hopf algebra \( O(\varpi_1(X, x)^{\rho,\text{Mal}}) \), and Proposition 6.3 puts a \( U_1^\delta \) action on \( \xi(O(\varpi_1(X, x)^{\rho,\text{Mal}}), \text{MTS}) \), satisfying the conditions of Lemma 1.35.
Now take an MHS $V$, and assume that we have an MTS representation $\varpi_1(X, x)^{\rho, \text{Mal}} \to \text{GL}(V)$, with the additional property that the corresponding morphism

$$\xi(V, \text{MHS}) \to \xi(V, \text{MHS}) \otimes \xi(O(\varpi_1(X, x)^{\rho, \text{Mal}}), \text{MTS})$$

of ind-MTS is equivariant for the $U_1^\delta$-action.

Now, $\text{gr}_n^W V$ is an MTS representation of $\text{gr}_0^W \varpi_1(X, x)^{\rho, \text{Mal}} = R$, giving a $U_1^\delta$-equivariant map

$$\text{gr}_n^W V \to \text{gr}_n^W V \otimes O(R).$$

If $\mathbb{V}$ is the local system associated to $V$, then this is equivalent to giving a compatible system of isomorphisms $\text{gr}_n^W \mathbb{V} \cong t \otimes \text{gr}_n^W \mathbb{V}$ for $t \in U_1$. Therefore Proposition 5.12 implies that $\text{gr}_n^W V$ is a representation of $\text{VHS} \varpi_1(X, x)$. Letting $R'$ be the largest common quotient of $R$ and $\text{VHS} \varpi_1(X, x)$, this means that $\text{gr}_n^W V$ is an $R'$-representation, so $V$ is a representation of $\varpi_1(X, x)^{\rho', \text{Mal}}$, for $\rho' : \pi_1(X, x) \to R'$.

Then we have a $U_1$-equivariant morphism

$$\xi(V, \text{MHS}) \to \xi(V, \text{MHS}) \otimes \xi(O(\varpi_1(X, x)^{\rho', \text{Mal}}), \text{MHS})$$

of ind-MTS, noting that $U_1$ now acts algebraically on both sides (using Corollary 4.20), so Lemma 1.35 implies that this is a morphism of ind-MHS, and therefore that $V$ is an MHS representation of $\varpi_1(X, x)^{\rho', \text{Mal}}$. Theorem 7.6 then implies that this amounts to $\mathbb{V}$ being a VMHS on $X$.

Combining this argument with Corollary 6.11 immediately gives:

**Proposition 7.11.** Under the conditions of Corollary 6.12, the local system associated to the $\pi_1(X, x)$-representation $\varpi_n(X, x)^{\rho, \text{Mal}}$ naturally underlies a VMHS, which is independent of the basepoint $x$.

8. **MONODROMY AT THE ARCHIMEDEAN PLACE**

Remark 4.22 shows that the mixed Hodge (resp. mixed twistor) structure on $G(X, x_0)^{\rho, \text{Mal}}$ can be recovered from a nilpotent monodromy operator $\beta : O(R \times \exp(g)) \to O(R \times \exp(g)) \otimes S(-1)$, where $g = G(H^*(X, O(B_\rho))$. In this section, we show how to calculate the monodromy operator in terms of standard operations on the de Rham complex.

**Definition 8.1.** If there is an algebraic action of $U_1$ on the reductive pro-algebraic group $R$, set $S' := S$. Otherwise, set $S' := G_m$. These two cases will correspond to mixed Hodge and mixed twistor structures, respectively.

We now show how to recover $\beta$ explicitly from the formality quasi-isomorphism of Theorem 5.14. By Corollary 4.12, $\beta$ can be regarded as an element of

$$W_{-1}^0 \text{Ext}_H^0(X, O(B_\rho)) \circlearrowleft \text{L}_H^\bullet \text{H}^0(X, O(B_\rho)); \text{H}^0(X, O(B_\rho)) \to O(R)) \otimes O(SL_2)^{-1})^{R \times S'}$$

**Definition 8.2.** Recall that we set $\tilde{D} = uD + vD_c$, and define $\tilde{D}_c := xD + yD_c$, for co-ordinates $(u \; v \; x \; y)$ on $SL_2$. Note that $\tilde{D}_c$ is of type $(0, 0)$ with respect to the $S$-action, while $\tilde{D}_c$ is of type $(1, 1)$.

As in the proof of Theorem 5.14, Corollary 2.9 adapts to give $R \times S'$-equivariant quasi-isomorphisms

$$H^0(X, O(B_\rho)) \otimes O(SL_2) \xrightarrow{Z_{\tilde{D}_c}} Z_{\tilde{D}_c} \to \text{row}^\mathbb{A}_c(X, O(B_\rho))$$

of DGAs, where $Z_{\tilde{D}_c} := \ker(\tilde{D}_c) \cap \text{row}^\mathbb{A}_c$ (so has differential $\tilde{D}$). These are moreover compatible with the augmentation maps to $O(B_\rho)_{x_0} \otimes O(SL_2) = O(R) \otimes O(SL_2)$. 
**Definition 8.3.** For simplicity of exposition, we denote these objects by $\mathcal{H}^*, \mathcal{Z}^*, \mathcal{A}^*$, so the quasi-isomorphisms become

$$\mathcal{H}^* \otimes \text{O}(\text{SL}_2) \xrightarrow{p} \mathcal{Z}^* \xrightarrow{i} \mathcal{A}^*.$$  

We also set $\mathcal{O} := \text{O}(\mathcal{R})$, $\mathcal{H}^* := \mathcal{H}^* \otimes \text{O}(\text{SL}_2)$ and $\mathcal{O} := \mathcal{O} \otimes \text{O}(\text{SL}_2)$.

This gives the following $R \times S'$-equivariant quasi-isomorphisms of Hom-complexes:

$$\text{RHom}_A(\mathbb{L}_A/\text{O}(C), A(-1) \xrightarrow{x_0^*} \mathcal{O}(-1)) \xrightarrow{\iota_*} \text{RHom}_Z(\mathbb{L}_Z/\text{O}(C), A(-1) \xrightarrow{x_0^*} \mathcal{O}(-1))$$

$$\text{RHom}_Z(\mathbb{L}_Z/\text{O}(C), Z(-1) \xrightarrow{\mathcal{O}(-1)} \xrightarrow{p_*} \text{RHom}_Z(\mathbb{L}_Z/\text{O}(C), \mathcal{H}(-1) \xrightarrow{\mathcal{O}(-1)})$$

(Note that, since $\mathcal{H}^0 = \mathbb{R}$ and $\mathcal{Z}^0 = \text{O}(\text{SL}_2)$, in both cases the augmentation maps to $\mathcal{O}$ are independent of the basepoint $x_0$.) The final expression simplifies, as

$$\mathbb{L}_{(\mathcal{H} \otimes \text{O}(\text{SL}_2))/\text{O}(C)} \cong (\mathbb{L}_{\mathcal{H}/\text{R}} \otimes \text{O}(\text{SL}_2)) \oplus (\mathcal{H} \otimes \Omega(\text{SL}_2/C)).$$

The derivation $N : \text{O}(\text{SL}_2) \to \text{O}(\text{SL}_2)(-1)$ has kernel $\mathcal{O}(C)$, so yields an $\mathcal{O}(C)$ derivation $A \to A$, and hence an element

$$(N, 0) \in \text{Hom}_A, R \times S' (\mathbb{L}_A, A(-1) \xrightarrow{x_0^*} \mathcal{O}(-1))^0 \quad \text{with} \quad d(N, 0) = (0, N \circ x_0^*).$$

The chain of quasi-isomorphisms then yields a homotopy-equivalent element $f$ in the final space, and we may choose the homotopies to annihilate $\text{O}(\text{SL}_2)(-1) = \Omega(\text{SL}_2/C) \subset \mathbb{L}_Z$, giving

$$\beta \in \text{RHom}_{\mathcal{H}, R \times S'}(\mathbb{L}_H, \mathcal{H}(-1) \otimes \text{O}(\text{SL}_2) \to \mathcal{O}(-1))^0 \quad \text{with} \quad d\beta = 0,$$

noting that $N \circ x_0^* = 0$ on $\mathcal{H} \subset \mathcal{H} \otimes \text{O}(\text{SL}_2)$, and that $f$ restricted to $\mathcal{H} \otimes \Omega(\text{SL}_2/C)$ is just the identification $\mathcal{H} \otimes \Omega(\text{SL}_2/C) \cong \mathcal{H}(-1) \otimes \text{O}(\text{SL}_2)$.

**8.1. Reformulation via $E_\infty$ derivations.**

**Definition 8.4.** Given a commutative DG algebra $B$ without unit, define $E(B)$ to be the real graded Lie coalgebra $\text{CoLie}(B[1])$ freely cogenerated by $B[1]$. Explicitly, $\text{CoLie}(V) = \bigoplus_{n \geq 1} \text{CoLie}^n(V)$, where $\text{CoLie}^n(V)$ is the quotient of $V^\otimes n$ by the elements

$$\text{sh}_{pq}(v_1 \otimes \ldots \otimes v_n) := \sum_{\sigma \in \text{Sh}(p,q)} \pm v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)},$$

for $p, q > 0$ with $p + q = n$. Here, $\text{Sh}(p,q)$ is the set of $(p,q)$ shuffle permutations, and $\pm$ is the Koszul sign.

$E(B)$ is equipped with a differential $d_{E(B)}$ defined on cogenerators $B[1]$ by

$$(q_B + d_B) : \bigwedge^2 (B[1]) \oplus B[1][-1] \to B[1],$$

where $q_B : \text{Symm}^2 B \to B$ is the product on $B$. Since $d^2_{E(B)} = 0$, this turns $E(B)$ into a differential graded Lie coalgebra.

Freely cogenerated differential graded Lie coalgebras are known as strong homotopy commutative algebras (SHCAs). A choice of cogenerators $V$ for and SHCA $E$ is then known as an $E_\infty$ or $C_\infty$ algebra. For more details, and analogies with $L_\infty$ algebras associated to DGLAs, see [Kon]. Note that when $B$ is concentrated in strictly positive degrees, $E(B)$ is dual to the dg Lie algebra $G(B \oplus \mathbb{R})$ of Definition 3.22.
**Definition 8.5.** The functor $E$ has a left adjoint $O(\bar{W}_+)$, given by $O(\bar{W}_+(C)) := \bigoplus_{n>0} \text{Symm}^n(C[-1])$, with differential as in Definition 3.22. In particular, if $C = \mathfrak{g}^\vee$, for $\mathfrak{g} \in dgVec$, then $\mathbb{R} \oplus O(\bar{W}_+(C)) = O(Wg)$.

For any dg Lie coalgebra $C$, we therefore define $O(\bar{W}C)$ to be the unital dg algebra $\mathbb{R} \oplus O(\bar{W}_+ C)$.

Now, the crucial property of this construction is that $O(\bar{W}_+ E(B))$ is a cofibrant replacement for $B$ in the category of non-unital dg algebras (as follows for instance from the proof of [Pri6] Theorem 4.55, interchanging the roles of Lie and commutative algebras). Therefore for any dg algebra $B$ over $A$, $O(\bar{W} E(B)) \otimes_{O(\bar{W}E(A))} A$ is a cofibrant replacement for $B$ over $A$, so

$$L_{B/A} \simeq \ker(\Omega(O(\bar{W} E(B))) \otimes_{O(\bar{W}E(B))} B \to \Omega(O(\bar{W}E(A))) \otimes_{O(\bar{W}E(A))} B).$$

Thus

$$R\text{Hom}_Z(L_Z, B) \simeq \text{Der}(O(\bar{W}E(Z)) \otimes_{O(\bar{W}E(R))} \mathbb{R}, B),$$

the complex of derivations over $\mathbb{R}$. This in turn is isomorphic to the complex

$$\text{Der}_{E(R)}(E(Z), E(B))$$

of dg Lie coalgebra derivations. The remainder of this section is devoted to constructing explicit homotopy inverses for the equivalences above, thereby deriving the element

$$\beta = (\alpha, \gamma) \in \text{Der}(E(H), E(H) \otimes O(SL_2))^0 \times \text{Der}(E(H), E(R) \otimes O(SL_2))^{-1}$$

required by Remark 4.22, noting that the second term can be rewritten to give $\gamma \in G(H)^0$.

### 8.2. Kähler identities.

By [Sim3] §, we have first-order Kähler identities

$$D^* = -[\Lambda, D^c], \quad (D^c)^* = [\Lambda, D]$$

(noting that our operator $D^c$ differs from Simpson’s by a factor of $-i$), with Laplacian

$$\Delta = [D, D^*] = [D^c, (D^c)^*] = -DAD^c + D^cAD + DD^c\Lambda + \Lambda DD^c.$$

Since $uy - vx = 1$, we also have

$$\Delta = -\hat{D}\Lambda\hat{D} + \hat{D}^c\Lambda\hat{D} + \hat{D}\hat{D}^c\Lambda + \Lambda\hat{D}\hat{D}^c.$$

**Definition 8.6.** Define a semilinear involution $*$ on $O(SL_2 \otimes \mathbb{C}$ by $u^* = y, v^* = -x$. This corresponds to the map $A \mapsto (A^\dagger)^{-1}$ on $SL_2(\mathbb{C})$. The corresponding involution on $S$ is given by $\lambda^* = \bar{\lambda}^{-1}$, for $\lambda \in S(\mathbb{R}) \cong \mathbb{C}^*.$

The calculations above combine to give:

**Lemma 8.7.**

$$\hat{D}^* = -[\Lambda, \hat{D}^c] \quad \hat{D}^c := [\Lambda, \hat{D}].$$

Note that this implies that $\hat{D}\hat{D}^c* + \hat{D}^c*\hat{D} = 0$. Also note that Green’s operator $G$ commutes with $\hat{D}$ and $\hat{D}^c$ as well as with $\Lambda$, and hence with $\hat{D}^*$ and $\hat{D}^c*$.

The working above yields the following.

**Lemma 8.8.**

$$\Delta = [\hat{D}, \hat{D}^*] = [\hat{D}^c, \hat{D}^c*] = -\hat{D}\Lambda\hat{D} + \hat{D}^c\Lambda\hat{D} + \hat{D}\hat{D}^c\Lambda + \Lambda\hat{D}\hat{D}^c.$$
8.3. Monodromy calculation. Given any operation \( f \) on \( A \) or \( Z \), we will simply denote the associated dg Lie coalgebra derivation on \( E(A) \) or \( E(Z) \) by \( f \), so \( d_{E(A)} = \bar{D} + q = d_{E(Z)} \).

Note that the complex \( \mathcal{D}er(C, C) \) of coderivations of a dg Lie coalgebra \( C \) has the natural structure of a DGLA, with bracket \( [f, g] = f \circ g - (-1)^{\deg f \deg g} g \circ f \). When \( C = E(B) \), this DGLA is moreover pro-nilpotent, since \( E(B) = \bigoplus_{n \geq 1} \text{CoLie}^n(B[1]) \), so

\[
\mathcal{D}er(E(B), E(B)) = \lim_{\longrightarrow} \mathcal{D}er\left( \bigoplus_{1 \leq m \leq n} \text{CoLie}^m(B[1]), \bigoplus_{1 \leq m \leq n} \text{CoLie}^m(B[1]) \right).
\]

Since \( \bar{D}^c, \bar{D}^* \) = 0 and

\[
\text{id} = \text{pr}_Z + G \Delta = \text{pr}_Z + G(\bar{D}^c \bar{D}^* + \bar{D}^* \bar{D}^c),
\]

it follows that \( \text{Im}(\bar{D}^*) \) is a subcomplex of \( Z = \text{ker}(\bar{D}^c) \), and \( Z = H \otimes O(SL_2) \oplus \text{Im}((\bar{D}^c)^*) \).

**Definition 8.9.** Decompose \( \text{Im}(\bar{D}^*) \) as \( B \oplus C \), where \( B = \text{ker}(\bar{D}) \cap \text{Im}(\bar{D}^*), \) and \( C \) is its orthogonal complement. Since \( i : Z \rightarrow A \) is a quasi-isomorphism, \( \bar{D} : C \rightarrow B \) is an isomorphism, and we may define \( h_i : A \rightarrow A[-1] \) by \( h_i(z + b + c) = D^{-1}b \in C \), for \( z \in Z, b \in B, c \in C \). Thus \( h_i^2 = 0 \), and id = \( \text{pr}_Z + Dh_i + h_i \bar{D} \). Explicitly,

\[
h_i := G \bar{D}^* \circ (1 - \text{pr}_Z) = G^2 \bar{D}^* \bar{D}^* \bar{D}^c = G^2 \bar{D}^* \bar{D}^c \bar{D}^c \bar{D}^*,
\]

where \( G \) is Green’s operator and \( \text{pr}_Z \) is orthogonal projection onto \( Z \). Since \( \bar{D} \bar{D}^c = D \bar{D}^c \), we can also rewrite this as \( G^2 D^* D^c D^c \).

**Lemma 8.10.** Given a derivation \( f \in \mathcal{D}er(E(Z), E(A))^0 \) with \([q, f] + [\bar{D}, f] = 0\), let

\[
\gamma^i(f) := \sum_{n \geq 0} (-1)^{n+1} h_i \circ [q, h_i]^n \circ (f + h_i \circ [q, f]).
\]

Then \( \gamma^i(f) \in \mathcal{D}er(E(Z), E(A))^{-1} \), and

\[
f + [d_E, \gamma^i(f)] = \text{pr}_Z \circ \left( \sum_{n \geq 0} (-1)^n \circ [q, h_i]^n \circ (f + h_i \circ [q, f]) \right),
\]

so lies in \( \mathcal{D}er(E(Z), E(A))^0 \).

**Proof.** First, observe that \( h_i \) is 0 on \( Z \), so \( g \circ h_i = 0 \) for all \( g \in \mathcal{D}er(E(Z), E(A)) \), and therefore \( h_i \circ g = [h_i, g] \) is a derivation. If we write \( \text{ad}_g^q(h_i) = [q, g] \), then \( \text{ad}_q^q(h_i \circ e) = [q, h_i] \circ e \), for any \( e \in \mathcal{D}er(E(Z), E(A))^0 \) with \([q, e] = 0\). Then

\[
\text{ad}_q^q(h_i \circ \text{ad}_q^q(h_i \circ e)) = [q, h_i \circ [q, h_i] \circ e] = [q, h_i] \circ [q, h_i] \circ e + h_i \circ 1 \cdot [q, h_i] \circ e,
\]

which is just \([q, h_i]^2 \circ e \), since \( q^2 = 0 \) (which amounts to saying that the multiplication on \( A \) is associative), so \( \text{ad}_q^2 = 0 \).

Now,

\[
\text{ad}_q^q(h_i \circ f) = [q, h_i] \circ f = f - h_i \circ [q, f],
\]

and this lies in \( \text{ker}(\text{ad}_q) \). Proceeding inductively, we get

\[
\gamma^i(f) := \sum_{n \geq 0} (-1)^{n+1} (\text{ad}_h^i \text{ad}_q)^n \text{ad}_h^i f,
\]

which is clearly a derivation. Note that the sum is locally finite because the \( n \)th term maps \( \text{CoLie}^n(Z) \) to \( \text{CoLie}^{m-n}(A) \).

Now, let \( y := \sum_{n \geq 0} (-1)^n (\text{ad}_q \text{ad}_h^i)^n f \), so \( \gamma^i(f) = -[h_i, y] = -h_i \circ y \). Set \( f' := f + [d_E, \gamma^i(f)] \); we wish to show that \([\bar{D}, h_i \circ f'] + h_i \circ [\bar{D}, f'] = 0 \). Note that \( f + [q, \gamma^i(f)] = y \), so

\[
f' = f - [q, h_i \circ y] - [\bar{D}, h_i \circ y] = y - [\bar{D}, h_i] \circ y - h_i \circ [\bar{D}, y].
\]
Since $pr_Z = (id - [\hat{D}, h_i])$, it only remains to show that $h_i \circ [\hat{D}, y] = 0$, or equivalently that $h_i \circ [\hat{D}, f'] = 0$.

Now, $0 = [d_E, f'] = [\hat{D}, f'] + [q, f']$. Since $[q, \mathcal{Z}] \subset \mathcal{Z}$, this means that
\[
h_i \circ [\hat{D}, f'] = -h_i \circ [q, h_i \circ [\hat{D}, f']].
\]
Since $h_i \circ \ad_q$ maps CoLie$^n(\mathcal{A}[1])$ to CoLie$^{n-1}(\mathcal{A}[1])$, this means that $h_i \circ [\hat{D}, f'] = 0$, since $h_i \circ [\hat{D}, f'] = (-h_i \circ \ad_q)^n \circ (h_i \circ [\hat{D}, f'])$ for all $n$, and this is 0 on CoLie$^n(\mathcal{A}[1])$.

**Definition 8.11.** On the complex $\mathcal{Z}$, define $h_p := G\hat{D}^*$, noting that this is also isomorphic to $G\hat{D}^c\Lambda$ here.

**Lemma 8.12.** Given a derivation $f \in \text{Der}(E(\mathcal{Z}), E(\mathcal{H}))^0$ with $[q, f] + [\hat{D}, f] = 0$, let
\[
\gamma^p(f) := \sum_{n \geq 0} (-1)^{n+1} (f + [q, f] \circ h_p) \circ [q, h_p]^n \circ h_p.
\]
Then $\gamma^p(f) \in \text{Der}(E(\mathcal{Z}), E(\mathcal{H}))^{-1}$, and
\[
f + [d_E, \gamma^p(f)] = \sum_{n \geq 0} (-1)^n (f + [q, f] \circ h_p) \circ [q, h_p]^n \circ \text{pr}_E,
\]
where $\text{pr}_E$ is orthogonal projection onto harmonic forms. Thus $f + [d_E, \gamma^p(f)]$ lies in $\text{Der}(E(\mathcal{H}), E(\mathcal{H}))^0$.

**Proof.** The proof of Lemma 8.10 carries over, since the section of $p : \mathcal{Z} \to \mathcal{H}$ given by harmonic forms corresponds to a decomposition $\mathcal{Z} = \mathcal{H} \oplus \text{Im}(\hat{D}^c)$. Then $h_p$ makes $p$ into a deformation retract, as $[h_p, \hat{D}] = \text{pr}_\mathcal{H}$ on $\mathcal{Z}$.

**Theorem 8.13.** For $\mathfrak{g} = G(\Pi^*(X, O(\mathcal{E}_p)))$, the monodromy operator
\[
\beta : O(R \times \exp(\mathfrak{g})) \to O(R \times \exp(\mathfrak{g})) \otimes O(\text{SL}_2)(-1)
\]
at infinity, corresponding to the MHS (or MTS) on the homotopy type $(X, x_0)^{\rho, \text{Mal}}$ is given by
\[
\beta = \alpha + \text{ad}_{\gamma_{x_0}}, \text{ where } \alpha : \mathfrak{g}^\vee \to \mathfrak{g}^\vee \otimes O(\text{SL}_2)(-1)
\]
is
\[
\alpha = \sum_{b > 0, a \geq 0} (-1)^{a+b+1} \text{pr}_{\mathcal{H}} \circ ([q, G^2 D^c D^c x] \circ \hat{D}^b) \circ (\hat{D} \circ [q, GA]) \circ (\hat{D}^c \circ [q, GA])^a \circ s
\]
\[
+ \sum_{b > 0, a > 0} (-1)^{a+b} [q, \text{pr}_{\mathcal{H}}] \circ ([q, G^2 D^c D^c x] \circ \hat{D}^b) \circ (\hat{D} \circ [q, GA]) \circ (\hat{D}^c \circ [q, GA])^a \circ s,
\]
for $s : \mathcal{H} \to A$ the inclusion of harmonic forms. Meanwhile, $\gamma_{x_0} \circ s = \sum_{a \geq 0, b \geq 0} (-1)^{a+b} x_0^a \circ h_i \circ ([q, G^2 D^c D^c x] \circ \hat{D}^b) \circ (\hat{D} \circ [q, GA]) \circ (\hat{D}^c \circ [q, GA])^a \circ s$
\[
+ \sum_{a \geq 0, b > 0} (-1)^{a+b} x_0^a \circ ([q, G^2 D^c D^c x] \circ \hat{D}^b) \circ (\hat{D} \circ [q, GA]) \circ (\hat{D}^c \circ [q, GA])^a \circ s.
\]

**Proof.** The derivation $N : A \to A(-1)$ yields a coderivation $N \in \text{Der}(E(\mathcal{Z}), E(A))^0$ with $[q, f] = [\hat{D}, f] = 0$. Lemma 8.10 then gives $\gamma^i(N) \in \text{Der}(E(\mathcal{Z}), E(A))^{-1}$ with $N + [d_E, \gamma^i(N)] \in \text{Der}(E(\mathcal{Z}), E(\mathcal{Z}))^0$. Therefore, in the cone complex $\text{Der}(E(\mathcal{Z}), E(A)) \xrightarrow{\gamma} \text{Der}(E(\mathcal{Z}), E(\mathcal{Q}))$, the derivation $N$ is homotopic to
\[
(N + [d_E, \gamma^i(N)], \gamma^i(N)_{x_0}) \in \text{Der}(E(Z), E(Z))^0 \oplus \text{Der}(E(Z), E(Q))^{-1}.
\]
Explicitly,
\[
\gamma^i(N) = \sum_{n \geq 0} (-1)^{n+1} h_i \circ [q, h_i]^n \circ N
\]
\[
N + [d_E, \gamma^i(N)] = \text{pr}_Z \circ (\sum_{n \geq 0} (-1)^n \circ [q, h_i]^n \circ N).
\]
Setting \( f := N + [d_E, \gamma^i(N)] \), we next apply Lemma 8.12 to the pair \((p \circ f, \gamma^i(N)_{x_0})\). If \( s : \mathcal{H} \to \mathcal{Z} \) denotes the inclusion of harmonic forms, we obtain

\[
\begin{align*}
\alpha \circ s &= p \circ f + [d_E, \gamma^p(p \circ f)], \\
\gamma_{x_0} \circ s &= \gamma^i(N)_{x_0} + \gamma^p(p \circ f)_{x_0} + [d_E, \gamma^p(\gamma^i(N)_{x_0}) + \gamma^p(p \circ f)_{x_0}].
\end{align*}
\]

Now,

\[
\alpha = \sum_{m \geq 0} (-1)^m (p \circ f + [q, p \circ f] \circ h_p) \circ [q, h_p]^m \circ s
\]

\[
= \sum_{m \geq 0, n \geq 0} (-1)^{m+n} \text{pr}_\mathcal{H} \circ [q, h_i]^n \circ N \circ [q, h_p]^m \circ s
\]

\[
+ \sum_{m \geq 0, n \geq 0} (-1)^{m+n} [q, \text{pr}_\mathcal{H}] \circ [q, h_i]^n \circ N \circ h_p \circ [q, h_p]^m \circ s
\]

since \( p \circ \text{pr}_\mathcal{Z} = \text{pr}_\mathcal{H}, [q, N] = 0 \) and \( \text{ad}_q^2 = 0 \).

Now, \( N \circ g = [N, g] + g \circ N \), but \( N = 0 \) on \( \mathcal{H} \subset \mathcal{H} \), while \( [N, s] = 0 \) (since \( s \) is \( SL_2 \)-linear).

Since \( h_i = G^2 D^* D^c \), \( h_p = D^c \circ [q, GA] \) and \( [q, D^c] = 0 \), we get \( [q, h_i] = [q, G^2 D^* D^c] \circ D^c \)

\[
\text{and } [q, h_p] = \tilde{D} \circ [q, GA].
\]

In particular, this implies that \( [q, h_i] \circ D^c = 0 \) and that \( [N, [q, h_p]] = \tilde{D} \circ [q, GA], \) since \([N, GA] = [N, q] = 0\).

Thus

\[
\alpha = \sum_{n, a, c \geq 0} (-1)^{n+a+c+1} \text{pr}_\mathcal{H} \circ [q, h_i]^n \circ [q, h_p]^c \circ (\tilde{D} \circ [q, GA]) \circ [q, h_p]^a \circ s
\]

\[
+ \sum_{m, n \geq 0} (-1)^{m+n} [q, \text{pr}_\mathcal{H}] \circ [q, h_i]^n \circ (\tilde{D} \circ GA) \circ [q, h_p]^m \circ s
\]

\[
+ \sum_{n, a, c \geq 0} (-1)^{m+n} [q, \text{pr}_\mathcal{H}] \circ [q, h_i]^n \circ D^c \circ GA \circ [q, h_p]^c \circ (\tilde{D} \circ [q, GA]) \circ [q, h_p]^a \circ s.
\]

When \( n = 0 \), all terms are 0, since \( \text{pr}_\mathcal{H} \circ \tilde{D} = \text{pr}_\mathcal{H} \circ D^c = 0 \), and \( [q, h_p] = \tilde{D} \circ [q, GA] \).

For \( n \neq 0 \), the first sum is 0 whenever \( c \neq 0 \), and the final sum is always 0 (since \( [q, h_i] \circ D^c = 0 \)). If \( m = 0 \), the second sum is also 0, as \( \tilde{D} \circ GA \) equals \( G \tilde{D}^c \) on \( \text{ker}(\tilde{D}) \), so is 0 on \( \mathcal{H} \). Therefore (writing \( b = n \)), we get

\[
\alpha = \sum_{b > 0, a \geq 0} (-1)^{a+b+1} \text{pr}_\mathcal{H} \circ [q, h_i]^b \circ (\tilde{D} \circ [q, GA]) \circ [q, h_p]^a \circ s
\]

\[
+ \sum_{b > 0, a > 0} (-1)^{a+b} [q, \text{pr}_\mathcal{H}] \circ [q, h_i]^b \circ (\tilde{D} \circ GA) \circ [q, h_p]^a \circ s,
\]

and substituting for \([q, h_i] \circ [q, h_p] \) gives the required expression.

Next, we look at \( \gamma_{x_0} \). First, note that \( \Lambda |_{\mathcal{Z}} = 0 \), so \( h_p |_{\mathcal{Z}} = 0 \), and therefore \( h_p \) (and hence \( \gamma^p(p \circ f) \)) restricted to \( \text{CoLie}^\gamma (\mathbb{Z}^1) \) is 0, so \( x_0^p \circ \gamma^p(p \circ f) = 0 \). Thus

\[
\gamma_{x_0} \circ s = \gamma^i(N)_{x_0} + [d_E, \gamma^p(\gamma^i(N)_{x_0})]
\]

\[
= \sum_{m \geq 0} (-1)^m (\gamma^i(N)_{x_0} + [q, \gamma^i(N)_{x_0}] \circ h_p) \circ [q, h_p]^m \circ s
\]

\[
= \sum_{m,n \geq 0} (-1)^{m+n+1} x_0^i \circ h_i \circ [q, h_i]^n \circ N \circ [q, h_p]^m \circ s
\]

\[
+ \sum_{m,n \geq 0} (-1)^{m+n+1} [q, x_0^i \circ h_i] \circ [q, h_i]^n \circ N \circ h_p \circ [q, h_p]^m \circ s.
\]

On restricting to \( \mathcal{H} \subset \mathcal{H} \), we may replace \( N \circ g \) with \([N, g] \) (using the same reasoning as for \( \alpha \)). Now, \([q, h_i]_{n+1} \circ h_p = 0 \), and on expanding out \( \tilde{D}^c \circ [N, [q, h_p]^m] \), all terms but
one vanish, giving
\[
\gamma_{x_0} \circ s = \sum_{m > n \geq 0} (-1)^{m+n+1} x^*_0 \circ h_i \circ [q, h_i]_n \circ (\tilde{D} \circ [q, GA]) \circ [q, h_p]^{m-1} \circ s + \sum_{m > n \geq 0} (-1)^{m+n+1} x^*_0 \circ [q, h_i]^{n+1} \circ (\tilde{D} \circ GA) \circ [q, h_p]^n \circ s,
\]
which expands out to give the required expression. □

**Remark 8.14.** This implies that the MHS \( O(\varpi_1(X, x_0)^{\rho,\text{Mal}}) \) is just the kernel of
\[
\beta \otimes \text{id} + \text{ad}_{\gamma_{x_0}} \otimes \text{id} + \text{id} \otimes N : O(R \times \exp(H_0(\mathfrak{g}))) \otimes \mathcal{S} \to O(R \times \exp(H_0(\mathfrak{g}))) \otimes \mathcal{S}(-1),
\]
where \( \beta, \gamma_{x_0} \) here denote the restrictions of \( \beta, \gamma_{x_0} \) in Theorem 8.13 to \( \text{Spec} \mathcal{S} = \frac{1}{K_1} 0 \subset \text{SL}_2 \).

Likewise, \( (\varpi_n(X, x_0)^{\rho,\text{Mal}})^{\vee} \) is the kernel of
\[
\beta \otimes \text{id} + \text{ad}_{\gamma_{x_0}} \otimes \text{id} + \text{id} \otimes N : (H_{n-1}\mathfrak{g})^{\vee} \otimes \mathcal{S} \to (H_{n-1}\mathfrak{g})^{\vee} \otimes \mathcal{S}(-1)
\]

**Examples 8.15.** Since \( q \) maps CoLie\(^n(\mathcal{H})\) to CoLie\(^{n-1}(\mathcal{H})\), we need only look at the truncations of the sums in Theorem 8.13 to calculate the MHS or MTS on \( G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_m \), where \([K]_1 = K\) and \([K]_{n+1} = [K, [K]_n]\).

1. Since all terms involve \( q \), this means that \( G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_2 \simeq R \ltimes H^0(X, O(\mathbb{B}_\rho))^{\vee} \)\[1\], the equivalence respecting the MHS (or MTS). This just corresponds to the quasi-isomorphism \( s : H^*(X, O(\mathbb{B}_\rho)) \to A^*(X, O(\mathbb{B}_\rho)) \) of cochain complexes, since the ring structure on \( A^*(X, O(\mathbb{B}_\rho)) \) is not needed to recover \( G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_2 \).

2. The first non-trivial case is \( G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_3 \). The only contributions to \( \beta \) here come from terms of degree 1 in \( q \). Thus \( \alpha \) vanishes on this quotient, which means that the obstruction to splitting the MHS is a unipotent inner automorphism.

The element \( \gamma_{x_0} \) becomes
\[
x^*_0 \circ h_i \circ \tilde{D} \circ [q, GA] \circ s = x^*_0 \circ G^2 D^* D^{ce} \tilde{D} \circ [q, GA] \circ s = x^*_0 \circ G^2 D^* D^{ce} D \circ [q, GA] \circ s,
\]
which we can rewrite as \( x^*_0 \circ \text{pr}_{\text{Im}(D^* D^{ce})} \circ [q, GA] \circ s \), where \( \text{pr}_{\text{Im}(D^* D^{ce})} \) is orthogonal projection onto \( \text{Im}(D^* D^{ce}) \). Explicitly, \( \gamma_{x_0} \in \left( [\mathfrak{g}]_2 / [\mathfrak{g}]_3 \right) \otimes O(\text{SL}_2) \) corresponds to the morphism \( \Lambda^2 \mathcal{H}_1 \to O(R) \otimes O(\text{SL}_2) \) given by
\[
v \otimes w \mapsto (\text{pr}_{\text{Im}(D^* D^{ce})} GA(s(v) \wedge s(w)))_{x_0},
\]
since \( A|_{\mathcal{H}_1} = 0 \).

Since \( [\mathfrak{g}]_2 / [\mathfrak{g}]_3 \) lies in the centre of \( \mathfrak{g} / [\mathfrak{g}]_3 \), this means that \( \text{ad}_{\gamma_{x_0}} \) acts trivially on \( \text{Ru}(G(X, x_0)^{\rho,\text{Mal}})/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_3 \), so \( G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_3 \) is an extension
\[
1 \to \text{Ru}(G(X, x_0)^{\rho,\text{Mal}})/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_3 \to G(X, x_0)^{\rho,\text{Mal}}/[\text{Ru} G(X, x_0)^{\rho,\text{Mal}}]_3 \to R \to 1
\]
of split MHS. Thus \( \gamma_{x_0} \) is the obstruction to any Levi decomposition respecting the MHS, and allowing \( x_0 \) to vary gives us the associated VMHS on \( X \).

In particular, taking \( R = 1 \), the MHS on \( G(X, x_0)^{1,\text{Mal}}/[G(X, x_0)^{1,\text{Mal}}]_3 \) is split, and specialising further to the case when \( X \) is simply connected,
\[
(\pi_3(X, x_0) \otimes \mathbb{R})^\vee \simeq H^3(X, \mathbb{R}) \oplus \ker(\text{Symm}^2 H^2(X, \mathbb{R}) \to H^4(X, \mathbb{R}))
\]
is an isomorphism of real MHS. This shows that the phenomena in [CCM] are entirely due to the lattice \( \pi_3(X, x_0) \) in \( \pi_3(X, x_0) \otimes \mathbb{R} \).
(3) The first case in which $\alpha$ is non-trivial is $R_u(G(X, x_0)^{R,\text{Mal}})/[R_uG(X, x_0)^{R,\text{Mal}}]_4$. We then have

$$\alpha = \text{pr}_H \circ [q, G^2D^*D^*] \circ D^cD \circ [q, GA] \circ s$$

and this determines the MHS on $G(X, x_0)^{R,\text{Mal}}$ up to pro-unipotent inner automorphism. In particular, if $X$ is simply connected, this determines the MHS on $\pi_4(X, x_0) \otimes \mathbb{R}$ as follows.

Let $V := \text{CoLie}^3(H^2(X, \mathbb{R})[1])[-2]$, i.e. the quotient of $H^2(X, \mathbb{R})^{\otimes 3}$ by the subspace generated by $a \otimes b \otimes c - a \otimes c \otimes b + c \otimes a \otimes b$ and $a \otimes b \otimes c - b \otimes a \otimes c + b \otimes c \otimes a$, then set $K$ to be the kernel of the map $q : V \to H^4(X, \mathbb{R}) \otimes H^2(X, \mathbb{R})$ given by $q(a \otimes b \otimes c) = (a \cup b) \otimes c - (b \cup c) \otimes a$. If we let $C := \text{coker}(\text{Symm}^2H^2(X, \mathbb{R}) \mathcal{U} H^4(X, \mathbb{R}))$ and $L := \ker(H^2(X, \mathbb{R}) \otimes H^3(X, \mathbb{R}) \mathcal{U} H^5(X, \mathbb{R}))$, then

$$\text{gr}^W(\pi_4(X) \otimes \mathbb{R})^\vee \cong C \oplus L \oplus K.$$

The MHS is then determined by $\alpha : K \to C(-1)$, corresponding to the restriction to $K$ of the map $\alpha' : V \to C(-1)$ given by setting $\alpha'(a \otimes b \otimes c)$ to be

$$\begin{align*}
\text{pr}_H(\text{pr}_I((GA\tilde{a}) \wedge \tilde{b}) \wedge \tilde{c}) &- \text{pr}_H(\text{pr}_I(GA\tilde{a}) \wedge (\tilde{b} \wedge \tilde{c})) \\
- \text{pr}_H(\text{pr}_I(\tilde{a} \wedge (G\tilde{b})) \wedge \tilde{c}) &- \text{pr}_H(\tilde{a} \wedge \text{pr}_I((G\tilde{b}) \wedge \tilde{c})) \\
- \text{pr}_H(\tilde{a} \wedge \tilde{b} \wedge (G\tilde{a} \wedge \tilde{c})) &+ \text{pr}_H(\tilde{a} \wedge \text{pr}_I(\tilde{b} \wedge (G\tilde{a} \wedge \tilde{c})))
\end{align*}$$

where $\tilde{a} := sa$, for $s$ the identification of cohomology with harmonic forms, while $\text{pr}_I$ and $\text{pr}_H$ are orthogonal projection onto $\text{Im}(d^*d^*)$ and harmonic forms, respectively.

Explicitly, the MHS $(\pi_4(X) \otimes \mathbb{R})^\vee$ is then given by the subspace

$$(c - x\alpha(k), l, k) \subset (C \oplus L \oplus K) \otimes \mathcal{S},$$

for $c \in C$, $l \in L$ and $k \in K$, with $\mathcal{S}$ the quasi-MHS of Lemma 1.18.

9. Simplicial and singular varieties

In this section, we will show how the techniques of cohomological descent allow us to extend real mixed Hodge and twistor structures to all proper complex varieties. By [SD] Remark 4.1.10, the method of [Gro1] §9 shows that a surjective proper morphism of topological spaces is universally of effective cohomological descent.

**Lemma 9.1.** If $f : X \to Y$ is a map of compactly generated Hausdorff topological spaces inducing an equivalence on fundamental groupoids, such that $R^if_*\mathcal{V} = 0$ for all local systems $\mathcal{V}$ on $X$ and all $i > 0$, then $f$ is a weak equivalence.

**Proof.** Without loss of generality, we may assume that $X$ and $Y$ are path-connected. If $\tilde{X} \overset{\pi}{\to} X, \tilde{Y} \overset{\pi'}{\to} Y$ are the universal covering spaces of $X, Y$, then it will suffice to show that $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a weak equivalence, since the fundamental groups are isomorphic.

As $\tilde{X}, \tilde{Y}$ are simply connected, it suffices to show that $R^i\tilde{f}_*\mathcal{Z} = 0$ for all $i > 0$. By the Leray-Serre spectral sequence, $R^i\pi_*\mathcal{Z} = 0$ for all $i > 0$, and similarly for $Y$. The result now follows from the observation that $\pi_*\mathcal{Z}$ is a local system on $X$. \hfill $\square$

**Proposition 9.2.** If $a : X_\bullet \to X$ is a morphism (of simplicial topological spaces) of effective cohomological descent, then $|a| : |X_\bullet| \to X$ is a weak equivalence, where $|X_\bullet|$ is the geometric realisation of $X_\bullet$. 

Proof. We begin by showing that the fundamental groupoids are equivalent. Since $H^0(|X_\bullet|, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$, we know that $\pi_0|X_\bullet| \cong \pi_0X$, so we may assume that $|X_\bullet|$ and $X$ are both connected.

Now the fundamental groupoid of $|X_\bullet|$ is isomorphic to the fundamental groupoid of the simplicial set diag $\text{Sing}(X_\bullet)$ (the diagonal of the bisimplicial complex given by the singular simplicial sets of the $X_n$). For any group $G$, the groupoid of $G$-torsors on $|X_\bullet|$ is thus equivalent to the groupoid of pairs $(T, \omega)$, where $T$ is a $G$-torsor on $X_0$, and the descent datum $\omega: \partial_0^{-1}T \to \partial_1^{-1}T$ is a morphism of $G$-torsors satisfying
\[
\partial_2^{-1}\omega \circ \partial_0^{-1}\omega = \partial_1^{-1}\omega, \quad \sigma_0^{-1}\omega = 1.
\]
Since $a$ is effective, this groupoid is equivalent to the groupoid of $G$-torsors on $X$, so the fundamental groups are isomorphic.

Given a local system $\mathbb{V}$ on $|X_\bullet|$, there is a corresponding $\text{GL}(\mathbb{V})$-torsor $T$, which therefore descends to $X$. Since $\mathbb{V} = T \times \text{GL}(\mathbb{V}) \mathbb{V}$ and $T = |a|^{-1}|a|_{\mathbb{S}}T$, we can deduce that $\mathbb{V} = |a|^{-1}|a|_{\mathbb{S}}\mathbb{V}$, so $\mathbb{R}^i|a|_{\mathbb{S}}\mathbb{V} = 0$ for all $i > 0$, as $a$ is of effective cohomological descent. Thus $|a|$ satisfies the conditions of Lemma 9.1, so is a weak equivalence.

Corollary 9.3. Given a proper complex variety $X$, there exists a smooth proper simplicial variety $X_\bullet$, unique up to homotopy, and a map $a: X_\bullet \to X$, such that $|X_\bullet| \to X$ is a weak equivalence.

In fact, we may take each $X_n$ to be projective, and these resolutions are unique up to simplicial homotopy.

Proof. Apply [Del2] 6.2.8, 6.4.4 and §8.2.

9.1. Semisimple local systems. From now on, $X_\bullet$ will be a fixed simplicial proper complex variety (a fortiori, this allows us to consider any proper complex variety).

In this section, we will define the real holomorphic $U_1$-action on a suitable quotient of the real reductive pro-algebraic fundamental group $\varpi_1(|X_\bullet|, x)^\text{red}$.

Recall that a local system on a simplicial complex $X_\bullet$ of topological spaces is equivalent to the category of pairs $(\mathbb{V}, \alpha)$, where $\mathbb{V}$ is a local system on $X_0$, and $\alpha: \partial_0^{-1}\mathbb{V} \to \partial_1^{-1}\mathbb{V}$ is an isomorphism of local systems satisfying
\[
\partial_2^{-1}\alpha \circ \partial_0^{-1}\alpha = \partial_1^{-1}\alpha, \quad \sigma_0^{-1}\alpha = 1.
\]

Definition 9.4. Given a simplicial complex $X_\bullet$ of smooth proper varieties and a point $x \in X_0$, define the fundamental group $\varpi_1(|X_\bullet|, x)^\text{norm}$ to be the quotient of $\varpi_1(|X_\bullet|, x)$ by the normal subgroup generated by the image of $R_0\varpi_1(X_0, x)$. We call its representations normally semisimple local systems on $|X_\bullet|$—these correspond to local systems $\mathbb{W}$ (on the connected component of $|X|$ containing $x$) for which $a_0^{-1}\mathbb{W}$ is semisimple, for $a_0: X_0 \to |X_\bullet|$.

Then define $\varpi_1(|X_\bullet|, x)^\text{norm,red}$ to be the reductive quotient of $\varpi_1(|X_\bullet|, x)^\text{norm}$. Its representations are semisimple and normally semisimple local systems on the connected component of $|X|$ containing $x$.

Lemma 9.5. If $f: X_\bullet \to Y_\bullet$ is a homotopy equivalence of simplicial smooth proper varieties, then $\varpi_1(|X_\bullet|, x)^\text{norm} \simeq \varpi_1(|Y_\bullet|, fx)^\text{norm}$.

Proof. Without loss of generality, we may assume that the matching maps
\[X_n \to Y_n \times_{\text{Hom}(|\partial\Delta^n, Y|)} \text{Hom}(|\partial\Delta^n, X)|
\] of $f$ are faithfully flat and proper for all $n \geq 0$ (since morphisms of this form generate all homotopy equivalences), and that $|X|$ is connected. Here, $S$ is the category of simplicial sets and $\partial\Delta^n$ is the boundary of $\Delta^n$, with the convention that $\partial\Delta^0 = \emptyset$.

Topological and algebraic effective descent then imply that $f^{-1}$ induces an equivalence on the categories of local systems, and that $f^*$ induces an equivalence on the categories of
quasi-coherent sheaves, and hence on the categories of Higgs bundles. Since representations of \( \varpi_1(|X|, x)^{\text{norm}} \) correspond to objects in the category of Higgs bundles on \( X \), this completes the proof.

**Definition 9.6.** If \( X \to Y \) is any resolution as in Corollary 9.3, with \( x_0 \in X_0 \) mapping to \( x \in X \), we denote the corresponding pro-algebraic group by \( \varpi_1(X, x)^{\text{norm}} := \varpi_1(|X|, x_0)^{\text{norm}} \), noting that this is independent of the choice of \( x_0 \), since \(|X| \) is a weak equivalence.

**Proposition 9.7.** If \( X \) is a proper complex variety with a smooth proper resolution \( a : X \to X \), then normally semisimple local systems on \( X \) correspond to local systems on \( X \) which become semisimple on pulling back to the normalisation \( \pi : X^{\text{norm}} \to X \) of \( X \).

**Proof.** First observe that \( \varpi_1(|X|, x_0)^{\text{norm}} = \varpi_1(X, x_0)/\langle a_0 \mathcal{R}u_1(\varpi_1(X_0, x_0)) \rangle \). Lemma 9.5 ensures that \( \varpi_1(|X|, x_0)^{\text{norm}} \) is independent of the choice of resolution \( X \) of \( X \), so can be defined as \( \varpi_1(X, x_0)/\langle f \mathcal{R}u_1(\varpi_1(Y, y)) \rangle \) for any smooth projective variety \( Y \) and proper faithfully flat \( f : Y \to X \), with \( fy = x \).

Now, since \( X^{\text{norm}} \) is normal, we may make use of an observation on pp. 9–10 of [ABC⁺] (due to M. Ramachandran). \( X^{\text{norm}} \) admits a proper faithfully flat morphism \( g \) from a smooth variety \( Y \) with connected fibres over \( X^{\text{norm}} \). If \( \tilde{x} \in X^{\text{norm}} \) is a point above \( x \in X \), and \( y \in Y \) is a point above \( \tilde{x} \), then this implies the morphism \( \pi_1 g : \pi_1(Y, y) \to \pi_1(X^{\text{norm}}, \tilde{x}) \) is surjective (from the long exact sequence of homotopy), and therefore \( \pi(R_u \varpi_1(Y, y)) = R_u \varpi_1(X^{\text{norm}}, \tilde{x}) \).

Taking \( f : Y \to X \) to be the composition \( Y \to X^{\text{norm}} \to X \), we see that \( \mathcal{R}u_1(\varpi_1(Y, y)) = \pi(R_u \varpi_1(X^{\text{norm}}, \tilde{x})) \). This shows that \( \varpi_1(X, x)^{\text{norm}} = \varpi_1(X, x)/\langle \pi(R_u \varpi_1(X^{\text{norm}}, \tilde{x})) \rangle \), as required.

**Proposition 9.8.** If \( X \) is a simplicial complex of compact Kähler manifolds, then there is a discrete action of the circle group \( U_1 \) on \( \varpi_1(|X|, x)^{\text{norm}} \), such that the composition \( U_1 \times \pi_1(X, x) \to \varpi_1(|X|, x)^{\text{norm}} \) is real analytic. We denote this last map by \( \sqrt{h} : \pi_1(|X|, x) \to \varpi_1(|X|, x)^{\text{norm}}(U_1^{\text{an}}) \).

This also holds if we replace \( X \) with any proper complex variety \( X \).

**Proof.** The key observation is that the \( U_1 \)-action defined in [Sim3] is functorial in \( X \), and that semisimplicity is preserved by pullbacks between compact Kähler manifolds (since Higgs bundles pull back to Higgs bundles), so there is a canonical isomorphism \( (\partial_t^{-1} V) \cong \partial_t^{-1}(tV) \) for \( t \in U_1 \); thus it makes sense for us to define

\[
\tau(V, \alpha) := (tV, \tau(\alpha)),
\]

whenever \( V \) is semisimple on \( X_0 \).

If \( \mathcal{C} \) is the category of finite-dimensional real local systems on \( X \), this defines a \( U_1 \)-action on the full subcategory \( \mathcal{C}' \subset \mathcal{C} \) consisting of those local systems \( \mathcal{V} \) on \( X \) whose restrictions to \( X_0 \) (or equivalently to all \( X_n \)) are semisimple. Now, the category of \( \varpi_1(|X|, x)^{\text{norm}} \)-representations is equivalent to \( \mathcal{C}' \) (assuming, without loss of generality, that \(|X| \) is connected). By Tannakian duality, this defines a \( U_1 \)-action on \( \varpi_1(|X|, x)^{\text{norm}} \).

Since \( X_0, X_1 \) are smooth and proper, the actions of \( U_1 \) on their reductive pro-algebraic fundamental groupoids are real analytic by Lemma 6.9, corresponding to maps

\[
\pi_1(X_1; T) \to \varpi_1(X_1; T)^{\text{red}}(U_1^{\text{an}})
\]

The morphisms \( \varpi_1(X_i; a_i^{-1}(x)) \to \varpi_1(|X|, x) \) (coming from \( a_i : X_i \to |X| \)) then give us maps

\[
\pi_1(X_i; a_i^{-1}(x)) \to \varpi_1(|X|, x)^{\text{norm}, \text{red}}(U_1^{\text{an}}),
\]
compatible with the simplicial operations on $X_\bullet$. Since
\[ \pi_1(X_1; a_i^{-1}(x)) \xrightarrow{\partial_0} \pi_1(X_0; a_0^{-1}(x)) \to \pi_1(|X_\bullet|, x) \]
is a coequaliser diagram in the category of groupoids, this gives us a map
\[ \pi_1(|X_\bullet|, x) \to \varpi_1(|X_\bullet|, x)^{\text{norm,red}}(U_1^\text{an}). \]

For the final part, replace a proper complex variety with a simplicial smooth proper resolution, as in Corollary 9.3.

\[ \square \]

9.2. The Malcev homotopy type. Now fix a simplicial complex $X_\bullet$ of compact Kähler manifolds, and take a full and essentially surjective representation $\rho : \varpi_1(|X_\bullet|, x)^{\text{norm,red}} \to R$. As in Definition 3.31, this gives rise to an $R$-torsor $\mathbb{B}_\rho$ on $X$.

**Definition 9.9.** Define the cosimplicial DGAs
\[ A^\bullet(X_\bullet, O(\mathbb{B}_\rho)), H^\bullet(X_\bullet, O(\mathbb{B}_\rho)) \in cDGA(R) \]
by $n \mapsto A^\bullet(X_n, O(\mathbb{B}_\rho))$ and $n \mapsto H^\bullet(X_n, O(\mathbb{B}_\rho))$.

**Definition 9.10.** Given a point $x_0 \in X_0$, define $x_0^! : A^\bullet(X_\bullet, O(\mathbb{B}_\rho)) \to O(R)$ to be given in cosimplicial degree $n$ by $(\sigma_0)^n x_0^! : A^\bullet(X_n, O(\mathbb{B}_\rho)) \to O(\mathbb{B}_\rho)(\sigma_0)^n x_0 \cong O(R)$.

**Lemma 9.11.** The relative Malcev homotopy type $|X_\bullet|^{\rho,\text{Mal}}$ is represented by the morphism
\[ (\text{Th}(A^\bullet(X_\bullet, O(\mathbb{B}_\rho)))) \xrightarrow{x_0^!} O(R) \in \text{Ho}(DGA(R)), \]
where $\text{Th} : cDGA(R) \to DGA(R)$ is the Thom-Sullivan functor (Definition 3.27) mapping cosimplicial DGAs to DG algebras.

**Proof.** This is true for any simplicial complex of manifolds, and follows by combining Propositions 3.28 and 3.34. \[ \square \]

9.3. Mixed Hodge structures. Retaining the hypothesis that $X_\bullet$ is a simplicial proper complex variety, observe that a representation of $\varpi_1(|X_\bullet|, x)^{\text{norm,red}}$ corresponds to a semisimple representation of $X_\bullet$ whose pullbacks to each $X_n$ are all semisimple. This follows because the morphisms $X_n \to X_0$ of compact Kähler manifolds all preserve semisimplicity under pullback, as observed in Proposition 9.8.

**Theorem 9.12.** If $R$ is any quotient of $\varpi_1(|X_\bullet|, x)^{\text{norm,red}}$ (resp. any quotient to which the $U_1^\text{an}$-action of Proposition 9.8 descends and acts algebraically), then there is an algebraic mixed twistor structure (resp. mixed Hodge structure) $(|X_\bullet|, x)^{\rho,\text{Mal}}_{\text{MHS}}$ (resp. $(|X_\bullet|, x)^{\rho,\text{Mal}}_{\text{MHS}}$) on the relative Malcev homotopy type $(|X_\bullet|, x)^{\rho,\text{Mal}}$, where $\rho$ denotes the quotient map.

There is also a $\mathbb{G}_m$-equivariant (resp. $S$-equivariant) splitting
\[ \mathbb{A}^1 \times (\text{gr}(X_\bullet)^{\rho,\text{Mal}}, 0)_{\text{MHS}} \times SL_2 \cong (|X_\bullet|, x)^{\rho,\text{Mal}}_{\text{MHS}} \times \mathbb{R}^{\rho,\text{row}_1} \times SL_2 \]
(resp.
\[ \mathbb{A}^1 \times (\text{gr}(X_\bullet)^{\rho,\text{Mal}}, 0)_{\text{MHS}} \times SL_2 \cong (|X_\bullet|, x)^{\rho,\text{Mal}}_{\text{MHS}} \times \mathbb{R}^{\rho,\text{row}_1} \times SL_2 \]
on pulling back along row_1 : $SL_2 \to C^*$, whose pullback over 0 $\in \mathbb{A}^1$ is given by the opposedness isomorphism.

**Proof.** We define the cosimplicial DGA $\tilde{A}(X_\bullet, O(\mathbb{B}_\rho))$ on $C$ by $n \mapsto \tilde{A}^\bullet(X_n, O(\mathbb{B}_\rho))$, observing that functoriality (similarly to Proposition 5.20) ensures that the simplicial and DGA structures are compatible. This has an augmentation $x^* : \tilde{A}(X_\bullet, O(\mathbb{B}_\rho)) \to O(R) \otimes O(C)$ given in level $n$ by $((\sigma_0)^n x)^*).

We then define the mixed Hodge structure by
\[ |X_\bullet|^{\rho,\text{Mal}}_{\text{MHS}} := (\text{Spec} \, \xi(\tilde{A}(X_\bullet, O(\mathbb{B}_\rho)), \tau_{\tilde{A}})) \times_C C^* \in d\mathbb{g}_{\mathbb{A}^1 \times C^*}(\mathbb{G}_m \times R \times S), \]
Proposition 9.15. Proposition 5.6 adapts to simplicial varieties, showing that our algebraic mixed algebras. Theorem 5.6.4, by applying the Thom-Sullivan functor to pass from cosimplicial to DG \(A^1 \times R \times C^* \rightarrow |X_\bullet|^{\rho,\text{Mal}}_{\text{MHS}}\), giving

\(|X_\bullet, x|^{\rho,\text{Mal}}_{\text{MHS}} \in d\mathbb{Z} \text{Aff}(A^1 \times C^*, (R \times S))\).

The proof of Theorem 5.14 now carries over. For a singular variety \(X\), apply Proposition 9.2 to substitute a simplicial smooth proper variety \(X_\bullet\).

Corollary 9.13. In the scenario of Theorem 9.12, the homotopy groups \(\varpi_n(|X_\bullet|^{\rho,\text{Mal}}, x)\) for \(n \geq 2\), and the Hopf algebra \(O(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x))\) carry natural ind-MTS (resp. ind-MHS), functorial in \((X_\bullet, x)\), and compatible with the action of \(\varpi_1\) on \(\varpi_n\), the Whitehead bracket and the Hurewicz maps \(\varpi_n(|X_\bullet|^{\rho,\text{Mal}}, x) \rightarrow H^n(|X_\bullet|, O(B_\rho))\).

Moreover, there are \(S\)-linear isomorphisms

\[\varpi_n(|X_\bullet|^{\rho,\text{Mal}}, x)^\vee \otimes S \cong \pi_n(\text{Th} H^*(X_\bullet, O(B_\rho)))^\vee \otimes S\]

\[O(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x)) \otimes S \cong O(R \times \pi_1(\text{Th} H^*(X_\bullet, O(B_\rho)))) \otimes S\]

of quasi-MTS (resp. quasi-MHS). The associated graded map from the weight filtration is just the pullback of the standard isomorphism \(\text{gr} W^* \varpi_*(|X_\bullet|^{\rho,\text{Mal}}) \cong \pi_* \text{Th} H^*(X_\bullet, O(B_\rho))\).

Here, \(\pi_*(B)\) are the homotopy groups \(\pi_n(G_1(B))\) associated to the \(R \times S\)-equivariant DG \(A^1 \times C^* \rightarrow \text{Spec} \mathbb{Z} \pi_0(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x))\) as constructed in Definition 3.22, with the induced real twistor (resp. Hodge) structure.

Furthermore, \(W_0 O(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x)) = O(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x)^{\text{norm}})\).

Proof. This is essentially the same as Corollary 5.16. Note that we may simplify the calculation of \(\pi_* \text{Th} H^*(X_\bullet, O(B_\rho))\) by observing that \(\pi_*(C^*) = \pi_* \text{Spec} (DC^*)\), where \(D\) denotes cosimplicial denormalisation, so \(\pi_*(\text{Th} H^*(X_\bullet, O(B_\rho))) = \pi_* \text{Spec} (\text{diag} D^* \text{Th} H^*(X_\bullet, O(B_\rho)))\)

For the final statement, note that representations of \(\text{gr} W^* \varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x) = \text{Spec} W_0 O(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x))\) correspond to representations of \(\varpi_1(|X_\bullet|^{\rho,\text{Mal}}, x)\) which annihilate the image of \(W_1 \varpi_1(X^n, x)\) for all \(n\). Since \(X_n\) is smooth and projective, we just have \(W_1 \varpi_1(X^n, x) = R^n \varpi_1(X^n, x)\), so these are precisely the normally semisimple representations.

Corollary 9.14. If \(\pi_1(|X_\bullet|, x)\) is algebraically good with respect to \(R\) and the homotopy groups \(\pi_n(|X_\bullet|, x)\) have finite rank for all \(n \geq 2\), with \(\pi_n(|X_\bullet|, x) \otimes \mathbb{R}\) an extension of \(R\)-representations, then Corollary 9.13 gives mixed twistor (resp. mixed Hodge) structures on \(\pi_n(|X_\bullet|, x) \otimes \mathbb{R}\) for all \(n \geq 2\), by Theorem 3.16.

Proposition 9.15. When \(R = 1\), the mixed Hodge structures of Corollary 9.12 agree with those defined in [Hai2] Theorem 6.3.1.

Proof. Proposition 5.6 adapts to simplicial varieties, showing that our algebraic mixed Hodge structure on the simplicial variety recovers the mixed Hodge complex of [Hai2] Theorem 5.6.4, by applying the Thom-Sullivan functor to pass from cosimplicial to DG algebras.

Since the reduced bar construction is just our functor \(\tilde{G}\), it follows from Theorem 3.28 that our characterisation of homotopy groups (Definition 3.7) is the same as that given in [Hai2], so our construction of Hodge structures on homotopy groups is essentially the same as [Hai2] Theorem 4.2.1.
9.4. Enriching twistor structures. For the remainder of this section, assume that $R$ is any quotient of $(\pi_1(|X|,x))^{red,norm}_\mathbb{R}$ to which the $U_1^\delta$-action descends, but does not necessarily act algebraically.

**Proposition 9.16.** There is a natural $U_1^\delta$-action on $\gr|X|^\rho_{\text{Mal}}_{\text{MTS}}$, giving a $U_1^\delta$-invariant map

$$h \in \text{Hom}_{\text{Ho}(\mathbb{S} \downarrow BR(U_{1an}^\delta))}(\text{Sing}(|X|,x), \text{R} \text{holim}(|X|,x)_{\rho,\text{Mal}}^{T}(U_{1an}^\delta)_{\text{C}^*})$$

where $(|X|,x)_{\rho,\text{Mal}}^{T}(U_{1an}^\delta)_{\text{C}^*} := \text{Hom}_{\text{C}^*}(U_{1an}^\delta, (|X|,x)^{\rho,\text{Mal}})$.

Moreover, for $1: \text{Spec } \mathbb{R} \rightarrow U_1^\delta$, the map

$$1^*h: \text{Sing}(|X|,x) \rightarrow \text{R} \text{holim}(|X|^{\rho,\text{Mal}},x)_{\text{C}^*} \times_{BR(U_{1an}^\delta)} BR(\mathbb{R})$$

in $\text{Ho}(\mathbb{S} \downarrow BR(\mathbb{R}))$ is just the canonical map

$$\text{Sing}(|X|,x) \rightarrow \text{R} \text{holim}(|X|^{\rho,\text{Mal}},x)(\mathbb{R}).$$

**Proof.** We first note that Proposition 6.3 adapts by functoriality to give a $U_1^\delta$-action on the mixed twistor structure $|X|^{\rho,\text{Mal}}_{\text{MTS}}$ of Theorem 9.12. It also gives a $U_1^\delta$-action on $\gr|X|^{\rho,\text{Mal}}_{\text{MTS}}$, for which the $G_m \times G_m$-equivariant splitting

$$A^1 \times \gr|X|^{\rho,\text{Mal}}_{\text{MTS}} \times SL_2 \cong (|X|^{\rho,\text{Mal}}_{\text{MTS}},x) \times_{C^*,\text{row}_1} SL_2$$

of Theorem 9.12 is also $U_1^\delta$-equivariant.

The proof of Proposition 6.10 also adapts by functoriality, with $h$ above extending the map $h: (|X|,x) \rightarrow BR(U_{1an}^\delta)$ corresponding to the group homomorphism $h: \pi_1(|X|,x) \rightarrow R(U_{1an}^\delta)$ given by $h(t) = \sqrt{h(t^2)}$, for $\sqrt{h}$ as in Proposition 9.8.

Thus (for $R$ any quotient of $\pi_1(|X|,x)^{red,norm}$ to which the $U_1^\delta$-action descends), we have:

**Corollary 9.17.** If the group $\pi_n(|X|^{\rho,\text{Mal}},x)$ is finite-dimensional and spanned by the image of $\pi_n(|X|,x)$, then the former carries a natural mixed Hodge structure, which splits on tensoring with $\mathbb{S}$ and extends the mixed twistor structure of Corollary 9.13. This is functorial in $X$ and compatible with the action of $\pi_1$ on $\omega_n$, the Whitehead bracket, the $R$-action, and the Hurewicz maps $\omega_n(|X|^{\rho,\text{Mal}},x) \rightarrow H^n(|X|,O(\mathbb{R}_\rho))^\vee$.

**Proof.** We first note that Corollary 6.11 adapts to show that for all $n$, the homotopy class of maps $\pi_n(|X|,x) \times U_1 \rightarrow \omega_n(|X|^{\rho,\text{Mal}},x)_{\text{T}}$, given by composing the maps $\pi_n(|X|,x) \rightarrow \omega_n(|X|^{\rho,\text{Mal}},x)$ with the $U_1^\delta$-action on $|(X|^{\rho,\text{Mal}},x)_{\text{T}}$, are analytic. The proof of Corollary 6.12 then carries over to this context.

**Remark 9.18.** Observe that if $\pi_1(|X|,x)$ is algebraically good with respect to $R$ and the homotopy groups $\pi_n(|X|,x)$ have finite rank for all $n \geq 2$, with $\pi_n(|X|,x) \otimes \mathbb{Z} \otimes \mathbb{R}$ an extension of $R$-representations, then Theorem 3.16 implies that $\omega_n(|X|^{\rho,\text{Mal}},x) \cong \pi_n(|X|,x) \otimes \mathbb{R}$, ensuring that the hypotheses of Corollary 6.12 are satisfied.

**References**


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